Regularity of Supersolutions

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These notes were written up after my lectures at Cetraro in June 2009 during the summer course "Regularity estimates for nonlinear elliptic and parabolic problems", organized by C.I.M.E. in Italy. Chapter 7 was written in 2011. New results were incorporated in a revision 2015. The topic is the sub- and supersolutions; they are like the stepchildren of Regularity Theory, since the proper solutions usually get most of the attention. Not now! My objective is the supersolutions of the p-Laplace Equation. The notes are a t o r s o: vital parts are missing. The fascinating story about the p-Laplace equation and its solutions is not told here, the text being focused on supersolutions. Generalizations to other equations are excluded. "Less is more."

1 Introduction

The regularity theory for *solutions* of certain parabolic differential equations of the type

$$\frac{\partial u}{\partial t} = \operatorname{div} \mathbf{A}(x,t,u,\nabla u) \tag{1}$$

is a well developed topic, but when it comes to (semicontinuous) *supersolutions* and *subsolutions* a lot remains to be done. Supersolutions are often auxiliary tools as in the celebrated Perron method, for example, but they are also interesting in their own right. They appear as solutions to obstacle problems and variational inequalities.

As a mnemonic rule

$$\frac{\partial v}{\partial t} \geq \operatorname{div} \mathbf{A}(x,t,v,\nabla v)$$

for smooth supersolutions. We shall restrict our exposition mainly to the basic equations

$$\frac{\partial v}{\partial t} \geq \operatorname{div}(|\nabla v|^{p-2}\nabla v) \qquad \text{and} \qquad \operatorname{div}(|\nabla u|^{p-2}\nabla u) \leq 0,$$

although the methods have a wider applicability. To avoid the splitting in different cases we usually keep $p > 2$.

Our supersolutions are required to be lower semicontinuous but are not assumed to be differentiable in any sense: part of the theory is to prove that they have Sobolev derivatives. If one instead studies weak supersolutions that by definition belong to a Sobolev space, then one has the task to prove that they are semicontinuos. Unfortunately, the weak supersolutions do not form a good closed class under monotone convergence. For *bounded* functions the definitions yield the same class of supersolutions.

The modern theory of viscosity solutions, created by Lions, Crandall, Evans, Ishii, Jensen, and others, relies on the appropriately defined *viscosity supersolutions*, which are merely lower semicontinuous functions by their definition. For second order equations, these are often the same functions as those supersolutions that are encountered in potential theory. The link enables one to study the regularity properties also of the viscosity supersolutions. This is the case for the so-called Evolutionary p-Laplace equation:

$$\frac{\partial v}{\partial t} = \operatorname{div}(|\nabla v|^{p-2}\nabla v).$$

We will restrict our exposition to this equation and we only treat the slow diffusion case $\mathbf{p} > \mathbf{2}$.

To sum up, we shall deal with three different definitions of supersolutions:

- **Weak supersolutions**. They belong to the natural Sobolev space and satisfy the equation in weak form with test functions under the integral sign.

- **Viscosity supersolutions**. The differential inequality is valid at points of contact for test functions touching from below.

- **p-supercaloric functions**. They are defined as in Potential Theory via the comparison principle.

The definitions are given later. As we will see in Chapter 7, the viscosity supersolutions and the p-supercaloric functions are the same functions. Therefore we often use the term "viscosity supersolution" as a label also for p-supercaloric functions.

As an example of what we have in mind, consider the Laplace equation $\Delta u = 0$ and recall that a superharmonic function is a lower semicontinuous function satisfying a comparison principle with respect to the harmonic functions. An analogous definition comes from the super meanvalue property. General superharmonic functions are not differentiable in the classical sense. Nonetheless, the following holds.

Proposition 1 *Suppose that v is a superharmonic function defined in $\mathbb{R}^n$. Then the Sobolev derivative ∇v exists and*

$$\int_{B_R} |\nabla v|^q \, dx < \infty$$

whenever $0 < q < \frac{n}{n-1}$. Moreover,

$$\int_{\mathbb{R}^n} \langle \nabla v, \nabla \eta \rangle \, dx \geq 0$$

for $\eta \geq 0$, $\eta \in C_0^\infty(\mathbb{R}^n)$.

The fundamental solution $v(x) = |x|^{2-n}$ $(= -\log(|x|)$, when $n = 2)$ is a superharmonic function showing that the summability exponent q is sharp. We seize the opportunity to mention that the superharmonic functions are exactly the same as the viscosity supersolutions of the Laplace equation. In other words, *a viscosity supersolution has a gradient in Sobolev's sense.* As an example, the Newtonian potential

$$v(x) = \sum \frac{c_j}{|x - q_j|^{n-2}},$$

where the rational points q_j are numbered and the c_j's are convergence factors, is a superharmonic function, illustrating that functions in the Sobolev space can be infinite in a dense set. —The proof of the proposition follows from Riesz's representation theorem, a classical result according to which we have a harmonic function plus a Newtonian potential. This was about the Laplace equation.

A similar theorem holds for the viscosity supersolutions (= the p-superharmonic functions) of the so-called p-Laplace equation

$$\nabla \cdot \left(|\nabla v|^{p-2} \nabla v \right) = 0$$

but now $0 < q < \frac{n(p-1)}{n-1}$ in the counterpart to Proposition 1. (Strictly speaking, we obtain a proper Sobolev space only for $p > 2 - \frac{1}{n}$, because $q < 1$ otherwise.) The principle of superposition is not valid and, in particular, Riesz's representation theorem is no longer available. The original proof in [L1] was based on the obstacle problem in the calculus of variations and on the so-called weak Harnack inequality. At present, the simplest proof seems to rely upon an approximation with so-called infimal convolutions

$$v_\varepsilon(x) = \inf_y \left\{ v(y) + \frac{|x-y|^2}{2\,\varepsilon} \right\}, \quad \varepsilon > 0.$$

At each point $v_\varepsilon(x) \nearrow v(x)$. They are viscosity supersolutions, if the original v is. Moreover, they are (locally) Lipschitz continuous and hence differentiable a.e. Therefore the approximants v_ε satisfy expedient *a priori* estimates, which, to some extent, can be passed over to the original function v itself.

Another kind of results is related to the *pointwise behaviour*. The viscosity supersolutions are pointwise defined. At *each* point we have

$$v(x) = \operatorname*{ess\,liminf}_{y \to x} v(y)$$

where *essential limes inferior* means that sets of measure zero are neglected in the calculation of the lower limit. In the linear case $p = 2$ the result seems to be due to Brelot, cf. [B]. So much about the p-Laplace equation for now. The theory extends to a wider class of elliptic equations of the type

$$\operatorname{div} \mathbf{A}(x, u, \nabla u) = 0.$$

For parabolic equations like

$$\frac{\partial u}{\partial t} = \sum_{i,j} \frac{\partial}{\partial x_i} \left(\Big| \sum_{k,m} a_{k,m} \frac{\partial u}{\partial x_k} \frac{\partial u}{\partial x_m} \Big|^{\frac{p-2}{2}} a_{i,j} \frac{\partial u}{\partial x_j} \right),$$

where the matrix $(a_{i,j})$ satisfies the ellipticity condition

$$\sum a_{i,j} \xi_i \xi_j \geq \gamma |\xi|^2,$$

the situation is rather similar, although technically much more demanding. Now the use of infimal convolutions as approximants offers considerable simplification, at least in comparison with the original proofs in [KL1]. We will study the Evolutionary p-Laplace Equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \tag{2}$$

where $u = u(x,t)$, restricting ourselves to the slow diffusion case $p > 2$.

We shall encounter two different classes of supersolutions, depending on whether they belong to $L^{p-1}_{loc}(\Omega_T)$ or not. Depending on this, seemingly little distinction, the classes are widely apart. Those that belong to $L^{p-1}_{loc}(\Omega_T)$ have been much studied, since they have many good properties and satisfy a differential equation where the right-hand side is a Riesz measure. The others are less known.

The celebrated Barenblatt solution[1]

$$\mathfrak{B}_p(x,t) = \begin{cases} t^{-n/\lambda}\left[C - \frac{p-2}{p}\lambda^{1/(1-p)}\left(\frac{|x|}{t^{1/\lambda}}\right)^{\frac{p}{p-1}}\right]_+^{\frac{p-1}{p-2}} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases} \tag{3}$$

where $\lambda = n(p-2)+p$, is the leading example of a viscosity supersolution (= p-supercaloric function). It plays the rôle of a fundamental solution, although the Principle of Superposition is naturally lost. It has a compact support in the $x-$variable for each fixed instance t. Disturbances propagate with finite speed and an interface (moving boundary) appears. Notice that

$$\int_0^T \int_{|x|<1} |\nabla \mathfrak{B}_p(x,t)|^p \, dx \, dt = \infty$$

due to the singularity at the origin. Thus $\mathfrak{B}_p$ fails to be a weak supersolution in a domain containing the origin.[2]

[1] "Einen wahren wissenschaftlichen Werth erkenne ich — auf dem Felde der *Mathematik* — nur in concreten mathematischen Wahrheiten, oder schärfer ausgedrückt, 'nur in mathematischen Formeln'. Diese allein sind, wie die Geschichte der Mathematik zeigt, das Unvergängliche. Die verschiedenen Theorien für die Grundlagen der Mathematik (so die von Lagrange) sind von der Zeit weggeweht, aber die Lagrangesche Resolvente ist geblieben!" Kronecker 1884

[2] In my opinion, a definition of "supersolutions" that excludes the fundamental solution cannot be regarded as entirely satisfactory.

A supersolution of a totally different kind is provided by the example

$$\mathfrak{M}(x,t) = \begin{cases} t^{-\frac{1}{p-2}}\mathfrak{U}(x) & \text{if} \quad t > 0, \\ 0, & \text{if} \quad t \leq 0, \end{cases} \tag{4}$$

where the function $\mathfrak{U} > 0$ is the solution to an auxiliary elliptic equation. Here $\mathbf{p} > \mathbf{2}$ (such a separable solution does not exist for $p = 2$). Our main theorem is:

Theorem 2 *Let $p > 2$ and suppose that $v = v(x,t)$ is a viscosity supersolution in the domain Ω_T in $\mathbb{R}^n \times \mathbb{R}$. There are two disjoint cases:*

Class $\mathfrak{B}$: $\quad \mathbf{v} \in \mathbf{L}^{\mathbf{p-2}}_{\mathbf{loc}}(\mathbf{\Omega_T})$. *Then*

$$v \in L^q_{loc}(\Omega_T) \text{ whenever } q < p - 1 + \frac{p}{n}$$

and the Sobolev derivative

$$\nabla v = \left(\frac{\partial v}{\partial x_1}, \cdots, \frac{\partial v}{\partial x_n} \right)$$

exists and

$$\nabla v \in L^q_{loc}(\Omega_T) \text{ whenever } q < p - 1 + \frac{1}{n+1}.$$

The summability exponents are sharp. Moreover,

$$\iint\limits_{\Omega_T} \left(-v\eta_t + \langle |\nabla v|^{p-2}\nabla v, \nabla \eta \rangle \right) dx\, dt \geq 0$$

for all $\eta \geq 0$, $\eta \in C_0^\infty(\Omega_T)$.

Class $\mathfrak{M}$: $\quad \mathbf{v} \notin \mathbf{L}^{\mathbf{p-2}}_{\mathbf{loc}}(\mathbf{\Omega_T})$. *Then there exists a time t_0, $0 < t_0 < T$, such that*

$$\liminf_{\substack{(y,t)\to(x,t_0)\\ t>t_0}} \left(v(y,t)(t-t_0)^{\frac{1}{p-2}} \right) > 0 \qquad \text{for all} \quad x \in \Omega.$$

In particular,

$$\lim_{\substack{(y,t)\to(x,t_0)\\ t>t_0}} v(y,t) \equiv +\infty \qquad \text{for all} \quad x \in \Omega.$$

The occurrence of the void gap

$$[p-2,\, p-1+\tfrac{p}{n})$$

is arresting: either the function belongs to $L_{loc}^{p-1+\frac{p}{n}-0}$ or does not even belong to L_{loc}^{p-2}. The two classes are deliberately labelled after their most representative members. The Barenblatt solution (3) belongs to class $\mathfrak{B}$ and it shows that the exponents are sharp. Notice that the time derivative is not included in the statement. Actually, the time derivative need not be a function, as the example $v(x,t) = 0$, when $t \leq 0$, and $v(x,t) = 1$, when $t > 0$ shows. Dirac's delta appears! It is worth our while to emphasize that the gradient ∇v is not present in the definitions of the viscosity supersolutions and the p-supercaloric functions.

The separable solution (4) belongs to class $\mathfrak{M}$. Notice that at time $t = 0$ it blows totally up, its infinities fill the whole space $\mathbb{R}^n$. A convenient criterion to guarantee that a viscosity supersolution $v : \Omega \times (t_1, t_2) \to (-\infty, \infty]$ is *not* of class $\mathfrak{M}$ is that on the lateral boundary

$$\limsup_{(x,\tau)\to(\xi,t)} v(x,t) < \infty \qquad \text{for all} \quad (\xi,t) \in \partial\Omega \times (t_1,t_2).$$

An important feature is that the viscosity supersolutions are defined at each point, not just almost everywhere in their domain. When it comes to the pointwise behaviour, one may even exclude all future times so that only the instances $\tau < t$ are used for the calculation of $v(x,t)$, as in the next theorem. (It is also, of course, valid without restriction to the past times.)

Theorem 3 *Let $p \geq 2$. A viscosity supersolution of the Evolutionary p-Laplace Equation satisfies*

$$v(x,t) = \operatorname*{ess\,liminf}_{\substack{(y,\tau)\to(x,t)\\ \tau<t}} v(y,\tau)$$

at each interior point (x,t).

In the calculation of *essential limes inferior*, sets of $(n+1)$-dimensional Lebesgue measure zero are neglected. We mention an immediate consequence, which does not seem to be easily obtained by other methods.

Corollary 4 *Two viscosity supersolutions that coincide almost everywhere are equal at* each *point.*

A general comment about the method employed in these notes is appropriate. We do not know about proofs for viscosity supersolutions that would totally stay within that framework. It must be emphasized that the proofs are carried out for those supersolutions that are defined as one does in Potential Theory, namely through comparison principles, and then the results are valid even for the viscosity supersolutions, just because, incidentally, they are the same functions. The identification[3] of these two classes of "supersolutions" is not a quite obvious fact. This limits the applicability of the method.

In passing, we also treat the measure data equation

$$\frac{\partial v}{\partial t} - \nabla \cdot (|\nabla v|^{p-2}\nabla v) = \mu$$

where the right-hand side is a Radon measure. It follows quite easily from Theorem 2 that each viscosity supersolution of class $\mathfrak{B}$ induces a measure and is a solution to the measure data equation. (The reversed problem, which starts with a given measure μ instead of a given function v, is a much investigated topic, cf. [BD].)

Some other equations that are susceptible of this kind of analysis are the Porous Medium Equation[4]

$$\frac{\partial u}{\partial t} = \Delta(|u|^{m-1}u)$$

and

$$\frac{\partial(|u|^{p-2}u)}{\partial t} = \nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right),$$

but it does not seem to be known which equations of the form

$$\frac{\partial u}{\partial t} = F(x, t, u, \nabla u, D^2 u)$$

enjoy the property of having their viscosity supersolutions in some local Sobolev x-space. I hope that this could be a fruitful research topic for the younger readers. —I thank T. Kuusi and M. Parviainen for a careful reading of the manuscript. The first version of these notes has appeared in [L3].

[3]A newer proof based on [JJ] of this fundamental identification is included in these notes in Chapter 7, replacing the reference [JLM].

[4]The Porous Medium Equation is not well suited for the viscosity theory (it is not "proper"), although the comparison principle works well. It is not ∇v but $\nabla(|v|^{m-1}v)$ that is guaranteed to exist.

2 The Stationary Equation

For reasons of exposition[5], we begin with the stationary equation

$$\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \tag{5}$$

which offers some simplifications not present in the time dependent situation. In principle, here we keep $p \geq 2$, although the theory often allows that $1 < p < 2$ at least with minor changes. Moreover, the cases $p > n$, $p = n$, and $p < n$ often require separate proofs. We sometimes skip the borderline case $p = n$.

The fundamental solution

$$c|x|^{\frac{p-n}{p-1}}$$

does not belong to the Sobolev space $W^{1,p}_{loc}(\mathbb{R}^n)$. The problem is the origin. It is good to keep this in mind, when learning the definition below.

Definition 5 *We say that $u \in W^{1,p}_{loc}(\Omega)$ is a* weak solution *in Ω, if*

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla \eta\rangle \, dx = 0$$

*for all $\eta \in C_0^\infty(\Omega)$. If, in addition, u is continuous, it is called a p-*harmonic function*.*

We say that $u \in W^{1,p}_{loc}(\Omega)$ is a weak supersolution *in Ω, if the integral is ≥ 0 for all nonnegative $\eta \in C_0^\infty(\Omega)$. It is a* weak subsolution *if the integral is ≤ 0.*

The terminology suggests that "super $\geq$ sub".

Lemma 6 (Comparison Principle) *Let u be a weak subsolution and v a weak supersolution, and $u, v \in W^{1,p}(\Omega)$. If*

$$\liminf_{x\to\xi} v(x) \geq \limsup_{x\to\xi} u(x) \quad \textit{when} \quad \xi \in \partial\Omega$$

then $v \geq u$ almost everywhere in Ω.

[5] Chapter 3 is pretty independent of the present chapter.

Proof: To see this, choose the test function $\eta = (u+\varepsilon-v)_+$ in the equations for v and u and subtract these:

$$\int_{\{\varepsilon+u>v\}} \langle |\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u, \nabla v - \nabla u\rangle \, dxdy \;\leq\; 0.$$

See Lemma 9. The integrand is strictly positive when $\nabla v \neq \nabla u$, since

$$\langle |b|^{p-2}b - |a|^{p-2}a, b-a\rangle \geq 2^{2-p}|b-a|^p, \quad p \geq 2$$

holds for vectors. The result follows. $\square$

The weak solutions can, in accordance with the elliptic regularity theory, be made continuous after a redefinition in a set of Lebesgue measure zero. The Hölder continuity estimate

$$|u(x)-u(y)| \leq L\,|x-y|^{\alpha} \tag{6}$$

holds when $x, y \in B(x_0, r)$, $B(x_0, 2r) \subset\subset \Omega$; here α depends on n and p while L also depends on the norm $\|u\|_{p,B(x_0,2r)}$. We omit the proof. The continuous weak solutions are called *p-harmonic functions*[6]. In fact, even the gradient is continuous. One has $u \in C^{1,\alpha}_{loc}(\Omega)$, where $\alpha = \alpha(n,p)$. This deep result of N. Ural'tseva will not be needed here. According to [T1] positive solutions obey the Harnack inequality.

Lemma 7 (Harnack's Inequality) *If the p-harmonic function u is non-negative in the ball* $B_{2r} = B(x_0, 2r)$, *then*

$$\max_{\overline{B_r}} u \leq C_{n,p} \min_{\overline{B_r}} u\,.$$

The p-Laplace equation is the *Euler-Lagrange equation* of a variational integral. Let us recall the *Dirichlet problem* in a bounded domain Ω. Let $f \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ represent the boundary values. Then there exists a unique function u in $C(\Omega) \cap W^{1,p}(\Omega)$ such that $u - f \in W^{1,p}_0(\Omega)$ and

$$\int_{\Omega} |\nabla u|^p \, dx \leq \int_{\Omega} |\nabla (u+\eta)|^p \, dx$$

[6]Thus the 2-harmonic functions are the familiar harmonic functions encountered in Potential Theory.

for all $\eta \in C_0^\infty(\Omega)$. The minimizer is p-harmonic. If the boundary $\partial\Omega$ is regular enough, the boundary values are attained in the classical sense:

$$\lim_{x\to\xi} u(x) = f(\xi), \quad \xi \in \Omega.$$

When it comes to the super- and subsolutions, several definitions are currently being used. We need the following ones:

(1) **weak supersolutions** (test functions under the integral sign)

(2) **p-superharmonic functions** (defined via a comparison principle)

(3) **viscosity supersolutions** (test functions evaluated at points of contact)

The p-superharmonic functions and the viscosity supersolutions are exactly the same functions, see Chapter 7, [JJ], or [JLM]. They are not assumed to have any derivatives. In contrast, the weak supersolutions are by their definition required to belong to the Sobolev space $W^{1,p}_{loc}(\Omega)$ and therefore their Caccioppoli estimates are at our disposal. As we will see, locally *bounded* p-superparabolic functions (= viscosity supersolutions) are, indeed, weak supersolutions, having Sobolev derivatives as they should. To this one may add that the weak supersolutions are p-superharmonic functions, provided that the issue of semicontinuity be properly handled.

Definition 8 *We say that a function $v : \Omega \to (-\infty, \infty]$ is* p-superharmonic *in Ω, if*

(i) v is finite in a dense subset

(ii) v is lower semicontinuous

(iii) in each subdomain $D \subset\subset \Omega$ v obeys the comparison principle: if $h \in C(\overline{D})$ is p-harmonic in D, then the implication

$$v|_{\partial D} \geq h|_{\partial D} \quad \Rightarrow \quad v \geq h$$

is valid.

Remarks. For $p = 2$ this is the classical definition of superharmonic functions due to F. Riesz. It is sufficient[7] to assume that $v \not\equiv \infty$ instead

[7]This is not quite that simple in the parabolic case.

of (i). The fundamental solution $|x|^{(p-n)/(p-1)}$, is not a weak supersolution in $\mathbb{R}^n$, merely because it fails to belong to the right Sobolev space, but it is p-superharmonic.

Some examples are the following functions.

$$(n-p)|x|^{-\frac{n-p}{p-1}} \quad (n \neq p), \quad V(x) = \int \frac{\varrho(y)\,dy}{|x-y|^{n-2}} \quad (p = 2, n \geq 3),$$
$$V(x) = \sum \frac{c_j}{|x-q_j|^{(n-p)/(p-1)}} \quad (2 < p < n),$$
$$V(x) = \int \frac{\varrho(y)\,dy}{|x-y|^{(n-p)/(p-1)}} \quad (2 < p < n), \quad v(x) = \min\{v_1, v_2, \cdots, v_m\}.$$

The first example is *the fundamental solution*, which fails to belong to the "natural" Sobolev space $W^{1,p}_{loc}(\mathbb{R}^n)$.[8] The second is the Newtonian potential. In the third example the c_j's are positive convergence factors and the q_j's are the rational points; the superposition of fundamental solutions is credited to M. Crandall and J. Zhang, cf. [CZ]. The last example says that one may take the pointwise minimum of a finite number of p-superharmonic functions, which is an essential ingredient in the celebrated Perron method, cf. [GLM].

The next definition is from the theory of viscosity solutions. One defines them as being both viscosity super- and subsolutions, since it is not practical to do it in one stroke.

Definition 9 *Let $p \geq 2$. A function $v : \Omega \to (-\infty, \infty]$ is called a* viscosity supersolution*, if*

(i) v is finite in a dense subset

(ii) v is lower semicontinuous

(iii) whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $v(x_0) = \phi(x_0)$ and $v(x) > \phi(x)$ when $x \neq x_0$, we have

$$\operatorname{div}\big(|\nabla\phi(x_0)|^{p-2}\nabla\phi(x_0)\big) \leq 0.$$

[8] Therefore it is not a weak supersolution, but it is a viscosity supersolution and a p-superharmonic function.

Remarks. The differential operator is evaluated only at the point of contact. The singular case $1 < p < 2$ requires a modification [9], if it so happens that $\nabla\phi(x_0) = 0$. Notice that each point has its own family of test functions. If there is no test function touching from below at x_0, then there is no requirement: the point passes for free. Please, notice that nothing is said about the gradient ∇v, it is $\nabla\phi(x_0)$ that appears.

Theorem 10 *A p-superharmonic function is a viscosity supersolution.*

Proof: Let v be a p-superharmonic function in the domain Ω. In order to prove that v is a viscosity supersolution, we use an indirect proof. Our antithesis is that there exist a point $x_0 \in \Omega$ and a test function ϕ touching v from below at x_0 and satisfying the inequality $\Delta_p\phi(x_0) > 0$. Replacing the test function with

$$\phi(x) - |x - x_0|^4$$

we may further assume that the strict inequality $\phi(x) < v(x)$ is valid when $x \neq x_0$. The subtracted fourth power does not affect $\Delta_p\phi(x_0)$. By continuity we can assure that the strict inequality $\Delta_p\phi(x) > 0$ holds in a small neighbourhood U of the point x_0. Now ϕ is p-*sub*harmonic in U and by adding a small positive constant m, say

$$2m = \max_{\partial U}\big(v(x) - \phi(x)\big)$$

we arrive at the following situation. The function $\phi + m$ is p-subharmonic in U, which contains x_0, and it is $\leq v$ on its boundary ∂U. By the comparison principle $\phi(x) + m \leq v(x)$ in U. This is a contradiction at the point $x = x_0$. This proves the claim. $\square$

The functions in the next lemma, the continuous weak supersolutions, form a more tractable subclass, when it comes to *a priori* estimates, since they are differentiable in Sobolev's sense.

Lemma 11 *Let $v \in C(\Omega) \cap W^{1,p}(\Omega)$. Then the following conditions are equivalent:*

(i) $\int_\Omega |\nabla v|^p \, dx \leq \int_\Omega |\nabla(v + \eta)|^p \, dx$ *when* $\eta \geq 0$, $\eta \in C_0^\infty(\Omega)$,

(ii) $\int_\Omega \langle |\nabla v|^{p-2}\nabla v, \nabla\eta\rangle \, dx \geq 0$ *when* $\eta \geq 0$, $\eta \in C_0^\infty(\Omega)$,

[9] There is no requirement when $\nabla\phi$ is 0, see [JLM].

(iii) v is p-superharmonic.

They imply that v is a viscosity supersolution.

Proof: The equivalence of (i) and (ii) is plain. So is the necessity of (iii), stating that the comparison principle must hold. The crucial part is the sufficiency of (iii), which will be established by the help of an *obstacle problem* in the calculus of variations. The function v itself will act as an obstacle for the admissible functions in the minimization of the p-energy $\int_D |\nabla v|^p \, dx$ and it also induces the boundary values in the subdomain D. If D is a regular subdomain of Ω, then there exists a unique minimizer, say w_v, in the class

$$\mathcal{F}_v = \{w \in C(\overline{D}) \cap W^{1,p}(D) | \ w \geq v, \ w = v \ on \ \partial D\}.$$

The crucial part is the continuity of w_v, cf. [MZ]. The solution of the obstacle problem automatically has the property (i), and hence also (ii). We claim that $w_v = v$ in D, from which the desired conclusion thus follows. The minimizer is a p-harmonic function in the open set $\{w_v > v\}$ where the obstacle does not hinder. On the boundary of this set $w_v = v$. Hence the comparison principle, which v is assumed to obey, can be applied. It follows that $w_v \leq v$ in the same set. To avoid a contradiction it must be the empty set. The conclusion is that $w_v = v$ in D, as desired. One can now deduce that (iii) is sufficient. $\square$

A function, whether continuous or not, belonging to $W^{1,p}(\Omega)$ and satisfying (ii) in the previous lemma is called a *weak supersolution.* For completeness we record below that weak supersolutions are semicontinuous "by nature".

Proposition 12 *A weak supersolution $v \in W^{1,p}(\Omega)$ is lower semicontinuous (after redefinition in a set of measure zero). We can define*

$$v(x) = \operatorname*{ess\,liminf}_{y \to x} v(y)$$

pointwise. This representative is a p-superharmonic function.

Proof: The case $p > n$ is clear, since then the Sobolev space contains only continuos functions (Morrey's inequality). In the range $p < n$ we claim that

$$v(x) = \operatorname*{ess\,liminf}_{y \to x} v(y)$$

at a.e. $x \in \Omega$. The proof follows from this, because the right-hand side is always lower semicontinuous. We omit two demanding steps. First, it is required to establish that v is locally bounded. This is standard regularity theory. Second, for non-negative functions we use "the weak Harnack estimate"[10]

$$\left(\frac{1}{|B_{2r}|}\int_{B_{2r}} v^q\,dx\right)^{\frac{1}{q}} \leq C \operatorname*{ess\,inf}_{B_r} v, \tag{7}$$

when $q < n(p-1)/(n-p)$, $C = C(n,p,q)$. This comes from the celebrated Moser iteration, cf. [T1]. Taking[11] $q = 1$ and using the non-negative function $v(x) - m(2r)$, where

$$m(r) = \operatorname*{ess\,inf}_{B_r} v,$$

we have

$$\begin{aligned} 0 &\leq \frac{1}{|B_{2r}|}\int_{B_{2r}} v\,dx \;-\; m(2r) \\ &= \frac{1}{|B_{2r}|}\int_{B_{2r}} (v(x) - m(2r))\,dx \leq C(m(r) - m(2r)). \end{aligned}$$

Since $m(r)$ is monotone, $m(r) - m(2r) \to 0$ as $r \to 0$. It follows that

$$\operatorname*{ess\,liminf}_{y \to x_0} v(y) = \lim_{r \to 0} m(2r) = \lim_{r \to 0} \frac{1}{|B_{2r}|}\int_{B(x_0,2r)} v(x)\,dx$$

---

[10]Harnack's inequality can be replaced by the more elementary estimate

$$\operatorname*{ess\,sup}_{B_r} (v(x_0) - v(x))_+ \leq \frac{C}{|B_{2r}|}\int_{B(x_0,2r)} (v(x_0) - v(x))_+\,dx$$

as a starting point for the proof. It follows immediately that also

$$\operatorname*{ess\,sup}_{B_r} (v(x_0) - v(x)) \leq \frac{C}{|B_{2r}|}\int_{B(x_0,2r)} |v(x_0) - v(x)|\,dx.$$

If x_0 is a Lebesgue point, the integral approaches zero as $r \to 0$ and it follows that

$$\operatorname*{ess\,liminf}_{x \to x_0} v(x) \geq v(x_0).$$

The opposite inequality holds for "arbitrary" functions at their Lebesgue points. (See the end of Chapter 4.)

[11]If $p \geq 2n/(n+1)$ does not hold, we need a larger q.

at *each* point x_0. Lebesgue's differentiation theorem states that the limit of the average on the right-hand side coincides with $v(x_0)$ at almost every point x_0.

Lemma 13 (Caccioppoli) *Let* $v \in C(\Omega) \cap W^{1,p}(\Omega)$ *be a* p*-superharmonic function. Then*

$$\int_\Omega \zeta^p |\nabla v|^p \, dx \leq p^p (\operatorname*{osc}_{\zeta \neq 0} v)^p \int_\Omega |\nabla \zeta|^p \, dx$$

holds for non-negative $\zeta \in C_0^\infty(\Omega)$*. If* $v \geq 0$*, then*

$$\int_\Omega \zeta^p v^{-1-\alpha} |\nabla v|^p \, dx \leq \left(\frac{p}{\alpha}\right)^p \int_\Omega v^{p-1-\alpha} |\nabla \zeta|^p \, dx$$

when $\alpha > 0$*.*

Proof: To prove the first estimate, fix ζ and let $L = \sup v$ taken over the set where $\zeta \neq 0$. Use the test function

$$\eta = (L - v(x))\zeta(x)^p$$

in Lemma 9(ii) and arrange the terms.

To prove the second estimate, first replace $v(x)$ by $v(x) + \varepsilon$, if needed, and use

$$\eta = v^{-\alpha}\zeta^p.$$

The rest is clear. $\square$

The special case $\alpha = p - 1$ is appealing, since the right-hand member of the inequality

$$\int_\Omega \zeta^p |\nabla \log v|^p \, dx \leq \left(\frac{p}{p-1}\right)^p \int_\Omega |\nabla \zeta|^p \, dx \tag{8}$$

is independent of the non-negative function v itself.

We aim at approximating v with functions for which Lemma 11 is valid. To this end, let v be lower semicontinuous and bounded in Ω:

$$0 \leq v(x) \leq L.$$

Define

$$v_\varepsilon(x) = \inf_{y \in \Omega} \left\{ v(y) + \frac{|x-y|^2}{2\,\varepsilon} \right\}, \quad \varepsilon > 0. \tag{9}$$

Then

- $v_\varepsilon(x) \nearrow v(x)$ as $\varepsilon \to 0+$
- $v_\varepsilon(x) - |x|^2/2\varepsilon$ is locally concave in Ω
- v_ε is locally Lipschitz continuous in Ω
- The Sobolev gradient ∇v_ε exists and belongs to $L^\infty_{loc}(\Omega)$
- The second Alexandrov derivatives $D^2 v_\varepsilon$ exist. See Section 7.

The next to last assertion follows from Rademacher's theorem about Lipschitz functions, cf. [EG]. Thus these "infimal convolutions" are rather regular. A most interesting property for a bounded viscosity supersolution is the following:

Proposition 14 *If v is a bounded viscosity supersolution in Ω, the approximant v_ε is a viscosity supersolution in the open subset of Ω where*

$$\operatorname{dist}(x, \partial\Omega) > \sqrt{2L\varepsilon}.$$

Similarly, if v is a p-superharmonic function, so is v_ε.

Proof: First, notice that for x as required above, the infimum is attained at some point $y = x^\star$ comprised in Ω. The possibility that $x^\star$ escapes to the boundary of Ω is prohibited by the inequalities

$$\frac{|x-x^\star|^2}{2\varepsilon} \leq \frac{|x-x^\star|^2}{2\varepsilon} + v(x^\star) = v_\varepsilon(x) \leq v(x) \leq L,$$
$$|x - x^\star| \leq \sqrt{2L\varepsilon} < \operatorname{dist}(x, \partial\Omega).$$

This explains why the domain shrinks a little. Now we give two proofs.

Viscosity proof: Fix a point x_0 so that also $x_0^\star \in \Omega$. Assume that the test function φ touches v_ε from below at x_0. Using

$$\begin{aligned} \varphi(x_0) &= v_\varepsilon(x_0) &&= \frac{|x_0 - x_0^\star|^2}{2\varepsilon} + v(x_0^\star) \\ \varphi(x) &\leq v_\varepsilon(x) &&\leq \frac{|x-y|^2}{2\varepsilon} + v(y) \end{aligned}$$

we can verify that the function

$$\psi(x) = \varphi(x + x_0 - x_0^\star) - \frac{|x_0 - x_0^\star|^2}{2\varepsilon}$$

touches the original function v from below at the point $x_0^\star$. Since $x_0^\star$ is an interior point, the inequality

$$\operatorname{div}\left(|\nabla\psi(x_0^\star)|^{p-2}\nabla\psi(x_0^\star)\right) \le 0$$

holds by assumption. Because

$$\nabla\psi(x_0^\star) = \nabla\varphi(x_0), \quad D^2\psi(x_0^\star) = D^2\varphi(x_0),$$

we also have that

$$\operatorname{div}\left(|\nabla\varphi(x_0)|^{p-2}\nabla\varphi(x_0)\right) \le 0$$

at the original point x_0, where φ was touching v_ε. Thus v_ε fulfills the requirement in the definition.

Proof by Comparison: Now we assume that v is p-superharmonic in Ω and show that v_ε is p-superharmonic in

$$\Omega_\varepsilon = \left\{x \in \Omega | \operatorname{dist}(x, \partial\Omega) > \sqrt{2L\varepsilon}\right\}.$$

We have to verify the comparison principle for v_ε. To this end, let $D \subset\subset \Omega_\varepsilon$ be a subdomain and suppose that $h \in C(\overline{D})$ is a p-harmonic function so that $v_\varepsilon(x) \ge h(x)$ on the boundary ∂D or, in other words,

$$\frac{|x-y|^2}{2\varepsilon} + v(y) \ge h(x) \text{ when } x \in \partial D, \ y \in \Omega.$$

Thus, writing $y = x + z$, we have

$$w(x) \equiv v(x+z) + \frac{|z|^2}{2\varepsilon} \ge h(x), \quad x \in \partial D$$

whenever z is a small fixed vector. But also $w = w(x)$ is a p-superharmonic function in Ω_ε. By the comparison principle $w(x) \ge h(x)$ in D. Given any point x_0 in D, we may choose $z = x_0^\star - x_0$. This yields $v_\varepsilon(x_0) \ge h(x_0)$. Since x_0 was arbitrary, we have verified that

$$v_\varepsilon(x) \ge h(x), \quad \text{when } x \in D.$$

This concludes the proof. We record the following result.

Corollary 15 *If v is a bounded p-superharmonic function, the approximant v_ε is a weak supersolution in Ω_ε, i.e.*

$$\int_{\Omega_\varepsilon} \langle |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon, \nabla \eta \rangle \, dx \geq 0 \tag{10}$$

when $\eta \geq 0$, $\eta \in C_0^\infty(\Omega_\varepsilon)$.

Proof: This is a combination of the Proposition and Lemma 11. $\square$

The Caccioppoli estimate for v_ε reads

$$\int_\Omega \zeta^p |\nabla v_\varepsilon|^p \, dx \leq (pL)^p \int_\Omega |\nabla \zeta|^p \, dx,$$

when ε is so small that the support of ζ is in Ω_ε. By a compactness argument (a subsequence of) ∇v_ε is locally weakly convergent in $L^p(\Omega)$. We conclude that ∇v exists in Sobolev's sense and that

$$\nabla v_\varepsilon \quad \rightharpoonup \quad \nabla v \quad \text{weakly in } L^p_{loc}(\Omega).$$

By the weak lower semicontinuity of the integral also

$$\int_\Omega \zeta^p |\nabla v|^p \, dx \leq (pL)^p \int_\Omega |\nabla \zeta|^p \, dx.$$

We have proved the first part of the next theorem.

Theorem 16 *Suppose that v is a* bounded *p-superharmonic function in Ω. Then the Sobolev gradient ∇v exists and $v \in W^{1,p}_{loc}(\Omega)$. Moreover,*

$$\int_\Omega \langle |\nabla v|^{p-2} \nabla v, \nabla \eta \rangle \, dx \geq 0 \tag{11}$$

for all $\eta \geq 0, \eta \in C_0^\infty(\Omega)$.

Proof: To conclude the proof, we show that the convergence $\nabla v_\varepsilon \to \nabla v$ is strong in $L^p_{loc}(\Omega)$, so that we may pass to the limit under the integral sign in (8). To this end, fix a function $\theta \in C_0^\infty(\Omega)$, $0 \leq \theta \leq 1$ and use the test function $\eta = (v - v_\varepsilon)\theta$ in the equation for v_ε. Then

$$\int_\Omega \langle |\nabla v|^{p-2} \nabla v - |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon, \nabla((v - v_\varepsilon)\theta) \rangle \, dx$$

$$\leq \int_\Omega \langle |\nabla v|^{p-2}\nabla v, \nabla((v-v_\varepsilon)\theta)\rangle\, dx \longrightarrow 0,$$

where the last integral approaches zero because of the weak convergence. The first integral splits into the sum

$$\int_\Omega \theta\langle |\nabla v|^{p-2}\nabla v - |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon, \nabla(v-v_\varepsilon)\rangle\, dx$$
$$+\int_\Omega (v-v_\varepsilon)\langle |\nabla v|^{p-2}\nabla v - |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon, \nabla\theta\rangle\, dx.$$

The last integral approaches zero because its absolute value is majorized by

$$\Big(\int_D (v-v_\varepsilon)^p\, dx\Big)^{1/p}\Bigg[\Big(\int_D |\nabla v|^p\, dx\Big)^{(p-1)/p} + \Big(\int_D |\nabla v_\varepsilon|^p\, dx\Big)^{(p-1)/p}\Bigg]\|\nabla\theta\|,$$

where D contains the support of θ and $\|v - v_\varepsilon\|_p$ approaches zero. Thus we have established that

$$\int_\Omega \theta\langle |\nabla v|^{p-2}\nabla v - |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon, \nabla(v-v_\varepsilon)\rangle\, dx$$

approaches zero. Now the strong convergence of the gradients follows from the vector inequality

$$2^{2-p}|b-a|^p \leq \langle |b|^{p-2}b - |a|^{p-2}a, b-a\rangle \tag{12}$$

valid for $p > 2$. □

It also follows that the Caccioppoli estimates in Lemma **??** are valid for locally bounded p-superharmonic functions. The case when v is unbounded can be reached via the truncations

$$v_k = \min\{v(x), k\}, \quad k = 1, 2, 3, \dots,$$

because Theorem 16 holds for these locally bounded functions. Aiming at a local result, we may just by adding a constant assume that $v \geq 0$ in Ω. The situation with $v = 0$ on the boundary $\partial\Omega$ offers expedient simplifications. We shall describe an iteration procedure, under this extra assumption. See [KM].

Lemma 17 *Assume that* $v \geq 0$ *and that* $v_k \in W_0^{1,p}(\Omega)$ *when* $k = 1, 2, \ldots$ *Then*

$$\int_\Omega |\nabla v_k|^p \, dx \leq k \int_\Omega |\nabla v_1|^p \, dx$$

and, in the case $1 < p < n$

$$\int_\Omega v^\alpha \, dx \leq C_\alpha \left(1 + \int_\Omega |\nabla v_1|^p \, dx \right)^{\frac{n}{n-p}}$$

whenever $\alpha < \frac{n(p-1)}{n-p}$.

Proof: Let j be a large index and use the test functions

$$\eta_k = (v_k - v_{k-1}) - (v_{k+1} - v_k), \quad k = 1, 2, \cdots, j-1$$

in the equation for v_j, i.e.

$$\int_\Omega \langle |\nabla v_j|^{p-2} \nabla v_j, \nabla \eta_k \rangle \, dx \geq 0.$$

Indeed, $\eta_k \geq 0$. We obtain

$$\begin{aligned} A_{k+1} &= \int_\Omega \langle |\nabla v_j|^{p-2} \nabla v_j, \nabla v_{k+1} - \nabla v_k \rangle \, dx \\ &\leq \int_\Omega \langle |\nabla v_j|^{p-2} \nabla v_j, \nabla v_k - \nabla v_{k-1} \rangle \, dx = A_k. \end{aligned}$$

Thus

$$A_{k+1} \leq A_1 = \int_\Omega |\nabla v_1|^p \, dx$$

and hence

$$A_1 + A_2 + \cdots + A_j \; \leq \; j A_1.$$

The "telescoping" sum becomes

$$\int_\Omega |\nabla v_j|^p \, dx \leq j \int_\Omega |\nabla v_1|^p \, dx.$$

This was the first claim.

If $1 < p < n$, it follows from Tshebyshev's and Sobolev's inequalities that

$$j |j \leq v \leq 2j|^{\frac{1}{p*}} \leq \left(\int_\Omega v_{2j}^{p*} \, dx \right)^{\frac{1}{p*}} \leq S \left(\int_\Omega |\nabla v_{2j}|^p \, dx \right)^{\frac{1}{p}} \leq S (2j)^{\frac{1}{p}} A_1^{\frac{1}{p}},$$

where $p* = np/(n-p)$. We arrive at the estimate

$$|j \leq v \leq 2j| \leq Cj^{-\frac{n(p-1)}{n-p}} A_1^{\frac{n}{n-p}}$$

for the measure of the level sets. To conclude the proof we write

$$\int_\Omega v^\alpha \, dx = \int_{v \leq 1} v^\alpha \, dx + \sum_{j=1}^{\infty} \int_{2^{j-1} < v \leq 2^j} v^\alpha \, dx.$$

Since

$$\int_{2^{j-1} < v \leq 2^j} v^\alpha \, dx \leq C\, 2^{j\alpha} 2^{-(j-1)\frac{n(p-1)}{n-p}} A_1^{\frac{n}{n-p}},$$

the series converges when α is as prescribed. $\square$

It remains to abandon the restriction about zero boundary values and to estimate

$$\int_\Omega |\nabla v_1|^p \, dx.$$

The reduction to zero boundary values is done locally in a ball $B_{2r} \subset\subset \Omega$. Suppose first that $v \in C(\overline{B_{2r}}) \cap W^{1,p}(B_{2r}),\ v \geq 0$, and define

$$w = \begin{cases} v \text{ in } \overline{B_r} \\ h \text{ in } B_{2r} \backslash \overline{B_r} \end{cases}$$

where h is the p-harmonic function in the annulus having outer boundary values zero and inner boundary values v. Now $h \leq v$. The so defined w is p-superharmonic in B_{2r}, which follows by comparison. It is quite essential that the original v was defined in a domain larger than B_r! We also have

$$\int_{B_{2r}} |\nabla w|^p \, dx \leq Cr^{n-p} (\max_{B_{2r}} w)^p$$

after some estimation.[12]

Finally, if $v \in W^{1,p}(B_{2r})$ is semicontinuous and bounded (but not necessarily continuous), then we first modify the approximants v_ε defined as in

[12] It is important to include the whole B_{2r}. Of course, the Caccioppoli estimate (Lemma 11) will do over any smaller ball B_ϱ, $\varrho < 2r$. To get the missing estimate, say over the boundary annulus $B_{2r} \setminus B_{3r/2}$, the test function $\eta = \zeta h$ works in Definition 5, where $\zeta = 1$ in the annulus and $= 0$ on ∂B_r. The zero boundary values of the weak solution h were essential.

(7) and obtain p-superharmonic functions w_ε. Since $0 \le w_\varepsilon \le v_\varepsilon \le v$, the previous estimate becomes

$$\int_{B_{2r}} |\nabla w_\varepsilon|^p \, dx \le C r^{n-p} (\max_{B_{2r}} v)^p$$

and, by the weak lower semicontinuity of the integral, we can pass to the limit as ε approaches zero. We end up with a p-superharmonic function $w \in W_0^{1,p}(B_{2r})$ such that $w = v$ in B_r and, in particular,

$$\int_{B_r} |\nabla v|^p \, dx \le \int_{B_{2r}} |\nabla w|^p \, dx \le C r^{n-p} (\max_{B_{2r}} v)^p.$$

This is the desired modified function. Now, repeat the procedure with every function $\min\{v(x), k\}$ in sight. We obtain

$$\int_{B_r} |\nabla v_1|^p \, dx \le \int_{B_{2r}} |\nabla w_1|^p \, dx \le C r^{n-p} 1^p$$

for the modification of $v_1 = \min\{v(x), 1\}$. We have achieved that the bounds in the previous lemma hold for the modified function over the domain B_{2r} and *a fortiori* for the original v, estimated only over the smaller ball B_r. Such a *local* estimate is all that is needed in the proof of the theorem below.

Theorem 18 *Suppose that v is a p-superharmonic function in Ω. Then*

$$v \in L^q_{loc}(\Omega), \quad \textit{whenever } q < \frac{n(p-1)}{n-p}$$

in the case $1 < p \le n$ and v is continuous if $p > n$. Moreover, ∇v exists in Sobolev's sense[13] *and*

$$\nabla v \in L^q_{loc}(\Omega), \quad \textit{whenever} \quad q < \frac{n(p-1)}{n-1}$$

in the case $1 < p \le n$. In the case $p > n$ we have $\nabla v \in L^p_{loc}(\Omega)$. Finally,

$$\int_\Omega \langle |\nabla v|^{p-2} \nabla v, \nabla \eta \rangle \, dx \ge 0 \tag{13}$$

when $\eta \ge 0$, $\eta \in C_0^\infty(\Omega)$.

[13] Strictly speaking, one needs $p > 2 - \frac{1}{n}$ so that $q \ge 1$. This can be circumvented.

Proof: In view of the local nature of the theorem we may assume that $v > 0$. According to the previous construction we can further reduce the proof to the case $v_k \in W_0^{1,p}(B_{2r})$ for each truncation at height k. The first part of the theorem is included in Lemma 17 when $1 < p < n$. We skip the borderline case $p = n$. The case $p > n$ is related to the fact that then all functions in the Sobolev space $W^{1,p}$ are continuos.

We proceed to the estimation of the gradient. First we keep $1 < p < n$ and write

$$\int_B |\nabla v_k|^q \, dx = \int_B v_k^{\frac{(1+\alpha)q}{p}} \left| \frac{\nabla v_k}{v_k^{(1+\alpha)/p}} \right|^q dx$$
$$\leq \left\{ \int_B v_k^{\frac{(1+\alpha)q}{p-q}} \, dx \right\}^{1-\frac{q}{p}} \left\{ \int_B v_k^{-1-\alpha} |\nabla v_k|^p \, dx \right\}^{\frac{q}{p}}.$$

Take $q < n(p-1)/(n-1)$ and fix α so that

$$\frac{(1+\alpha)q}{p-q} < \frac{n(p-1)}{n-p}.$$

Continuing, the Caccioppoli estimate yields the majorant

$$\leq \left\{ \int_B v_k^{\frac{(1+\alpha)q}{p-q}} \, dx \right\}^{1-\frac{q}{p}} C \left\{ \int_{2B} v_k^{p-1-\alpha} \, dx \right\}^{\frac{q}{p}}. \tag{14}$$

We can take $v \geq 1$. Then let $k \longrightarrow \infty$. Clearly, the resulting majorant is finite (Lemma 17). This concludes the case $1 < p < n$.

If $p > n$ we obtain that

$$\int_{B_r} |\nabla \log v_k|^p \, dx \leq C r^{n-p}$$

from (8), where C is independent of k. Hence $\log v_k$ is continuous. So is v itself. Now

$$\int_{B_r} |\nabla v_k|^p \, dx = \int_{B_r} v_k^p \, |\nabla \log v_k|^p \, dx \leq C \|v\|_\infty^p r^{n-p}$$

implies the desired p-summability of the gradient. $\square$

It stands to reason that the lower semicontinuous solutions of (16) are p-superharmonic functions. However, this is not known under the summability assumption $\nabla v \in L^q_{loc}(\Omega)$ accompanying the differential equation, if $q < p$ and $p < n$. In fact, an example of J. Serrin indicates that even for solutions to linear equations strange phenomena occur, cf. [S]. False solutions appear, when the *a priori* summability of the gradient is too poor. About this topic there is nowadays a theory credited to T. Iwaniec, cf. [L].[14]

3 The Evolutionary Equation

This chapter is rather independent of the previous one. After some definitions we first treat bounded supersolutions and then the unbounded ones. As a mnemonic rule, $v_t \geq \Delta_p v$ for smooth supersolutions, $u_t \leq \Delta_p u$ for smooth subsolutions. We need the following classes of supersolutions:

(1) **weak supersolutions** (test functions under the integral sign)

(2) p**-supercaloric functions** (defined via a comparison principle)

(3) **viscosity supersolutions** (test functions evaluated at points of contact)

The weak supersolutions do not form a good closed class under monotone convergence.

3.1 Definitions

We first define the concept of solutions, p-supercaloric functions and viscosity supersolutions. The section ends with an outline of the procedure for the proof.

Suppose that Ω is a bounded domain in $\mathbb{R}^\ltimes$ and consider the space-time cylinder $\Omega_T = \Omega \times (0,T)$. Its *parabolic boundary* consists of the portions $\Omega \times \{0\}$ and $\partial\Omega \times [0,T]$.

Definition 19 *In the case*[15] $p \geq 2$ *we say that* $u \in L^p(0,T;W^{1,p}(\Omega))$ *is a* weak solution *of the Evolutionary p-Laplace Equation, if*

$$\int_0^T \int_\Omega \left(-u\phi_t + \langle |\nabla u|^{p-2}\nabla u, \nabla \phi\rangle\right) dx\, dt = 0 \tag{15}$$

[14] The "pathological solutions" of Serrin are now called "very weak solutions".

[15] The singular case $1 < p < 2$ requires an extra *a priori* assumption, for example, $u \in L^\infty(0,T;L^2(\Omega))$ will do.

for all $\phi \in C_0^1(\Omega_T)$. If the integral is ≥ 0 for all test functions $\phi \geq 0$, we say that u is a weak supersolution*.*

In particular, one has the requirement

$$\int_0^T \int_\Omega (|u|^p + |\nabla u|^p)\, dx\, dt < \infty.$$

Sometimes it is enough to require that $u \in L^p_{loc}(0,T; W^{1,p}_{loc}(\Omega))$. By the regularity theory one may regard a weak solution $u = u(x,t)$ as continuous.[16] For simplicity, we call the continuous weak solutions for *p-caloric functions*[17].

The *interior Hölder estimate*[18] takes the following form for solutions according to [dB]. In the subdomain $D \times (\delta, T-\delta)$

$$|u(x_1,t_1) - u(x_2,t_2)| \leq \gamma \|u\|_{L^\infty(\Omega_T)} \left(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/p}\right), \tag{16}$$

where the positive exponent α depends only on n and p, while the constant γ depends, in addition, on the distance to the subdomain. Also an *intrinsic Harnack inequality* is valid, see Lemma 31 below.

Recall that the *parabolic boundary* of the domain $\Omega_T = \Omega \times (0,T)$ is

$$\Omega \times \{0\} \ \cup \ \partial\Omega \times [0,T].$$

The part $\Omega \times \{T\}$ is excluded.

Proposition 20 (Comparison Principle) *Suppose that v is a weak supersolution and u a weak subsolution, $u, v \in L^p(0,T; W^{1,p}(\Omega))$, satisfying*

$$\liminf v \geq \limsup u$$

on the parabolic boundary. Then $v \geq u$ almost everywhere in the domain Ω_T.

Proof: This is well-known and we only give a formal proof. For a non-negative test function $\varphi \in C_0^\infty(\Omega_T)$ the equations

$$\int_0^T \int_\Omega (-v\varphi_t + \langle |\nabla v|^{p-2}\nabla v, \nabla\varphi\rangle)\, dx\, dt \geq 0$$
$$\int_0^T \int_\Omega (+u\varphi_t - \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle)\, dx\, dt \geq 0$$

[16] The weak supersolutions are lower semicontinuous according to [K], see Chapter 4.

[17] One may argue that this is more adequate than "p-parabolic functions", which is in use.

[18] This is weaker than the estimate in [dB]. See also [U] for intrinsic scaling.

can be added. Thus

$$\int_0^T \int_\Omega \big((u-v)\varphi_t + \langle |\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle\big)\, dx\, dt \geq 0.$$

These equations remain true if v is replaced by $v+\varepsilon$, where ε is any positive constant. To complete the proof we choose (formally) the test function to be

$$\varphi = (u-v-\varepsilon)_+\eta,$$

where $\eta = \eta(t)$ is a cut-off function; the plain choice $\eta(t) = T-t$ will do here. We arrive at

$$\begin{aligned}
&\int_0^T \int_{\{u \geq v+\varepsilon\}} \eta(\langle |\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u, \nabla v - \nabla u\rangle)\, dx\, dt \\
&\leq \int_0^T \int_\Omega (u-v-\varepsilon)_+^2 \eta'\, dx\, dt + \frac{1}{2}\int_0^T \int_\Omega \eta \frac{\partial}{\partial t}(u-v-\varepsilon)_+^2\, dx\, dt \\
&= \frac{1}{2}\int_0^T \int_\Omega (u-v-\varepsilon)_+^2 \eta'\, dx\, dt \\
&= -\frac{1}{2}\int_0^T \int_\Omega (u-v-\varepsilon)_+^2\, dx\, dt \;\leq\; 0.
\end{aligned}$$

Since the first integral is non-negative by the structural inequality (10), the last integral is, in fact, zero. Hence the integrand $(u-v-\varepsilon)_+ = 0$ almost everywhere. But this means that

$$u \leq v + \varepsilon$$

almost everywhere. Since $\varepsilon > 0$ was arbitrary, we have the desired inequality $v \geq u$ a.e.. $\square$

For continuous functions the Comparison Principle is especially appealing. Then the conclusion is valid at every point. As we will see later, the redefined functions

$$v_* = \operatorname{ess\,liminf} v, \qquad u^* = \operatorname{ess\,limsup} u$$

are weak super- and subsolutions, and $v_* \geq u^*$ at each point.

Definition 21 *We say that a function $v : \Omega_T \to (-\infty, \infty]$ is p-*supercaloric *in Ω_T, if* *(i)* $\quad$ *v is finite in a dense subset*

(ii) v is lower semicontinuous

(iii) in each cylindrical subdomain $D \times (t_1, t_2) \subset\subset \Omega_T$ v obeys the comparison principle:
if $h \in C(\overline{D} \times [t_1, t_2])$ is p-caloric in $D \times (t_1, t_2)$, then $v \geq h$ on the parabolic boundary of $D \times (t_1, t_2)$ implies that $v \geq h$ in the whole subdomain.

As a matter of fact, every weak supersolution has a semicontinuous representative which is a p-supercaloric function. (This is postponed till Section 5) The leading example is the Barenblatt solution, which is a p-supercaloric function in the whole $\mathbb{R}^{\ltimes+\nVdash}$. Another example is any function of the form

$$v(x,t) = g(t),$$

where $g(t)$ is an arbitrary monotone increasing lower semicontinuous function. We also mention

$$v(x,t) + \frac{\varepsilon}{T-t}, \qquad 0 < t < T,$$

$$v(x,t) = \min\{v_1(x,t), \ldots, v_j(x,t)\}.$$

The pointwise minimum of (finitely many) p-supercaloric functions is employed in Perron's Method. Finally, if $v \geq 0$ is a p-supercaloric function, so is the function obtained by redefining $v(x,t) = 0$ when $t \leq 0$.

A Separable Minorant. Separation of variables suggests that there are p-caloric functions of the type

$$v(x,t) = (t-t_0)^{-\frac{1}{p-2}} u(x).$$

Indeed, if Ω is a domain of finite measure, there exists a p-caloric function of the form

$$\mathfrak{V}(x,t) = \frac{\mathfrak{U}(x)}{(t-t_0)^{\frac{1}{p-2}}}, \quad \text{when} \quad t > t_0 \tag{17}$$

where $\mathfrak{U} \in C(\Omega) \cap W_0^{1,p}(\Omega)$ is a weak solution to the elliptic equation

$$\nabla \cdot \left(|\nabla \mathfrak{U}|^{p-2} \nabla \mathfrak{U}\right) \; + \; \tfrac{1}{p-2} \, \mathfrak{U} \; = \; 0 \tag{18}$$

and $\mathfrak{U} > 0$ in Ω. The solution $\mathfrak{U}$ is unique[19]. (Actually, $\mathfrak{U} \in C^{1,\alpha}_{loc}(\Omega)$ for some exponent $\alpha = \alpha(n,p) > 0$.) The extended function

$$\mathfrak{V}(x,t) = \begin{cases} \dfrac{\mathfrak{U}(x)}{(t-t_0)^{\frac{1}{p-2}}}, & \text{when} \quad t > t_0 \\ 0 \qquad \text{when} \quad t \leq t_0. \end{cases} \tag{19}$$

is p-supercaloric in $\Omega \times \mathbb{R}$. The existence of $\mathfrak{U}$ follows by the direct method in the Calculus of Variations, when the quotient

$$J(w) = \frac{\int_\Omega |\nabla w|^p \, dx}{\left(\int_\Omega w^2 \, dx\right)^{\frac{p}{2}}}$$

is minimized among all functions w in $W^{1,p}_0(\Omega)$, $w \not\equiv 0$. Replacing w by its absolute value $|w|$, we may assume that all functions are non-negative. Notice that $J(\lambda w) = J(w)$ for $\lambda =$ constant. Sobolev's and Hölder's inequalities yield

$$J(w) \geq c(p,n)|\Omega|^{1-\frac{p}{n}-\frac{p}{2}}, \qquad c(p,n) > 0$$

and so

$$J_0 = \inf_w J(w) > 0.$$

Choose a minimizing sequence of admissible normalized functions w_j :

$$\lim_{j\to\infty} J(w_j) = J_0, \qquad \|w_j\|_{L^p(\Omega)} = 1.$$

By compactness, we may extract a subsequence such that

$$\begin{aligned} \nabla w_{j_k} &\rightharpoonup \nabla w \qquad \text{weakly in} \quad L^p(\Omega) \\ w_{j_k} &\longrightarrow w \qquad \text{strongly in} \quad L^p(\Omega) \end{aligned}$$

for some function w. The weak lower semicontinuity of the integral implies that

$$J(w) \leq \liminf_{k\to\infty} J(w_{j_k}) = J_0.$$

[19]Unfortunately, the otherwise reliable paper [J. Garci'a Azorero, I. Peral Alonso: *Existence and nonuniqueness for the p-Laplacian: Nonlinear eigenvalues*, Communications in Partial Differential Equations **12**, 1987, pp. 1389–1430], contains a misprint exactly for those parameter values that would yield this function.

Since $w \in W_0^{1,p}(\Omega)$ this means that w is a minimizer. We have $w \geq 0$, $w \not\equiv 0$.

It follows that w has to be a weak solution of the Euler–Lagrange Equation

$$\nabla \cdot \left(|\nabla w|^{p-2} \nabla w \right) + J_0 \|w\|_{L^p(\Omega)}^{p-2} w = 0$$

where $\|w\|_{L^p(\Omega)} = 1$. By elliptic regularity theory $w \in C(\Omega)$, see [T1] and [G]. Finally, since $\nabla \cdot \left(|\nabla w|^{p-2} \nabla w \right) \leq 0$ in the weak sense and $w \geq 0$ we have that $w > 0$ by the Harnack inequality (7). A normalization remains to be done. The function

$$\mathfrak{U} = Cu, \quad \text{where} \quad J_0 C^{p-2} = \tfrac{1}{p-2},$$

will do.

The next definition is from the theory of viscosity solutions. One defines them as being both viscosity super- and subsolutions, since it is not practical to do it in one stroke.

Definition 22 *Let $p \geq 2$. A function $v : \Omega_T \to (-\infty, \infty]$ is called a* viscosity supersolution*, if*

(i) v is finite in a dense subset

(ii) v is lower semicontinuous

(iii) whenever $(x_0, t_0) \in \Omega_T$ and $\phi \in C^2(\Omega_T)$ are such that $v(x_0, t_0) = \phi(x_0, t_0)$ and $v(x, t) > \phi(x, t)$ when $(x, t) \neq (x_0, t_0)$, we have

$$\phi_t(x_0, t_0) \geq \nabla \cdot \left(|\nabla \phi(x_0, t_0)|^{p-2} \nabla \phi(x_0, t_0) \right).$$

The p-supercaloric functions are exactly the same as the viscosity supersolutions. For a proof of this fundamental equivalence we refer to Section 7.2 or [JLM]. However, the following implication is easyly obtained.

Proposition 23 *A viscosity supersolution of the Evolutionary p-Laplace Equation is a p-supercaloric function.*

Proof: Similar to the elliptic case in Theorem 10.

Continuous weak supersolutions are p-supercaloric functions according to the Comparison Principle (Proposition 20).

We aim at proving the summability results (Theorem 2) for a general p-supercaloric function v. An outline of the procedure is the following.

- *Step 1.* Assume first that v is bounded.
- *Step 2.* Approximate v locally with infimal convolutions v_ε. These are differentiable.
- *Step 3.* The infimal convolutions are p-supercaloric functions and they are shown to be weak supersolutions of the equation (with test functions under the integral sign).
- *Step 4.* Estimates of the Caccioppoli type for v_ε are extracted from the equation.
- *Step 5.* The Caccioppoli estimates are passed over from v_ε to v. This concludes the proof for bounded functions.
- *Step 6.* The unbounded case is reached via the bounded p-supercaloric functions $v_k = \min\{v, k\}, k = 1, 2, \cdots$, for which the results in Step 5 already are available.
- *Step 7.* An iteration with respect to the index k is designed so that the final result does not blow up as $k \to \infty$. This works well when the parabolic boundary values (in the subdomain studied) are zero.
- *Step 8.* An extra construction is performed to reduce the proof to the situation of zero parabolic boundary values (so that the iterated result in Step 7 is at our disposal). This is not possible for class $\mathfrak{M}$, which is singled out.

3.2 Bounded Supersolutions

We aim at proving Theorem 2, which was given in the Introduction. The first step is to consider *bounded* p-supercaloric functions. We want to prove that they are weak supersolutions. First we approximate them with their infimal convolutions. Then estimates mainly of the Caccioppoli type are proved for these approximants. Finally, the so obtained estimates are passed over to the original functions. Assume therefore that

$$0 \leq v(x,t) \leq L, \quad (x,t) \in \Omega_T = \Omega \times (0,T).$$

The approximants

$$v_\varepsilon(x,t) = \inf_{(y,\tau)\in\Omega_T} \left\{ v(y,\tau) + \frac{|x-y|^2 + |t-\tau|^2}{2\,\varepsilon} \right\}, \quad \varepsilon > 0,$$

have the properties

- $v_\varepsilon(x,t) \nearrow v(x,t)$ as $\varepsilon \to 0+$
- $v_\varepsilon(x,t) - \frac{|x|^2+t^2}{2\varepsilon}$ is locally concave in Ω_T
- v_ε is locally Lipschitz continuous in Ω_T
- The Sobolev derivatives $\frac{\partial v_\varepsilon}{\partial t}$ and ∇v_ε exist and belong to $L^\infty_{loc}(\Omega_T)$
- The second Alexandrov derivatives of v_ε exist[20]

The next to last assertion follows from Rademacher's theorem about Lipschitz functions. Thus these "infimal convolutions" are differentiable almost everywhere. The existence of the time derivative is very useful. A most interesting property for a bounded viscosity supersolution is the following:

Proposition 24 *Suppose that v is a viscosity supersolution in Ω_T. The approximant v_ε is a viscosity supersolution in the open subset of Ω_T where*

$$\mathrm{dist}((\mathrm{x},\mathrm{t}),\partial\Omega_\mathrm{T}) > \sqrt{2\mathrm{L}\varepsilon}.$$

Similarly, if v is p-supercaloric, so is v_ε.

Proof: First, notice that for (x,t) as required above, the infimum is attained at some point $(y,\tau) = (x^\star,t^\star)$ comprised in Ω_T. The possibility that $(x^\star,t^\star)$ escapes to the boundary of Ω is prohibited by the inequalities

$$\begin{aligned} \frac{|x-x^\star|^2+|t-t^\star|^2}{2\varepsilon} &\leq \frac{|x-x^\star|^2+|t-t^\star|^2}{2\varepsilon} + v(x^\star,t^\star) \\ &= v_\varepsilon(x,t) \leq v(x,t) \leq L, \\ \sqrt{|x-x^\star|^2+|t-t^\star|^2} &\leq \sqrt{2L\varepsilon} < \mathrm{dist}((\mathrm{x},\mathrm{t}),\partial\Omega_\mathrm{T}). \end{aligned}$$

Thus the domain shrinks a little. Again there are two proofs.

Viscosity proof: Fix a point (x_0,t_0) so that also $(x_0^\star,t_0^\star) \in \Omega_T$. Assume that the test function φ touches v_ε from below at $(x_0^\star,t_0^\star)$. Using

$$\begin{aligned} \varphi(x_0,t_0) &= v_\varepsilon(x_0,t_0) &= \frac{|x_0-x_0^\star|^2+|t_0-t_0^\star|^2}{2\varepsilon} + v(x_0^\star,t_0^\star) \\ \varphi(x,t) &\leq v_\varepsilon(x,t) &\leq \frac{|x-y|^2+|t-\tau|^2}{2\varepsilon} + v(y,\tau) \end{aligned}$$

[20] See Section 7.

we can verify that the function

$$\psi(x,t) = \varphi(x + x_0 - x_0^\star, t + t_0 - t_0^\star) - \frac{|x_0 - x_0^\star|^2 + |t_0 - t_0^\star|^2}{2\varepsilon}$$

touches the original function v from below at the point $(x_0^\star, t_0^\star)$. Since $(x_0^\star, t_0^\star)$ is an interior point, the inequality

$$\operatorname{div}\big(|\nabla\psi(x_0^\star, t_0^\star)|^{p-2}\nabla\psi(x_0^\star, t_0^\star)\big) \le \psi_t(x_0^\star, t_0^\star)$$

holds by assumption. Because

$$\psi_t(x_0^\star, t_0^\star) = \varphi_t(x_0, t_0), \quad \nabla\psi(x_0^\star, t_0^\star) = \nabla\varphi(x_0, t_0), \quad D^2\psi(x_0^\star, t_0^\star) = D^2\varphi(x_0, t_0)$$

we also have that

$$\operatorname{div}\big(|\nabla\varphi(x_0, t_0)|^{p-2}\nabla\varphi(x_0, t_0)\big) \le \varphi_t(x_0, t_0)$$

at the original point (x_0, t_0), where φ was touching v_ε. Thus v_ε fulfills the requirement in the definition.

Proof by Comparison: We have to verify the comparison principle for v_ε in a subcylinder D_{t_1,t_2} having at least the distance $\sqrt{2L\varepsilon}$ to the boundary of Ω_T. To this end, assume that $h \in C(\overline{D_{t_1,t_2}})$ is a p-caloric function such that $v_\varepsilon \ge h$ on the parabolic boundary. It follows that the inequality

$$\frac{|x-y|^2 + |t-\tau|^2}{2\varepsilon} + v(y,\tau) \ge h(x,t)$$

is available when $(y,\tau) \in \Omega_T$ and (x,t) is on the parabolic boundary of D_{t_1,t_2}. Fix an arbitrary point (x_0, t_0) in D_{t_1,t_2}. Then we can take $y = x + x_0^\star - x_0$ and $\tau = t + t_0^\star - t_0$ in the inequality above so that

$$w(x,t) \equiv v(x + x_0^\star - x_0, t + t_0^\star - t_0) + \frac{|x_0 - x_0^\star|^2 + |t_0 - t_0^\star|^2}{2\varepsilon} \ge h(x,t)$$

when (x,t) is on the parabolic boundary. But the translated function w is p-supercaloric in the subcylinder D_{t_1,t_2}. By the comparison principle $w \ge h$ in the whole subcylinder. In particular,

$$v_\varepsilon(x_0, t_0) = w(x_0, t_0) \ge h(x_0, t_0).$$

This proves the comparison principle for v_ε. $\square$

The "viscosity proof" did not contain any explicite comparison principle while the "proof by comparison" required the piece of knowledge that the original v obeys the principle. This *parabolic* comparison principle allows comparison in space-time cylinders. We will encounter domains of a more general shape, but the following *elliptic* version of the principle turns out to be enough for our purpose. Instead of the expected parabolic boundary, the whole boundary (the "Euclidean" boundary) appears.

Proposition 25 *Given a domain $\Upsilon \subset\subset \Omega_\varepsilon$ and a p-caloric function $h \in C(\overline{\Upsilon})$, then $v_\varepsilon \geq h$ on the whole boundary $\partial\Upsilon$ implies that $v_\varepsilon \geq h$ in Υ.*

Now Υ does not have to be a space-time cylinder and $\partial\Upsilon$ is the total boundary in $\mathbb{R}^{n+1}$.

Proof: It is enough to realize that the proof is immediate when Υ is a finite union of space-time cylinders $D_j \times (a_j, b_j)$. To verify this, just start with the earliest cylinder(s) and pay due attention to the passages of t over the a_j's and b_j's. Then the general case follows by exhausting Υ with such unions. Indeed, given $\alpha > 0$ the compact set $\{h(x,t) \geq v_\varepsilon(x,t)\}$ is contained in an open finite union

$$\bigcup D_j \times (a_j, b_j)$$

comprised in Υ so that $h < v_\varepsilon + \alpha$ on the (Euclidean) boundary of the union. It follows that $h \leq v_\varepsilon + \alpha$ in the whole union. Since α was arbitrary, we conclude that $v_\varepsilon \geq h$ in Υ. $\square$

The above *elliptic* comparison principle does not acknowledge the presence of the parabolic boundary. The reasoning above can easily be changed so that the latest portion of the boundary is exempted. For this improvement, suppose that $t < T^\star$ for all $(x,t) \in \Upsilon$; in this case $\partial\Upsilon$ may have a plane portion with $t = T^\star$. It is now sufficient to verify that

$$v_\varepsilon \geq h \quad \text{on} \quad \partial\Upsilon \quad \text{when} \quad t < T^\star$$

in order to conclude that $v_\varepsilon \geq h$ in Υ. To see this, just use

$$v_\varepsilon + \frac{\sigma}{T^\star - t}$$

in the place of v_ε and then let $\sigma \to 0+$. This variant of the comparison principle is convenient for the proof of the following conclusion.

Lemma 26 *The approximant v_ε is a weak supersolution in the shrunken domain, i.e.*

$$\int_0^T\int_\Omega \Big(-v_\varepsilon\frac{\partial\phi}{\partial t}+\langle|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon,\nabla\phi\rangle\Big)\,dx\,dt\geq 0 \tag{20}$$

whenever $\phi\in C_0^\infty(\Omega_\varepsilon\times(\varepsilon,T-\varepsilon)),\ \phi\geq 0.$

Proof: We show that in a given subdomain $D_{t_1,t_2}=D\times(t_1,t_2)$ of the "shrunken domain" our v_ε coincides with the solution of an obstacle problem. The solutions of the obstacle problem are *per se* weak supersolutions. Hence, so is v_ε. Consider the class of all functions

$$\begin{cases} w\in C(\overline{D_{t_1,t_2}})\cap L^p(t_1,t_2;W^{1,p}(D)),\\ w\geq v_\varepsilon \text{ in } D_{t_1,t_2},\ \text{and}\\ w=v_\varepsilon \text{ on the parabolic boundary of } D_{t_1,t_2}.\end{cases}$$

The function v_ε itself acts as an obstacle and induces the boundary values. There exists a (unique) weak supersolution w_ε in this class satsfying the variational inequality

$$\int_{t_1}^{t_2}\int_D\Big((\psi-w_\varepsilon)\frac{\partial\psi}{\partial t}+\langle|\nabla w_\varepsilon|^{p-2}\nabla w_\varepsilon,\nabla(\psi-w_\varepsilon)\rangle\Big)dx\,dt$$
$$\geq\frac{1}{2}\int_D(\psi(x,t_2)-w_\varepsilon(x,t_2))^2\,dx$$

for all smooth ψ in the aforementioned class. Moreover, w_ε is p-caloric in the open set $A_\varepsilon=\{w_\varepsilon>v_\varepsilon\}$, where the obstacle does not hinder. We refer to [C]. On the boundary ∂A_ε we know that $w_\varepsilon=v_\varepsilon$ except possibly when $t=t_2$. By the elliptic comparison principle we have $v_\varepsilon\geq w_\varepsilon$ in A_ε. On the other hand $w_\varepsilon\geq v_\varepsilon$. Hence $w_\varepsilon=v_\varepsilon$.

To finish the proof, let $\varphi\in C_0^\infty(D_{t_1,t_2})$, $\varphi\geq 0$, and choose $\psi=v_\varepsilon+\varphi=w_\varepsilon+\varphi$ above. —An easy manipulation yields (20). $\Box$

Recall that $0\leq v\leq L$. Then also $0\leq v_\varepsilon\leq L$. An estimate for ∇v_ε is provided in the well-known lemma below.

Lemma 27 (Caccioppoli) *The inequality*

$$\int_0^T\int_\Omega\zeta^p|\nabla v_\varepsilon|^p\,dx\,dt\leq CL^p\int_0^T\int_\Omega|\nabla\zeta|^p\,dx\,dt+CL^2\int_0^T\int_\Omega\Big|\frac{\partial\zeta^p}{\partial t}\Big|\,dx\,dt$$

holds whenever $\zeta\in C_0^\infty(\Omega_\varepsilon\times(\varepsilon,T-\varepsilon)),\ \zeta\geq 0.$

Proof: Use the test function

$$\phi(x,t) = (L - v_\varepsilon(x,t))\zeta^p(x,t). \quad \square$$

The Caccioppoli estimate above leads to the conclusion that, keeping $0 \le v \le L$, the Sobolev gradient $\nabla v \in L^p_{loc}$ exists and

$$\nabla v_\varepsilon \rightharpoonup \nabla v \text{ weakly in } \ L^p_{loc},$$

at least for a subsequence. For v the Caccioppoli estimate

$$\int_0^T\!\!\int_\Omega \zeta^p |\nabla v|^p \, dx\, dt \le CL^p \int_0^T\!\!\int_\Omega |\nabla \zeta|^p \, dx\, dt + CL^2 \int_0^T\!\!\int_\Omega \left|\frac{\partial \zeta^p}{\partial t}\right| dx\, dt,$$

is immediate, because of the lower semicontinuity of the integral under weak convergence. However the corresponding passage to the limit under the integral sign of

$$\int_0^T\!\!\int_\Omega \Big(-v_\varepsilon \frac{\partial \phi}{\partial t} + \langle |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon, \nabla \phi\rangle\Big)\, dx\, dt \ge 0$$

requires a justification, as $\varepsilon \to 0$. The elementary vector inequality

$$\big||b|^{p-2}b - |a|^{p-2}a\big| \le (p-1)|b-a|\,(|b|+|a|)^{p-2}, \tag{21}$$

$p > 2$, and Hölder's inequality show that it is sufficient to establish that

$$\nabla v_\varepsilon \longrightarrow \nabla v \ \text{ strongly in } \ L^{p-1}_{loc},$$

to ackomplish the passage. Notice the exponent $p-1$ in place of p. This strong convergence is given in the next theorem, where the sequence is renamed to v_k.

Theorem 28 *Suppose that $v_1, v_2, v_3, \ldots$ is a sequence of Lipschitz continuous weak supersolutions, such that*

$$0 \le v_k \le L \ \ in \ \ \Omega_T = \Omega \times (0,t), \quad v_k \to v \ in \ \ L^p(\Omega_T).$$

Then

$$\nabla v_1, \nabla v_2, \nabla v_3, \ldots$$

is a Cauchy sequence in $L^{p-1}_{loc}(\Omega_T)$.

Proof: The central idea is that the measure of the set where $|v_j - v_k| > \delta$ is small. Given $\delta > 0$, we have, in fact,

$$\operatorname{mes}\{|v_j - v_k| > \delta\} \le \delta^{-p}\|v_j - v_k\|_p^p \tag{22}$$

according to Tshebyshef's inequality. Fix a test function $\theta \in C_0^\infty(\Omega_T),\ 0 \le \theta \le 1$. From the Caccioppoli estimate we can extract a bound of the form

$$\iint_{\{\theta \neq 0\}} |\nabla v_k|^p \, dx\, dt \le A^p, \quad k = 1, 2, \ldots,$$

since the support is a compact subset. Fix the indices k and j and use the test function

$$\varphi = (\delta - w_{jk})\theta$$

where

$$w_{jk} = \begin{cases} \delta, & \text{if } v_j - v_k > \delta \\ v_j - v_k, & \text{if } |v_j - v_k| \le \delta \\ -\delta, & \text{if } v_j - v_k < -\delta \end{cases}$$

in the equation

$$\int_0^T \int_\Omega \Big(-v_j \frac{\partial \phi}{\partial t} + \langle |\nabla v_j|^{p-2} \nabla v_j, \nabla \phi \rangle \Big) \, dx\, dt \ge 0.$$

Since $|w_{jk}| \le \delta$, we have $\varphi \ge 0$. In the equation for v_k use

$$\varphi = (\delta + w_{jk})\theta.$$

Add the two equations and arrange the terms:

$$\begin{aligned}
\iint_{|v_j - v_k| \le \delta} & \theta \langle |\nabla v_j|^{p-2}\nabla v_j - |\nabla v_k|^{p-2}\nabla v_k, \nabla v_j - \nabla v_k \rangle \, dx\, dt \\
&\le \delta \int_0^T \int_\Omega \langle |\nabla v_j|^{p-2}\nabla v_j + |\nabla v_k|^{p-2}\nabla v_k, \nabla\theta \rangle \, dx\, dt \\
&- \int_0^T \int_\Omega w_{jk} \langle |\nabla v_j|^{p-2}\nabla v_j - |\nabla v_k|^{p-2}\nabla v_k, \nabla\theta \rangle \, dx\, dt \\
+ \int_0^T \int_\Omega (v_j - v_k) & \frac{\partial}{\partial t}(\theta w_{jk}) \, dx\, dt - \delta \int_0^T \int_\Omega (v_j + v_k) \frac{\partial \theta}{\partial t} \, dx\, dt \\
&= I - II + III - IV.
\end{aligned}$$

The left-hand side is familiar from inequality (12). As we will see, the right-hand side is of the magnitude $O(\delta)$. We begin with term III, which contains time derivatives that ought to be avoided. Integration by parts yields

$$\begin{aligned} III &= \int_0^T \int_\Omega \theta \frac{\partial \theta}{\partial t} (\frac{w_{jk}^2}{2})\, dx\, dt + \int_0^T \int_\Omega (v_j - v_k) w_{jk} \frac{\partial \theta}{\partial t}\, dx\, dt \\ &= -\frac{1}{2} \int_0^T \int_\Omega w_{jk}^2 \frac{\partial \theta}{\partial t}\, dx\, dt + \int_0^T \int_\Omega (v_j - v_k) w_{jk} \frac{\partial \theta}{\partial t}\, dx\, dt. \end{aligned}$$

We obtain the estimate

$$|III| \le \frac{1}{2}\delta^2 \|\theta_t\|_1 + 2L\delta \|\theta_t\|_1 \le \delta C_3.$$

For the last term we immediately have

$$|IV| \le 2\delta L \|\theta_t\|_1 = \delta C_4.$$

The two first terms are easy,

$$|I| \le \delta C_1, \quad |II| \le \delta C_2.$$

Summing up,

$$|I| + |II| + |III| + |IV| \le C\delta.$$

Using the vector inequality (12) to estimate the left hand side, we arrive at

$$\begin{aligned} \iint_{|v_j - v_k| \le \delta} \theta |\nabla v_j - \nabla v_k|^p\, dx\, dt &\le 2^{p-2}\delta C, \\ \iint_{|v_j - v_k| \le \delta} \theta |\nabla v_j - \nabla v_k|^{p-1}\, dx\, dt &= O(\delta^{1-\frac{1}{p}}). \end{aligned}$$

We also have in virtue of (22)

$$\begin{aligned} &\iint_{|v_j - v_k| > \delta} \theta |\nabla v_j - \nabla v_k|^{p-1}\, dx\, dt \\ &\le \delta^{-1} \|v_j - v_k\|_p \left(\|\nabla v_j\|_p + \|\nabla v_k\|_p\right)^{p-1} \le (2A)^{p-1}\delta^{-1}\|v_j - v_k\|_p. \end{aligned}$$

Finally, combining the estimates over the sets $|v_j - v_k| \le \delta$ and $|v_j - v_k| > \delta$, we have an integral over the whole domain:

$$\int_0^T \int_\Omega \theta |\nabla v_j - \nabla v_k|^{p-1}\, dx\, dt \le O(\delta^{1-\frac{1}{p}}) + C_5 \delta^{-1} \|v_j - v_k\|_p.$$

Since the left-hand side is independent of δ, we can make it as small as we please, by first fixing δ small enough and then taking the indices large enough. $\square$

We have arrived at the following result for bounded supersolutions.

Theorem 29 *Let v be a* bounded *p-supercaloric function. Then*

$$\nabla v = \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_n}\right)$$

exists in Sobolev's sense, $\nabla v \in L^p_{loc}(\Omega_T)$, and

$$\int_0^T \int_\Omega \left(-v\frac{\partial \phi}{\partial t} + \langle |\nabla v|^{p-2}\nabla v, \nabla \phi\rangle\right) dx\, dt \geq 0$$

for all non-negative compactly supported test functions φ.

Notice that so far the exponent is p, as it should for v bounded.

Remark: It was established that the p-supercaloric functions are also weak supersolutions. In the opposite direction, according to [K] every weak supersolution is lower semicontinuous upon a redefinition in a set of (n+1)-dimensional measure 0. Moreover, the representative obtained as

$$\operatorname*{ess\,liminf}_{(y,\tau)\to(x,t)} v(y,\tau)$$

will do. A proof is given in Chapter 4.

We need a few auxiliary results.

Lemma 30 (Sobolev's inequality) *If $u \in L^p(0,T; W_0^{1,p}(\Omega))$, then*

$$\int_0^T \int_\Omega |u|^{p(1+\frac{2}{n})}\, dx\, dt \leq \; C \int_0^T \int_\Omega |\nabla u|^p\, dx\, dt \left\{ \mathop{ess\,sup}_{0<t<T} \int_\Omega |u(x,t)|^2\, dx \right\}^{\frac{p}{n}}.$$

.

Proof: See for example [dB, Chapter 1, Proposition 3.1].

If the test function ϕ is zero on the lateral boundary $\partial\Omega \times [t_1, t_2]$, then the differential inequality for the weak supersolution takes the form

$$\int_{t_1}^{t_2} \int_\Omega \Big(-v\frac{\partial \phi}{\partial t} + \langle |\nabla v|^{p-2}\nabla v, \nabla \phi\rangle\Big)\, dx\, dt$$
$$+ \int_\Omega v(x,t_2)\phi(x,t_2)\, dx \geq \int_\Omega v(x,t_1)\phi(x,t_1)\, dx.$$

Thus, if v is zero on the lateral boundary, we may take $\phi = v$ above. We obtain

$$\frac{1}{2}\int_\Omega v(x,t_1)^2\,dx \leq \int_{t_1}^{t_2}\int_\Omega |\nabla v|^p\,dx\,dt + \frac{1}{2}\int_\Omega v(x,t_2)^2\,dx, \tag{23}$$

which estimates the *past* in terms of the *future* and an "energy term".

3.3 Harnack's Convergence Theorem

The classical convergence theorem of Harnack states that the limit of an increasing sequence of harmonic functions is either a harmonic function itself or identically $+\infty$. The convergence is locally uniform. The situation is similar for many elliptic equations. However, the Evolutionary p-Laplace Equation exhibits a more delicate behaviour. The limit function can be finite at each point without being even locally bounded![21] This is a characteristic feature for the class $\mathfrak{M}$ previously defined. Consider a sequence of nonnegative p-caloric functions

$$0 \leq h_1 \leq h_2 \leq h_3 \leq \ \dots \qquad h = \lim_{k\to\infty} h_k$$

in Ω_T. There are two different possibilities, depending on whether the limit function h is locally bounded or not. The basic tool is an intrinsic version of Harnack's inequality, which is due to E. DiBenedetto, see [dB2, pp. 157–158] or [dB1].

Lemma 31 (Harnack) *Let $p > 2$. There are constants C and γ, depending only on n and p, such that if $u > 0$ is a continuous weak solution in*

$$B(x_0,4R)\times(t_0-4\theta,t_0+4\theta), \quad \textit{where} \quad \theta = \frac{CR^p}{u(x_0,t_0)^{p-2}}$$

then the inequality

$$u(x_0,t_0) \ \leq\ \gamma \inf_{B(x_0,R)} u(x,t_0+\theta) \tag{24}$$

is valid.

[21] This was unfortunately overlooked in [KL] and [KL2]. Corrections appear in [KP] and [KL3].

Notice how the waiting time θ depends on the solution itself. It is very short, if the solution is large.

Proposition 32 *Suppose that we have an increasing sequence*

$$0 \leq h_1 \leq h_2 \leq h_3 \leq \ \ldots \qquad h = \lim_{k\to\infty} h_k$$

of p-caloric functions h_k. If for some sequence

$$h_k(x_k, t_k) \to +\infty \quad \textit{and} \quad (x_k, t_k) \to (x_0, t_0)$$

where $x_0 \in \Omega$, $0 < t_0 < T$, then

$$\liminf_{\substack{(y,t)\to(x,t_0)\\ t>t_0}} h(y,t)(t-t_0)^{\frac{1}{p-2}} \;>\; 0 \quad \textit{for all} \quad x \in \Omega.$$

Thus, at time t_0,

$$\lim_{\substack{(y,t)\to(x,t_0)\\ t>t_0}} h(y,t) \;\equiv\; \infty \quad \textit{in} \quad \Omega.$$

Proof: Let $B(x_0, 4R) \subset\subset \Omega$. With

$$\theta_k = \frac{CR^p}{h_k(x_k,t_k)^{p-2}} \longrightarrow 0$$

we have by Harnack's inequality (24)

$$h_k(x_k,t_k) \leq \gamma h_k(x, t_k + \theta_k) \tag{25}$$

when $x \in B(x_k, R)$ provided that $B(x_k, 4R) \times (t_k - 4\theta_k, t_k + 4\theta_k) \subset\subset \Omega_T$. The center is moving, but since $x_k \to x_0$, equation (25) holds for sufficiently large indices k. Let $\Gamma > 1$. We want to compare the two solutions

$$\frac{\mathfrak{U}^R(x)}{\bigl(t - t_k + (\Gamma-1)\theta_k\bigr)^{\frac{1}{p-2}}} \qquad \text{and} \qquad h_k(x,t)$$

when $t = t_k + \theta_k$ and $x \in B(x_0, R)$. Here $\mathfrak{U}^R$ is the positive solution of the elliptic equation (18) with boundary values zero on $\partial B(x_0, R)$. By 25 we have

$$\begin{aligned}
&\frac{\mathfrak{U}^R(x)}{\big(t - t_k + (\Gamma-1)\theta_k\big)^{\frac{1}{p-2}}}\Bigg|_{t=t_k+\theta_k} = \frac{\mathfrak{U}^R(x)}{(\Gamma C R^p)^{\frac{1}{p-2}}}\, h_k(x_k, t_k)\\
&\leq \frac{\mathfrak{U}^R(x)}{(\Gamma C R^p)^{\frac{1}{p-2}}}\,\gamma\, h_k(x, t_k+\theta_k) \;\leq\; h_k(x, t_k+\theta_k)
\end{aligned}$$

if we fix Γ so large that

$$\gamma \|\mathfrak{U}^R\|_{L^\infty(B(x_0,R))} \;\leq\; (\Gamma C R^p)^{\frac{1}{p-2}}.$$

By the Comparison Principle

$$\frac{\mathfrak{U}^R(x)}{\big(t - t_k + (\Gamma-1)\theta_k\big)^{\frac{1}{p-2}}} \leq h_k(x,t) \leq h(x,t)$$

when $t \geq t_k + \theta_k$ and $x \in B(x_0, R)$. Sending k to ∞, we arrive at

$$\frac{\mathfrak{U}^R(x)}{(t-t_0)^{\frac{1}{p-2}}} \leq h(x,t) \quad \text{when} \quad t_0 < t < T.$$

This yields the desired estimate, though only in a subdomain. repeat the procedure in suitably chosen balls, thus extending the estimate to the entire domain Ω. $\square$

Proposition 33 *Suppose that we have an increasing sequence*

$$0 \leq h_1 \leq h_2 \leq h_3 \leq \ldots \qquad h = \lim_{k\to\infty} h_k$$

of p-caloric functions h_k in Ω_T. If the sequence is locally bounded, then the limit function h is p-caloric in Ω_T.

Remark: The situation is delicate. It is not enough to assume that h is finite at every point. This is different for elliptic equations! So is it for the Heat Equation.

Proof: In a fixed strict subdomain we have Hölder continuity

$$|h_k(x_1,t_1) - h_k(x_2,t_2)| \le C\|h_k\| \left(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{p}} \right).$$

Here $\|h_k\| \le \|h\| < \infty$ so that the family is locally equicontinuous. Hence the convergence $h_k \to h$ is locally uniform in Ω_T and, consequently, the limit function h is continuous.

From the usual Caccioppoli estimate

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega \zeta^p |\nabla h_k|^p \,dxdt \\
&\le C(p)\int_{t_1}^{t_2}\int_\Omega h_k^p |\nabla\zeta|^p \,dxdt + C(p)\int_\Omega \zeta(x)^p h_k(x,t)^2 \Big|_{t_1}^{t_2} dx \\
&\le C(p)\int_{t_1}^{t_2}\int_\Omega h^p |\nabla\zeta|^p \,dxdt + C(p)\int_\Omega \zeta(x)^p h(x,t)^2 \Big|_{t_1}^{t_2} dx
\end{aligned}$$

we can, in a standard way, conclude that $h \in L^p_{loc}(0,T;W^{1,p}_{loc}(\Omega))$. It is easy to see that h satisfies the Comparison Principle in Ω_T, since each h_k does it and the convergence is uniform. From Theorem 24 we conclude that the equation

$$\int_0^T\int_\Omega \Big(-h\frac{\partial\varphi}{\partial t} + \langle |\nabla h|^{p-2}\nabla h, \nabla\varphi\rangle \Big)\,dxdt \;=\; 0$$

is valid. $\square$

3.4 Unbounded Supersolutions

We proceed to study an *un*bounded p-supercaloric function v. Let us briefly describe the method. The starting point is to apply Theorem 29 on the functions $v_k = \min\{v, k\}$ so that estimates depending on $k = 1, 2, \cdots$ are obtained. To begin with, it is crucial that

$$v_k \in L^p(0,T;W^{1,p}_0(\Omega)).$$

Then an iterative procedure is used to gradually increase the summability exponent of v. First, we achieve that v^α is locally summable for some small exponent $\alpha < p - 2$. That result is iterated, again using the v_k's till we come close to the exponent $\alpha = p - 1 - 0$. Then the passage over $p - 1$ requires a special, although simple, device. At the end we will reach the desired summability for the function v itself. From this it is not difficult to obtain the corresponding result also for the gradient ∇v. Again the v_k's are employed. Finally, one has the problem to remove the restriction about zero lateral boundary values. This is not possible for functions of class $\mathfrak{M}$[22]. For class $\mathfrak{B}$, this is done in Section 4.

The considerations are in a bounded subdomain, which we again call $\Omega_T = \Omega \times (0, T)$, for simplicity. The situation is much easier when the function is zero on the whole parabolic boundary:

$$v(x, 0) = 0 \text{ when } x \in \Omega, \quad v = 0 \text{ on } \partial\Omega \times [0, T].$$

We assume that $v \geq 0$. The functions

$$v_k(x, t) = \min\{v(x, t), k\}, \quad k = 0, 1, 2, \dots,$$

cut off at the height k are bounded, whence the previous results in Section 3.2 apply for them. Fix a large index j. We may use the test functions

$$\phi_k = (v_k - v_{k-1}) - (v_{k+1} - v_k), \quad k = 1, 2, \dots, j$$

in the equation

$$\int_0^\tau \int_\Omega \langle |\nabla v_j|^{p-2} \nabla v_j, \nabla \phi_j \rangle \, dx \, dt + \int_0^\tau \int_\Omega \phi_k \frac{\partial v_j}{\partial t} \, dx \, dt \geq 0,$$

where $0 < \tau \leq T$. Indeed, $\phi_k \geq 0$. The "forbidden" time derivative can be avoided through an appropriate regularization. In principle v_j is first replaced by its convolution with a mollifier and later the limit is to be taken. We postpone this complication in order to keep the exposition more transparent. The insertion of the test function yields

$$\int_0^\tau \int_\Omega \Big(\langle |\nabla v_j|^{p-2} \nabla v_j, \nabla (v_{k+1} - v_k) \rangle \; + (v_{k+1} - v_k) \frac{\partial v_j}{\partial t} \Big) dx \, dt$$
$$\leq \int_0^\tau \int_\Omega \Big(\langle |\nabla v_j|^{p-2} \nabla v_j, \nabla (v_k - v_{k-1}) \rangle \; + (v_k - v_{k-1}) \frac{\partial v_j}{\partial t} \Big) dx \, dt,$$

succinctly written as

$$a_{k+1}(\tau) \leq a_k(\tau).$$

[22] Their infinities always hit the lateral boundary.

It follows that
$$\sum_{k=1}^{j} a_k(\tau) \le j a_1(\tau)$$
and, since the sum is "telescoping", we have the result below.

Lemma 34 *If each* $v_k \in L^p(0,T;W_0^{1,p}(\Omega))$ *and* $v_k(x,0) = 0$ *when* $x \in \Omega$, *then*
$$\begin{aligned}&\int_0^\tau \int_\Omega |\nabla v_j|^p \, dx \, dt + \frac{1}{2}\int_\Omega v_j^2(x,\tau)\, dx\\ \le j &\int_0^\tau \int_\Omega |\nabla v_1|^p \, dx \, dt + j\int_\Omega v_j(x,\tau)\, dx\end{aligned}$$
holds for a.e. τ *in the range* $0 < \tau \le T$. *Consequently,*
$$\begin{aligned}&\int_0^T \int_\Omega |\nabla v_j|^p \, dx \, dt + \frac{1}{2}\sup_{0<t<T}\int_\Omega v_j^2(x,t)\, dx\\ &\le 2j^2\Big(\int_0^T \int_\Omega |\nabla v_1|^p \, dx \, dt + |\Omega|\Big).\end{aligned}$$

Before continuing, we justify the use of the time derivative in the previous reasoning.

Regularisation of the equation. We use the convolution
$$(f \star \rho_\varepsilon)(x,t) = \int_{-\infty}^{\infty} f(x,t-s)\rho_\varepsilon(s)\, ds, \tag{26}$$
where ρ_ε is, for instance, Friedrich's mollifier defined as
$$\rho_\varepsilon(t) = \begin{cases} \frac{C}{\varepsilon} e^{-\varepsilon^2/(\varepsilon^2 - t^2)}, & |t| < \varepsilon,\\ 0, & |t| \ge \varepsilon.\end{cases}$$
If the function v_j is extended as 0 when $t \le 0$, $x \in \Omega$, the new function is p-supercaloric in $\Omega \times (-\infty, T)$, indeed. To see this, one has only to verify the comparison principle.

We have, when $\tau \le T - \varepsilon$
$$\int_{-\infty}^{\tau}\int_\Omega \Big(\langle (|\nabla v_j|^{p-2}\nabla v_j)\star \rho_\varepsilon, \nabla\varphi\rangle + \varphi \frac{\partial}{\partial t}(v_j \star \rho_\varepsilon)\Big)\, dx\, dt \ge 0$$

for all test functions $\varphi \geq 0$ vanishing on the lateral boundary. Replace v_k in the previous proof by

$$\tilde{v}_k = \min\{v_j \star \rho_\varepsilon, k\}$$

and choose

$$\varphi_k = (\tilde{v}_k - \tilde{v}_{k-1}) - (\tilde{v}_{k+1} - \tilde{v}_k).$$

Since the convolution with respect to the time variable does not affect the zero boundary values on the lateral boundary, we conclude that $\tilde{v}_k$ vanishes on the parabolic boundary of $\Omega \times (-\delta/2, T - \delta/2)$, when $\varepsilon < \delta/2$ and δ can be taken as small as we wish. (The functions $v_k \star \rho_\varepsilon$ instead of the employed $(v \star \rho_\varepsilon)_k$ do not work well in this proof.) The same calculations as before yield

$$\tilde{a}_{k+1}(\tau) \leq \tilde{a}_k(\tau) \quad \text{and} \quad \sum_{k=1}^{j} \tilde{a}_k(\tau) \leq j\, \tilde{a}_1(\tau),$$

where

$$\begin{aligned} \tilde{a}_k(\tau) = & \int_{-\delta/2}^{\tau} \int_\Omega \langle (|\nabla v_j|^{p-2} \nabla v_j) \star \rho_\varepsilon, \nabla(\tilde{v}_k - \tilde{v}_{k-1}) \rangle \, dx\, dt \\ & + \int_{-\delta/2}^{\tau} \int_\Omega (\tilde{v}_k - \tilde{v}_{k-1}) \frac{\partial}{\partial t} (v_j \star \rho_\varepsilon) \, dx\, dt. \end{aligned}$$

Summing up, we obtain

$$\begin{aligned} \sum_{k=1}^{j} \tilde{a}_k(\tau) = & \int_{-\delta/2}^{\tau} \int_\Omega \langle (|\nabla v_j|^{p-2} \nabla v_j) \star \rho_\varepsilon, \nabla \tilde{v}_j \rangle \, dx\, dt \\ & + \int_{-\delta/2}^{\tau} \int_\Omega \tilde{v}_j \frac{\partial}{\partial t} (v_j \star \rho_\varepsilon) \, dx\, dt, \end{aligned}$$

where the last integral can be written as

$$\frac{1}{2} \int_\Omega (v_j \star \rho_\varepsilon)^2 (x, \tau) \, dx.$$

Also for $\tilde{a}_1(\tau)$ we get an expression free of time derivatives. Therefore we can safely first let $\varepsilon \to 0$ and then $\delta \to 0$. This leads to the lemma.

Let us return to the lemma. Provided that we already have a majorant for the term

$$\int_0^T \int_\Omega |\nabla v_1|^p \, dx\, dt,$$

we see that

$$\int_0^T \int_\Omega |\nabla v_j|^p \, dx \, dt = O(j^2).$$

Yet, the right magnitude is $O(j)$.

Lemma 35 *Suppose that* $v_j \in L^p(0, T; W_0^{1,p}(\Omega))$ *and*

$$\int_0^T \int_\Omega |\nabla v_j|^p \, dx \, dt \le K j^2, \qquad j = 1, 2, 3, \dots$$

Then $v \in L^q(\Omega_T)$, *whenever* $q < p - 2$. *(Here* $p > 2$.*)*

Proof: (Recall that the desired bound is $p - 1 + \frac{p}{n}$ and not the above $p - 2$.) The assumption and Sobolev's inequality (Lemma 30) will give us a bound on the measure of the level sets

$$E_j = \{(x,t) | \, j \le v(x,t) \le 2j\}$$

so that the integral can be controlled. To this end, denote

$$\kappa = 1 + \frac{2}{n}.$$

We have

$$j^{\kappa p} |E_j| \le \iint_{E_j} v_{2j}^{\kappa p} \, dx \, dt \le \int_0^T \int_\Omega v_{2j}^{\kappa p} \, dx \, dt$$

$$\le C \int_0^T \int_\Omega |\nabla v_{2j}|^p \, dx \, dt \cdot \left(\operatorname*{ess\,sup}_{0<t<T} \int_\Omega v_{2j}^2 \, dx \right)^{\frac{p}{n}} \le C K j^2 \left(4 |\Omega| j^2 \right)^{\frac{p}{n}}.$$

It follows that

$$|E_j| \le \text{Constant} \, j^{2-p}.$$

We use this to estimate the L^q-norm using a dyadic division of the domain. Thus

$$\begin{aligned} \int_0^T \int_\Omega v^q \, dx \, dt &\le T \, |\Omega| + \sum_{j=1}^{\infty} \int \int_{E_{2^{j-1}}} v^q \, dx \, dt \\ &\le T \, |\Omega| + \sum_{j=1}^{\infty} 2^{jq} |E_{2^{j-1}}| \\ &\le T \, |\Omega| + C \sum_{j=1}^{\infty} 2^{j(q+2-p)}, \end{aligned}$$

which is a convergent majorant when $q < p - 2$. $\Box$

Remark: If the majorant Kj^2 in the assumption is replaced by a better Kj^γ, then the procedure yields that $|E_j| \approx j^{\gamma - p}$ resulting in $q < p - \gamma$.

The previous lemma guarantees that v^ε is summable for some small positive power ε, since[23] $p > 2$. To improve the exponent, we start from Lemma 34 and write the estimate in the form

$$\int_0^{t_1}\int_\Omega |\nabla v_j|^p \, dx\, dt \leq j \int_0^T \int_\Omega |\nabla v_1|^p \, dx\, dt + j \int_\Omega v_j(x,\tau)\, dx,$$

where $0 < t_1 \leq \tau \leq T$. Integrate with respect to τ over the interval $[t_1, T]$:

$$\begin{aligned}(T - t_1) \int_0^{t_1}\int_\Omega |\nabla v_j|^p \, dx\, dt &\leq j(T - t_1)K + j \int_{t_1}^T \int_\Omega v_j(x,t)\, dx\, dt \\ &\leq j(T-t_1)K + j^{2-\varepsilon} \int_{t_1}^T \int_\Omega v^\varepsilon(x,t)\, dx\, dt.\end{aligned}$$

Thus we have reached the estimate

$$\int_0^{t_1}\int_\Omega |\nabla v_j|^p \, dx\, dt \leq j^{2-\varepsilon} K_1, \qquad (t_1 < T). \tag{27}$$

This is an improvement from j^2 to $j^{2-\varepsilon}$, but we have to obey the restriction that $\varepsilon \leq 1$, because the term $j(T - t_1)K$ was absorbed. Estimating again the measures $|E_j|$, but this time starting with the bound $j^\gamma K$, $\gamma = 2 - \varepsilon$ in place of $j^2 K$, yields

$$|E_j| \approx j^{2-p-\varepsilon}, \qquad 0 < \varepsilon \leq 1.$$

The result is that

$$\int_0^{t_1}\int_\Omega v^q \, dx\, dt < \infty \quad \text{when} \quad 0 < q < p - \gamma = p - 2 + \varepsilon.$$

Iterating, we have the scheme

$$\begin{array}{ll} q_0 = \varepsilon & T \\ q_1 = p - 2 + \varepsilon & t_1 \\ q_2 = 2(p-2) + \varepsilon & t_2 \end{array}$$

[23] It does not work for the Heat Equation!

We can continue till we reach

$$q_k = k(p-2) + \varepsilon > p - 1.$$

We have to stop, because the previous exponent $(k-1)(p-2)+\varepsilon$ has to obey the rule not to become larger than 1. This way we can reach that

$$v \in L^1(\Omega_{T'}), \tag{28}$$

with a $T' < T$, which will do to proceed. In fact, adjusting we can reach any exponent strictly below $p-1$, but the passage over the exponent $p-1$ requires a special device. Since only a finite number of steps were involved, we can take T' as close to T as we wish.

We use inequality (23) in the form

$$\frac{1}{2}\int_\Omega v_j(x,t)^2\,dx \le \int_0^\tau\int_\Omega |\nabla v_j|^p\,dx\,dt + \frac{1}{2}\int_\Omega v_j(x,\tau)^2\,dx, \tag{29}$$

where $t < \tau$. For $t_1 < \tau < T$ it follows that

$$\begin{aligned}
\operatorname*{ess\,sup}_{0<t<t_1}\int_\Omega v_j(x,t)^2\,dx &\le 2\int_0^\tau\int_\Omega |\nabla v_j|^p\,dx\,dt + \int_\Omega v_j(x,\tau)^2\,dx\\
&\le 2j\int_0^\tau\int_\Omega |\nabla v_1|^p\,dx\,dt + 2j\int_\Omega v_j(x,\tau)\,dx\\
&\le 2j\int_0^T\int_\Omega |\nabla v_1|^p\,dx\,dt + 2j\int_\Omega v_j(x,\tau)\,dx,
\end{aligned}$$

where the middle step is from Lemma 34. We integrate the resulting inequality with respect to τ over the interval $[t_1, T]$, which affects only the last integral. Upon division by $T - t_1$, the last term is replaced by

$$\frac{2j}{T-t_1}\int_{t_1}^T\int_\Omega v_j\,dx\,dt.$$

We can combine this and the earlier estimate

$$\int_0^{t_1}\int_\Omega |\nabla v_j|^p\,dx\,dt \le j\int_0^T\int_\Omega |\nabla v_1|^p\,dx\,dt + \frac{j^{2-\varepsilon}}{T-t_1}\int_{t_1}^T\int_\Omega v^\varepsilon\,dx\,dt,$$

taking $\varepsilon = 1$, so that we finally arrive at

$$\begin{aligned}
&\int_0^{t_1}\int_\Omega |\nabla v_j|^p\,dx\,dt + \operatorname*{ess\,sup}_{0<t<t_1}\int_\Omega v_j(x,t)^2\,dx\\
&\le 3j\int_0^T\int_\Omega |\nabla v_1|^p\,dx\,dt + \frac{3j}{T-t_1}\int_{t_1}^T\int_\Omega v\,dx\,dt.
\end{aligned} \tag{30}$$

The majorant is now $O(j)$, which is of the right order, as the following lemma shows with its sharp exponents.

Lemma 36 *If $v_j \in L^p(0,T;W_0^{1,p}(\Omega))$ and*

$$\int_0^T\int_\Omega |\nabla v_j|^p\,dx\,dt + \mathop{ess\,sup}_{0<t<T}\int_\Omega v_j(x,t)^2\,dx \le jK$$

when $j = 1, 2, \ldots$ then ∇v exists and

$$\begin{aligned} v \in L^q(\Omega_T) \quad &\textit{whenever} \quad 0<q<p-1+\frac{p}{n},\\ \nabla v \in L^q(\Omega_T) \quad &\textit{whenever} \quad 0<q<p-1+\frac{1}{n+1}. \end{aligned}$$

Proof: The first part is a repetition of the proof of Lemma 35. Denote again

$$E_j = \{(x,t)\,|\, j\le v(x,t)\le 2j\}, \qquad \kappa = 1+\frac{2}{n}.$$

We have as before

$$\begin{aligned} j^{\kappa p}|E_j| &\le \iint_{E_j} v_{2j}^{\kappa p}\,dx\,dt \le \int_0^T\int_\Omega v_{2j}^{\kappa p}\,dx\,dt\\ &\le C\int_0^T\int_\Omega |\nabla v_{2j}|^p\,dx\,dt\cdot\left(\operatorname*{ess\,sup}_{0<t<T}\int_\Omega v_{2j}^2\,dx\right)^{\frac{p}{n}} \le CK^{1+\frac{p}{n}}(2j)^{1+\frac{p}{n}}. \end{aligned}$$

It follows that

$$|E_j| \le \text{Constant}\times j^{1-p-\frac{p}{n}}.$$

We estimate the L^q-norm using the subdivision of the domain. Thus

$$\begin{aligned} \int_0^T\int_\Omega v^q\,dx\,dt &\le T\,|\Omega| + \sum_{j=1}^\infty \iint_{E_{2^{j-1}}} v^q\,dx\,dt\\ &\le T\,|\Omega| + \sum_{j=1}^\infty 2^{jq}|E_{2^{j-1}}|\\ &\le T\,|\Omega| + C\sum_{j=1}^\infty 2^{j(q+1-p-\frac{p}{n})}, \end{aligned}$$

which converges in the desired range for q. Thus the first part is proved.

For the summability of the gradient, we use the bound on the measure of the level sets E_j and also the growth assumed for the energy of the truncated functions. Fix a large index k and write, using that $|\nabla v_k| \leq |\nabla v_{2^j}|$ on $E_{2^{j-1}}$:

$$\begin{aligned}
\int_0^T \int_\Omega |\nabla v_k|^q \, dx \, dt &\lesssim \sum_{j=1}^{\infty} \int\int_{E_{2^{j-1}}} |\nabla v_k|^q \, dx \, dt \\
&\leq \sum_{j=1}^{\infty} \left(\int\int_{E_{2^{j-1}}} |\nabla v_k|^p \, dx \, dt \right)^{\frac{q}{p}} |E_{2^{j-1}}|^{1-\frac{q}{p}} \\
&\leq \sum_{j=1}^{\infty} \left(\int_0^T \int_\Omega |\nabla v_{2^j}|^p \, dx \, dt \right)^{\frac{q}{p}} 2^{(j-1)(1-\frac{q}{p})(1-p-\frac{p}{n})} \\
&\leq \sum_{j=1}^{\infty} 2^{(j-1)(1-\frac{q}{p})(1-p-\frac{p}{n})} (2^j K)^{\frac{q}{p}},
\end{aligned}$$

where the geometric series converges provided that $q < p - 1 + 1/(n+1)$. Strictly speaking, the "first term" $K^{q/p}(T|\Omega|)^{1-q/p}$ ought to be added to the sum, since the integral over the set $\{0 < v < 1\}$ was missing. Now we may let k go to infinity. $\square$

A combination of the results in this section (formula 28, equation (30), and Lemma 36) yields the following

Lemma 37 *Suppose that $v \geq 0$ is a p-supercaloric function in Ω_T with initial values $v(x,0) = 0$ in Ω. If every $v_k \in L^p(0,T; W_0^{1,p}(\Omega))$, then*

$$\begin{aligned}
v \in L^q(\Omega_{T_1}) \quad &\textit{whenever} \quad 0 < q < p - 1 + \frac{p}{n}, \\
\nabla v \in L^q(\Omega_{T_1}) \quad &\textit{whenever} \quad 0 < q < p - 1 + \frac{1}{n+1}
\end{aligned}$$

when $T_1 < T$. In particular, v is of class $\mathfrak{B}$.

In principle, this lemma is Theorem 2 in the special case with zero lateral boundary values.

4 Proof of the Theorem

For the proof of Theorem 2 we start with a non-negative p-supercaloric function v defined in Ω_T. A simple device is used for the initial values: fix a small $\delta > 0$ and redefine v so that $v(x,t) \equiv 0$ when $t \leq \delta$. This function is p-supercaloric, since it obviously satisfies the comparison principle. This does not affect the statement of the theorem, since we can take δ as small as we please. The initial condition $v(x,0) = 0$ required in Lemma 37 is now in order.

Let $Q_{2l} \subset\subset \Omega$ be a cube with side length $4l$ and consider the concentric cube

$$Q_l = \{x \big| \, |x_i - x_i^0| < l, \, i = 1, 2, \ldots n\}$$

of side length $2l$. The center is at x^0. The main difficulty is that v is not zero on the lateral boundary, neither does v_j obey Lemma 37. We aim at correcting v outside $Q_l \times (0,T)$ so that also the new function is p-supercaloric and, in addition, satisfies the requirements of zero boundary values in Lemma 37. Thus we study the function

$$w = \begin{cases} v & \text{in} \quad Q_l \times (0,T) \\ h & \text{in} \quad (Q_{2l} \setminus Q_l) \times (0,T) \end{cases} \tag{31}$$

where the function h is, in the outer region, the weak solution to the boundary value problem

$$\begin{cases} h = 0 & \text{on} \quad \partial Q_{2l} \times (0,T) \\ h = v & \text{on} \quad \partial Q_l \times (0,T) \\ h = 0 & \text{on} \quad (Q_{2l} \setminus Q_l) \times \{0\} \end{cases} \tag{32}$$

An essential observation is that the solution h does not always exist! This counts for the dichotomy described in the main Theorem 2. If it exists, the truncations w_j satisfy the assumptions in Lemma 37, as we shall see.

For the construction we use the infimal convolutions

$$v^\varepsilon(x,t) = \inf_{(y,\tau) \in \Omega_T} \Big\{ v(y) + \frac{1}{2\varepsilon}(|x-y|^2 + |t-\tau|^2) \Big\}$$

described in Section 3.2. They are Lipschitz continuous in $\overline{Q_{2l}} \times [0,T]$. They are weak supersolutions when ε is small enough according to Proposition 24 and Theorem 29. Then we define the solution h^ε as in formula (32) above, but with v^ε in place of v. Then we construct

$$w^\varepsilon = \begin{cases} v^\varepsilon & \text{in} \quad Q_l \times (0,T) \\ h^\varepsilon & \text{in} \quad (Q_{2l} \setminus Q_l) \times (0,T) \end{cases}$$

and $w^\varepsilon(x,0) = 0$ in Ω. Now $h^\varepsilon \leq v^\varepsilon$, and when $t \leq \delta$ we have $0 \leq h^\varepsilon \leq v^\varepsilon = 0$ so that $h^\varepsilon(x,t) = 0$ when $t \leq \delta$. The function w^ε satisfies the comparison principle and is therefore a p-supercaloric function. Here it is essential that $h^\varepsilon \leq v^\varepsilon$! The function w^ε is also (locally) bounded; thus we have arrived at the conclusion that w^ε is a weak supersolution in $Q_{2l} \times (0,T)$. Se Theorem 29.

There are two possibilities, depending on whether the sequence $\{h^\varepsilon\}$ is bounded or not, when $\varepsilon \searrow 0$ through a sequence of values.

Bounded case. Assume that there does not exist any sequence of points such that

$$\lim_{\varepsilon\to 0} h^\varepsilon(x_\varepsilon, t_\varepsilon) = \infty, \quad (x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$$

where $x_0 \in Q_{2l} \setminus \overline{Q_l}$ and $0 < t_0 < T$ (that is an *interior* limit point). By Proposition 33

$$h = \lim_{\varepsilon\to 0} h^\varepsilon$$

is a p-caloric function in its domain. The function $w = \lim w^\varepsilon$ itself is p-supercaloric and agrees with formula (31).

By Theorem 29 the truncated functions

$$w_j = w_j(x,t) = \min\{w(x,t), j\}, \qquad j = 1, 2, 3, \ldots,$$

are weak supersolutions in $Q_{2l}\times(0,T)$. We claim that $w_j \in L^p(0,T'; W_0^{1,p}(Q_{2l}))$ when $T' < T$. This requires an estimation where we use

$$L = \sup\{h\} \quad \text{over} \quad (Q_{2l} \setminus Q_{5l/4}) \times (0, T').$$

Let $\zeta = \zeta(x)$ be a smooth function such that

$$0 \leq \zeta \leq 1, \quad \zeta = 1 \quad \text{in} \quad Q_{2l} \setminus Q_{3l/2}, \quad \zeta = 0 \quad \text{in} \quad Q_{5l/4}.$$

Using the test function $\zeta^p h$ when deriving the Caccioppoli estimate we get

$$\begin{aligned}
&\int_0^{T'}\int_{Q_{2l}\setminus Q_{3l/2}} |\nabla w_j|^p \mathrm{d}x\mathrm{d}t \\
&\leq \int_0^{T'}\int_{Q_{2l}\setminus Q_{3l/2}} |\nabla h|^p \mathrm{d}x\mathrm{d}t \ \leq \ \int_0^{T'}\int_{Q_{2l}\setminus Q_{5l/4}} \zeta^p |\nabla h|^p \mathrm{d}x\mathrm{d}t \\
&\leq C(p)\Big\{\int_0^{T'}\int_{Q_{2l}\setminus Q_l} h^p|\nabla\zeta|^p \mathrm{d}x\mathrm{d}t + \int_{Q_{2l}\setminus Q_{5l/4}} h(x,T')^2 \mathrm{d}x\Big\} \\
&\leq C(n,p)\Big(L^p l^{n-p}T + L^2 l^n\Big),
\end{aligned}$$

where we first used

$$|\nabla w_j| = |\nabla \min\{h,j\}| \leq |\nabla h|$$

in the outer region. Thus we have an estimate over the outer region $Q_{2l}\setminus Q_{3l/2}$. Concerning the inner region $Q_{3l/2}$, we first choose a test function $\eta = \eta(x,t)$

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \quad \text{in} \quad Q_{3l/2}, \quad \eta = 0 \quad \text{in} \quad Q_{2l}\setminus Q_{9l/4}.$$

Then the Caccioppoli estimate for the truncated functions

$$w_j = \min\{w, j\}, \qquad j = 1,2,3,\dots,$$

takes the form

$$\begin{aligned}
\int_0^{T'}\int_{Q_{3l/2}} |\nabla w_j|^p \mathrm{d}x\mathrm{d}t &\leq \int_0^{T'}\int_{Q_{2l}} \eta^p |\nabla w_j|^p \mathrm{d}x\mathrm{d}t \\
\leq Cj^p \int_0^{T'}\int_{Q_{2l}} |\nabla \eta|^p \mathrm{d}x\mathrm{d}t &+ Cj^p \int_0^{T'}\int_{Q_{2l}} |\eta_t|^p \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Therefore we have obtained an estimate over the whole domain $Q_{2l}\times(0,T')$:

$$\int_0^{T'}\int_{Q_{2l}} |\nabla w_j|^p \mathrm{d}x\mathrm{d}t \ \leq \ Cj^p$$

and it follows that $w_j \in L(0,T'; W_0^{1,p}(Q_{2l}))$. In particular, the crucial estimate

$$\int_0^{T'}\int_{Q_{2l}} |\nabla w_1|^p \mathrm{d}x\mathrm{d}t < \infty,$$

which was "assumed" in [KL], is now established.[24]

From Lemma 37 we conclude that $v \in L^q(Q_l \times (0,T'))$ and $\nabla v \in L^{q'}(Q_l \times (0,T'))$ with the correct summability exponents. Either we can proceed like this for all interor cubes, or then the following case happens.

Unbounded case[25]. If

$$\lim_{\varepsilon \to 0} h^\varepsilon(x_\varepsilon, t_\varepsilon) = \infty, \quad (x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$$

for some $x_0 \in Q_{2l} \setminus \overline{Q_l},\ 0 < t_0 < T,$ then

$$v(x,t) \geq h(x,t) \geq (t-t_0)^{-\frac{1}{p-2}}\mathfrak{U}(x), \quad \text{when} \quad t > t_0,$$

according to Proposition 32. Therefore

$$v(x, t_o+) \;=\; \infty$$

in $Q_{2l} \setminus \overline{Q_l}$. But in this construction we can replace the outer cube Q_{2l} with Ω, that is, a new h is defined in $\Omega \setminus \overline{Q_l}$. The proof is the same as above. Then by comparison

$$v \;\geq\; h^\Omega \;\geq\; h^{Q_{2l}}$$

and so $v(x, t_0+) = \infty$ in the whole boundary zone $\Omega \setminus \overline{Q_l}$.

It remains to include the inside, the cube Q_l. This is easy. Reflect $h = h^{Q_{2l}}$ in the plane $x_1 = x_1^0 + l$, which contains one side of the small cube:

$$h^*(x_1, x_2 \dots, x_n) = h(2x_1^0 + 2l - x_1, x_2, \dots, x_n),$$

so that

$$\frac{x_1 + \big(2(x_1^0 + l) - x_1\big)}{2} = x_1^0 + l$$

as it should. Recall that x^0 was the center of the cube. (The same can be done earlier for all the h^ε.) The reflected function h^* is p-caloric. Clearly, $v \geq h^*$ by comparison. This forces $v(x, t_0+) = 0$ when $x \in Q_l,\ x_1 > x_1^0$. A similar reflexion in the plane $x_1 = x_1^0 - l$ includes the other half $x_1 < x_1^0$. We have achieved that $v(x, t_0+) = \infty$ also in the inner cube Q_l. This proves that

$$v(x, t_o+) \;\equiv\; \infty \quad \text{in the whole} \quad \Omega.$$

[24]The class $\mathfrak{M}$ passed unnoticed in [KL2].

[25]This case is described in [KP] and [KL3].

$\square$

From the proof we can extract that

$$v(x,t) \geq \frac{\mathfrak{U}(x)}{(t-t_0)^{\frac{1}{p-2}}} \quad \text{in} \quad \Omega \times (t_0, T), \tag{33}$$

where $\mathfrak{U}$ is from equation (18).

5 Weak Supersolutions Are Semicontinuous

Are the weak supersolutions p-supercaloric functions (= viscosity supersolutions)? The question is seemingly trivial, but there is a requirement. To qualify they have to obey the comparison principle and to be semicontinuos. The comparison principle is rather immediate. The semicontinuity is a delicate issue. For a weak supersolution defined in the classical way with test functions under the integral sign (Definition 16) the Sobolev derivative is assumed to exist, but the semicontinuity, which now is not assumed, has to be established. The proof requires parts of the classical regularity theory[26]. We will use a variant of the Moser iteration, for practical reasons worked out for weak *sub*solutions bounded from below. Our proof of the theorem below is essentially the same as in [K], but we avoid the use of infinitely stretched infinitesimal space-time cylinders.

Theorem 38 *Suppose that $v = v(x,t)$ is a weak supersolution of the Evolutionary p-Laplace equation. Then it is locally bounded from below and at almost every point (x_0, t_0) it holds that*

$$v(x_0,t_0) = \operatorname{ess} \liminf_{(x,t)\to(x_0,t_0)} v(x,t).$$

In particular, v is lower semicontinuous after a redefinition in a set of measure zero.

Functions like $\operatorname{ess}\liminf v(x,t)$ are lower semicontinuous, if they are bounded from below. Thus the problem is the formula. The hardest part of the proof is to establish that the supremum norm of a non-negative weak subsolution is 1^0) bounded (Lemma 42) and 2^0) bounded in terms of quantities

[26]The preface of Giuseppe Mingione's work [M] is worth reading as an enlightment.

that can carry information from the Lebesgue points (Theorem 44). With such estimates the proof follows easily (at the end of this section). Before entering into the semicontinuity proof we address the comparison principle.

Proposition 39 (Comparison Principle) *Let Ω be bounded. Suppose that v is a weak supersolution and u a weak subsolution, $u, v \in L^p(0,T; W^{1,p}(\Omega))$, satisfying*

$$\liminf v \geq \limsup u$$

on the parabolic boundary. Then $v \geq u$ almost everywhere in the domain Ω_T.

Proof: This is well-known and we only give a formal proof. For a non-negative test function $\varphi \in C_0^\infty(\Omega_T)$ the equations

$$\int_0^T \int_\Omega (-v\varphi_t + \langle |\nabla v|^{p-2}\nabla v, \nabla\varphi\rangle)\, dx\, dt \geq 0$$
$$\int_0^T \int_\Omega (+u\varphi_t - \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle)\, dx\, dt \geq 0$$

can be added. Thus

$$\int_0^T \int_\Omega \big((u-v)\varphi_t + \langle |\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle\big)\, dx\, dt \geq 0.$$

These equations remain true if v is replaced by $v+\varepsilon$, where ε is any constant. To complete the proof we choose (formally) the test function to be

$$\varphi = (u - v - \varepsilon)_+ \eta,$$

where $\eta = \eta(t)$ is a cut-off function; even $\eta(t) = T - t$ will do here. We arrive at

$$\begin{aligned}
&\int_0^T \int_{u \geq v+\varepsilon} \eta(\langle |\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u, \nabla v - \nabla u\rangle)\, dx\, dt \\
&\leq \int_0^T \int_\Omega (u-v-\varepsilon)_+^2 \eta'\, dx\, dt + \frac{1}{2}\int_0^T \int_\Omega \eta \frac{\partial}{\partial t}(u-v-\varepsilon)_+^2\, dx\, dt \\
&= \frac{1}{2}\int_0^T \int_\Omega (u-v-\varepsilon)_+^2 \eta'\, dx\, dt \\
&= -\frac{1}{2}\int_0^T \int_\Omega (u-v-\varepsilon)_+^2\, dx\, dt \leq 0.
\end{aligned}$$

Since the first integral is non-negative by the vector inequality (12), the last integral is, in fact, zero. Hence the integrand $(u - v - \varepsilon)_+^2 = 0$ almost everywhere. But this means that

$$u \leq v + \varepsilon$$

almost everywhere. Since $\varepsilon > 0$ we have the desired inequality $v \geq u$ a.e.. $\square$

We need some estimates for the semicontinuity proof and begin with the well-known Caccioppoli estimates, which are extracted directly from the differential equation.

Lemma 40 (Caccioppoli estimates) *For a non-negative weak subsolution u in $\Omega \times (t_1, t_2)$ we have the estimates*

$$\begin{aligned}\operatorname*{ess\,sup}_{t_1<t<t_2} \int_\Omega \zeta^p u^{\beta+1}\, dx \leq & \int_{t_1}^{t_2}\int_\Omega u^{\beta+1}\Big|\frac{\partial}{\partial t}\zeta^p\Big|\, dx\, dt \\ & +2p^{p-1}\beta^{2-p}\int_{t_1}^{t_2}\int_\Omega u^{p-1+\beta}\,|\nabla\zeta|^p\; dx\, dt\end{aligned}$$

and

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega \Big|\nabla(\zeta u^{\frac{p-1+\beta}{p}})\Big|^p\, dx\, dt \leq & \; C\beta^{p-2}\int_{t_1}^{t_2}\int_\Omega u^{\beta+1}\Big|\frac{\partial}{\partial t}\zeta^p\Big|\, dx\, dt \\ & +C\int_{t_1}^{t_2}\int_\Omega u^{p-1+\beta}\,|\nabla\zeta|^p\; dx\, dt,\end{aligned}$$

where the exponent $\beta \geq 1$, $C = C(p)$, and $\zeta \in C^\infty(\Omega \times [t_1, t_2))$, $\zeta(x, t_1) = 0$, $\zeta \geq 0$.

Proof: Use the test function $\varphi = u^\beta \zeta^p$ in the equation

$$\begin{aligned}&\int_{t_1}^{\tau}\int_\Omega \big(-u\varphi_t + \langle|\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle\big)\, dx\, dt \\ &+\int_\Omega u(x,\tau)\varphi(x,\tau)\, dx \leq \int_\Omega u(x,t_1)\varphi(x,t_1)\, dx = 0,\end{aligned}$$

where $t_1 < \tau \leq t_2$. (The intermediate τ is needed to match the supremum in the first estimate.) Strictly speaking, the "forbidden" time derivative

u_t is required at the intermediate steps. This can be handled through a regularization, which we omit. Proceeding, integration by parts leads to

$$\int_{t_1}^{\tau}\int_{\Omega} -u\varphi_t \, dx \, dt + \int_{\Omega} u(x,\tau)\varphi(x,\tau)\, dx$$
$$= \frac{1}{\beta+1}\int_{\Omega}\zeta(x,\tau)^p u(x,\tau)^{\beta+1}\, dx - \frac{1}{\beta+1}\int_{t_1}^{\tau}\int_{\Omega} u^{\beta+1}\Big|\frac{\partial}{\partial t}\zeta^p\Big|\, dx\, dt$$

valid for a.e. τ. To treat the "elliptic term", we use

$$\nabla\varphi = \beta\zeta^p u^{\beta-1}\nabla u + p\zeta^{p-1}u^{\beta}\nabla\zeta$$

and obtain

$$\frac{1}{\beta+1}\int_{\Omega}\zeta(x,\tau)^p u(x,\tau)^{\beta+1}\, dx + \beta\int_{t_1}^{\tau}\int_{\Omega}\zeta^p u^{\beta-1}|\nabla u|^p\, dx\, dt$$
$$\leq \frac{1}{\beta+1}\int_{t_1}^{\tau}\int_{\Omega} u^{\beta+1}\Big|\frac{\partial}{\partial t}\zeta^p\Big|\, dx\, dt + p\int_{t_1}^{\tau}\int_{\Omega}\zeta^{p-1}u^{\beta}|\nabla u|^{p-1}|\nabla\zeta|\, dx\, dt.$$

As much as possible of the last integral must be absorbed by the double integral in the left-hand member. It is convenient to employ Young's inequality

$$ab \leq \frac{a^q}{q} + \frac{b^p}{p}$$

to achieve the splitting

$$\zeta^{p-1}u^{\beta}|\nabla u|^{p-1}|\nabla\zeta|$$
$$= \overbrace{\left(\frac{\beta}{p}\right)^{\frac{p-1}{p}}\zeta^{p-1}u^{(\beta-1)\frac{p-1}{p}}|\nabla u|^{p-1}}^{a} \times \overbrace{\left(\frac{p}{\beta}\right)^{\frac{p-1}{p}} u^{\frac{p-1+\beta}{p}}|\nabla\zeta|}^{b}$$
$$\leq \frac{p-1}{p}\left(\frac{\beta}{p}\right)\zeta^p u^{\beta-1}|\nabla u|^p + \frac{1}{p}\left(\frac{p}{\beta}\right)^{p-1} u^{p-1+\beta}|\nabla\zeta|^p,$$

which has to be multiplied by p and integrated. Absorbing one integral into the left-hand member, we arrive at the fundamental estimate

$$\frac{1}{\beta+1}\int_{\Omega}\zeta(x,\tau)^p u(x,\tau)^{\beta+1}\, dx + \frac{\beta}{p}\int_{t_1}^{\tau}\int_{\Omega}\zeta^p u^{\beta-1}|\nabla u|^p\, dx\, dt$$
$$\leq \frac{1}{\beta+1}\int_{t_1}^{\tau}\int_{\Omega} u^{\beta+1}|\frac{\partial}{\partial t}\zeta^p|\, dx\, dt + \left(\frac{p}{\beta}\right)^{p-1}\int_{t_1}^{\tau}\int_{\Omega} u^{p-1+\beta}|\nabla\zeta|^p\, dx\, dt.$$

Since the integrands are positive it follows that

$$\frac{1}{\beta+1}\int_\Omega \zeta(x,\tau)^p u(x,\tau)^{\beta+1}\,dx$$
$$\leq \frac{1}{\beta+1}\int_{t_1}^{t_2}\int_\Omega u^{\beta+1}\Big|\frac{\partial}{\partial t}\zeta^p\Big|\,dx\,dt + \left(\frac{p}{\beta}\right)^{p-1}\int_{t_1}^{t_2}\int_\Omega u^{p-1+\beta}|\nabla\zeta|^p\,dx\,dt,$$

where the majorant now is free from τ. Taking the supremum over τ we obtain the first Caccioppoli inequality.

To derive the second Caccioppoli inequality, we start from

$$\frac{\beta}{p}\int_{t_1}^{t_2}\int_\Omega \zeta^p u^{\beta-1}|\nabla u|^p\,dx\,dt$$
$$\leq \frac{1}{\beta+1}\int_{t_1}^{t_2}\int_\Omega u^{\beta+1}\Big|\frac{\partial}{\partial t}\zeta^p\Big|\,dx\,dt + \left(\frac{p}{\beta}\right)^{p-1}\int_{t_1}^{t_2}\int_\Omega u^{p-1+\beta}|\nabla\zeta|^p\,dx\,dt$$

and notice that

$$\zeta^p u^{\beta-1}|\nabla u|^p = \left(\frac{p}{p-1+\beta}\right)^p |\zeta\nabla u^{\frac{p-1+\beta}{p}}|^p.$$

Then the triangle inequality

$$|\nabla(\zeta u^{\frac{p-1+\beta}{p}})| \leq |\zeta\nabla u^{\frac{p-1+\beta}{p}}| + |u^{\frac{p-1+\beta}{p}}\nabla\zeta|$$

and a simple calculation yield the desired result. □

In the following version of Sobolev's inequality the exponents are adjusted to our need. For a proof, see [dB, Chapter 1].

Proposition 41 (Sobolev) *For $\zeta \in C^\infty(\Omega_T)$ vanishing on the lateral boundary $\partial\Omega \times [0,T]$ we have*

$$\int_0^T\int_\Omega \zeta^{p\gamma}|u|^{p-2+(\beta+1)\gamma}\,dx\,dt$$
$$\leq S\int_0^T\int_\Omega |\nabla(\zeta|u|^{\frac{p-1+\beta}{p}})|^p\,dx\,dt\left\{\operatorname*{ess\,sup}_{0<t<T}\int_\Omega \zeta^p|u|^{\beta+1}\,dx\right\}^{\frac{p}{n}},$$

where $\gamma = 1 + \frac{p}{n}$.

Now we can control the right-hand member in the Sobolev inequality by the quantities in the Caccioppoli estimates for the weak subsolution:

$$\left(\int_{t_1}^{t_2}\int_\Omega \zeta^{p\gamma} u^{p-2+(\beta+1)\gamma}\,dx\,dt\right)^{\frac{1}{\gamma}}$$
$$\leq C\beta^{\frac{(2-p)p}{n+p}}\left(\beta^{p-2}\int_{t_1}^{t_2}\int_\Omega u^{\beta+1}\left|\frac{\partial}{\partial t}\zeta^p\right|dx\,dt+\int_{t_1}^{t_2}\int_\Omega u^{p-1+\beta}\left|\nabla\zeta\right|^p\,dx\,dt\right).$$

We select the test function ζ so that it is equal to 1 in the cylinder $B_{R-\Delta R}\times(T+\Delta T,t_2)$, $\zeta(x,T)=0$, and so that $\zeta(x,t)=0$ when x is outside B_R. Then we can write

$$\left(\int_{T+\Delta T}^{t_2}\int_{B_{R-\Delta R}} u^{p-2+(\beta+1)\gamma}\,dx\,dt\right)^{\frac{1}{\gamma}}$$
$$\leq C\beta^{\frac{(2-p)p}{n+p}}\left(\frac{\beta^{p-2}}{\Delta T}\int_T^{t_2}\int_{B_R} u^{\beta+1}\,dx\,dt+\frac{1}{(\Delta R)^p}\int_T^{t_2}\int_{B_R} u^{p-1+\beta}\,dx\,dt\right),$$

where C is a new constant. Recall that $\gamma>1$. This is a *reverse Hölder inequality*, which is most transparent for $p=2$. It will be important to keep $\Delta T=(\Delta R)^p$. This is the basic inequality for the celebrated *Moser iteration*, which we will employ. The power of u increases to $p-2+(\beta+1)\gamma$, but the integral is taken over a smaller cylinder. In order to iterate over a chain of shrinking cylinders $U_k=B(x_0,R_k)\times(T_k,t_2)$, starting with

$$U_0=B(x_0,2R)\times(\frac{T}{2},t_2)$$

and ending up with an estimate over the cylinder

$$U_\infty=B(x_0,R)\times(T,t_2),$$

we introduce the quantities

$$R_k=R+\frac{R}{2^k},\qquad R_k-R_{k+1}=\frac{R}{2^{k+1}}$$
$$T_k=T-\frac{T}{2^{kp+1}},\qquad T_{k+1}-T_k=\frac{T}{2^{(k+1)p}}s,$$
$$\omega=\frac{R^p}{Ts}=\frac{(\Delta R_k)^p}{\Delta T_k}\qquad s=\frac{2^{p-1}-1}{2}.$$

We remark that ω *is independent of the index* k. Further, we write $\alpha = \beta + 1$, so that $\alpha \geq 2$. Thus

$$\left(\iint\limits_{U_{k+1}} u^{p-2+\alpha\gamma}\,dx\,dt\right)^{\frac{1}{\gamma}} \tag{34}$$

$$\leq C\,\frac{2^{(k+1)p}\beta^{\frac{(2-p)p}{n+p}}}{R^p}\left(\beta^{p-2}\omega\iint\limits_{U_k} u^{\alpha}\,dx\,dt + \iint\limits_{U_k} u^{p-2+\alpha}\,dx\,dt\right).$$

It is inconvenient to deal with *two* different integrals in the majorant. For simplicity we will perform two iteration procedures, depending on which integral is dominating. For the first procedure we assume that

$$\omega \leq u^{p-2}.$$

Then we have the simpler expression

$$\left(\iint\limits_{U_{k+1}} u^{p-2+\alpha\gamma}\,dx\,dt\right)^{\frac{1}{\gamma}} \leq C_1\,\frac{2^{kp}\alpha^{\frac{(p-2)}{\gamma}}}{R^p}\iint\limits_{U_k} u^{p-2+\alpha}\,dx\,dt.$$

We start the iteration with $\alpha = 2$ and $k = 0$. Thus

$$\left(\iint\limits_{U_1} u^{p-2+2\gamma}\,dx\,dt\right)^{\frac{1}{\gamma}} \leq C_1\,\frac{2^{0p}2^{\frac{(p-2)}{\gamma}}}{R^p}\iint\limits_{U_0} u^{p}\,dx\,dt.$$

Then take $\alpha = 2\gamma$ and $k = 1$ so that

$$\left(\iint\limits_{U_2} u^{p-2+2\gamma^2}\,dx\,dt\right)^{\frac{1}{\gamma^2}} \leq \left(C_1\,\frac{2^{1p}(2\gamma)^{\frac{(p-2)}{\gamma}}}{R^p}\iint\limits_{U_1} u^{p-2+2\gamma}\,dx\,dt\right)^{\frac{1}{\gamma}}$$

$$\leq \left(C_1\,\frac{2^{1p}(2\gamma)^{\frac{(p-2)}{\gamma}}}{R^p}\right)^{\frac{1}{\gamma}} \times C_1\,\frac{2^{0p}2^{\frac{(p-2)}{\gamma}}}{R^p}\iint\limits_{U_0} u^{p}\,dx\,dt.$$

The result of the next step is

$$\left(\iint\limits_{U_3} u^{p-2+2\gamma^3}\,dx\,dt\right)^{\frac{1}{\gamma^3}}$$
$$\leq \left(\frac{C_1 2^{\frac{p-2}{\gamma}}}{R^p}\right)^{1+\frac{1}{\gamma}+\frac{1}{\gamma^2}} 2^{p(\frac{1}{\gamma}+\frac{2}{\gamma^2})}\gamma^{(p-2)(\frac{1}{\gamma^2}+\frac{2}{\gamma^3})}\iint\limits_{U_0} u^p\,dx\,dt.$$

Continuing the chain and noticing that the geometric series

$$1+\frac{1}{\gamma}+\frac{1}{\gamma^2}+\frac{1}{\gamma^3}+\cdots = 1+\frac{n}{p}$$

and the series $\sum k\gamma^{-k}$ appearing in the exponents converge, since $\gamma > 1$, we arrive at

$$\left(\iint\limits_{U_{k+1}} u^{p-2+2\gamma^{k+1}}\,dx\,dt\right)^{\frac{1}{\gamma^{k+1}}} \leq KR^{-p(1+\frac{1}{\gamma}+\frac{1}{\gamma^2}+\cdots+\frac{1}{\gamma^k})}\iint\limits_{U_0} u^p\,dx\,dt.$$

Here K is a numerical constant. As $k \to \infty$, we obtain the final estimate

$$\operatorname*{ess\,sup}_{B_R\times(T,t_2)} (u^2) \leq \frac{K}{R^{n+p}}\int_{\frac{T}{2}}^{t_2}\int_{B_{2R}} u^p\,dx\,dt = \frac{K}{\omega s\,TR^n}\int_{\frac{T}{2}}^{t_2}\int_{B_{2R}} u^p\,dx\,dt,$$

where the square came from the factor 2 in $2\gamma^{k+1}$. The sum of the geometric series determined the power of R.

Finally, if the assumption $\omega \leq u^{p-2}$ is relaxed to $u \geq 0$, we can apply the previous estimate to the function

$$u(x,t)+\omega^{\frac{1}{p-2}} = u(x,t)+\left(\frac{R^p}{Ts}\right)^{\frac{1}{p-2}}.$$

A simple calculation gives us the bound in the next lemma.

Lemma 42 *Suppose that $u \geq 0$ is a weak subsolution in the cylinder $B_{2R}\times(\frac{T}{2},t_2)$. Then*

$$\operatorname*{ess\,sup}_{B_R\times(T,t_2)} \{u^2\} \leq C\left\{\left(\frac{R^p}{T}\right)^{\frac{2}{p-2}}+\frac{T}{R^p}\left(\frac{1}{TR^n}\int_{\frac{T}{2}}^{t_2}\int_{B_{2R}} u^p\,dx\,dt\right)\right\},$$

where $C = C(n,p)$.

We can extract the following piece of information.

Corollary 43 *A weak supersolution that is bounded from above, is locally bounded from below.*

Proof: Use $u(x,t) = L - v(x,t)$. $\Box$

The estimate in the lemma suffers from the defect that it is not sharp when $u \approx 0$ because of the presence of the constant term. Our remedy is a second iteration procedure, this time under the assumption that

$$0 \le u \le j,$$

where we take j so large that also

$$j^{p-2} \ge \omega.$$

Read j^{p-2} as $\max\{\omega, j^{p-2}\}$. The previous lemma shows that j is finite, but the point now is that u is not bounded away from zero. Then the first integral in the majorant of (34) is dominating and we can begin with the bound

$$\left(\iint\limits_{U_{k+1}} u^{p-2+\alpha\gamma}\, dx\, dt \right)^{\frac{1}{\gamma}} \le C\, j^{p-2} \frac{2^{kp} \alpha^{\frac{(p-2)}{\gamma}}}{R^p} \iint\limits_{U_k} u^{\alpha}\, dx\, dt.$$

We start the iteration with $\alpha = p$ and $k = 0$. Thus

$$\left(\iint\limits_{U_1} u^{p-2+p\gamma}\, dx\, dt \right)^{\frac{1}{\gamma}} \le C j^{p-2}\, \frac{2^{0p} p^{\frac{(p-2)}{\gamma}}}{R^p} \iint\limits_{U_0} u^{p}\, dx\, dt.$$

Then take $\alpha = p - 2 + p\gamma$, which is $< n\gamma^2$, and $k = 1$ so that

$$\begin{aligned} \left(\iint\limits_{U_2} u^{(p-2)(1+\gamma)+p\gamma^2}\, dx\, dt \right)^{\frac{1}{\gamma^2}} &\le \left(C j^{p-2}\, \frac{2^{1p} (n\gamma^2)^{\frac{(p-2)}{\gamma}}}{R^p} \iint\limits_{U_1} u^{p-2+p\gamma}\, dx\, dt \right)^{\frac{1}{\gamma}} \\ &\le \left(C j^{p-2}\, \frac{2^{1p} (n\gamma^2)^{\frac{(p-2)}{\gamma}}}{R^p} \right)^{\frac{1}{\gamma}} \times C j^{p-2}\, \frac{2^{0p} (n\gamma)^{\frac{(p-2)}{\gamma}}}{R^p} \iint\limits_{U_0} u^{p}\, dx\, dt. \end{aligned}$$

At the next step $\alpha = (p-2)(1+\gamma) + p\gamma^2 < n\gamma^3$ and $k = 2$. The result is

$$\left(\iint\limits_{U_3} u^{(p-2)(1+\gamma+\gamma^2)+p\gamma^3}\, dx\, dt \right)^{\frac{1}{\gamma^3}}$$
$$\leq \left(\frac{Cj^{p-2} n^{\frac{p-2}{\gamma}}}{R^p} \right)^{1+\frac{1}{\gamma}+\frac{1}{\gamma^2}} 2^{p(\frac{1}{\gamma}+\frac{2}{\gamma^2})} \gamma^{(p-2)(\frac{1}{\gamma}+\frac{2}{\gamma^2}+\frac{3}{\gamma^3})} \iint\limits_{U_0} u^p\, dx\, dt.$$

Continuing like this we end up with an estimate integrated over U_{k+1} with the power $\alpha_{k+1} = p - 2 + \alpha_k \gamma$, where

$$\begin{aligned}
\alpha_k &= (p-2)(1+\gamma+\gamma^2+\cdots+\gamma^{k-1}) + p\gamma^k \\
&= \frac{n(p-2)}{p}(\gamma^k - 1) + p\gamma^k \approx \Big(n + p - \frac{2n}{p}\Big)\gamma^k
\end{aligned}$$

and $\alpha_k < n\gamma^{k+1}$. As $k \to \infty$ we find that

$$\operatorname*{ess\,sup}_{B_R \times (T, t_2)} \{u^{n+p-\frac{2n}{p}}\} \leq C \frac{j^{\frac{(p-2)(n+p)}{p}}}{R^{n+p}} \int_{\frac{T}{2}}^{t_2} \int_{B_{2R}} u^p\, dx\, dt.$$

We can summarize the result.

Theorem 44 *A weak subsolution u that is non-negative in the cylinder $U = B(x_0, 2R) \times (t_0 - 3T/2, t_0 + T)$ has the bound*

$$\operatorname*{ess\,sup}_{B_R \times (t_0 - T, t_0 + T)} \{u^{n+p-\frac{2n}{p}}\} \leq K \frac{\left(\frac{R^p}{T} + \|u\|_\infty^{p-2} \right)^{1+\frac{n}{p}}}{TR^n} \int_{t_0 - \frac{3T}{2}}^{t_0+T} \int_{B_{2R}} u^p\, dx\, dt, \tag{35}$$

where $0 \leq u \leq \|u\|_\infty$ in U.

We need the fact that the positive part $(u)_+$ of a weak subsolution is again a weak subsolution. Here the proof has to avoid the comparison principle, which is not yet available. It reduces to the following lemma.

Lemma 45 *If v is a weak supersolution, so is $v_L = \min\{v, L\}$.*

Proof: Formally, the test function[27]

$$\varphi = \min\{k(L-v)_+, 1\}\zeta = \chi_k\zeta$$

inserted into

$$\int_0^T \int_\Omega \big(-v\varphi_t + \langle|\nabla v|^{p-2}\nabla v, \nabla\varphi\rangle\big)\, dx\, dt \geq 0$$

implies the desired inequality

$$\int_0^T \int_\Omega \big(-v_L\zeta_t + \langle|\nabla v_L|^{p-2}\nabla v_L, \nabla\zeta\rangle\big)\, dx\, dt \geq 0$$

at the limit $k = \infty$. As usual, $\zeta \in C_0^\infty(\Omega_T), \quad \zeta \geq 0$. The explanation is that $\lim \chi_k =$ the characteristic function of the set $\{v < L\}$. Under the assumption that the "forbidden" time derivative u_t is available at the intermediate steps we have

$$\begin{aligned}
&\int_0^T \int_\Omega \chi_k\big(-v\zeta_t + \langle|\nabla v|^{p-2}\nabla v, \nabla\zeta\rangle\big)\, dx\, dt \\
&\geq k \iint\limits_{L-\frac{1}{k}<v<L} \zeta|\nabla v|^p\, dx\, dt + \int_0^T \int_\Omega v\zeta\frac{\partial}{\partial t}\chi_k\, dx\, dt \\
&\geq \int_0^T \int_\Omega v\zeta\frac{\partial}{\partial t}\chi_k\, dx\, dt = -\frac{1}{2k}\int_0^T \int_\Omega \zeta\frac{\partial}{\partial t}\left(\chi_k\right)^2\, dx\, dt \\
&= +\frac{1}{2k}\int_0^T \int_\Omega \left(\chi_k\right)^2\zeta_t\, dx\, dt \longrightarrow 0.
\end{aligned}$$

The formula $\partial\chi_k/\partial t = -vk$ or $= 0$ was used. The result follows.

Finally, to handle the problem with the time derivative, one has first to regularize the equation and then to use the test function

$$\varphi^\varepsilon = \min\{k(L-v^\varepsilon)_+, 1\}\zeta = \chi_k\zeta,$$

where v^ε is the convolution in (26). The term

$$\int_0^T \int_\Omega -v^\varepsilon\frac{\partial\varphi^\varepsilon}{\partial t}dx\, dt$$

[27]This is from Lemma 2.109 on page 122 of J. Maly & W. Ziemer:"Fine Regularity of Solutions of Elliptic Partial Differential Equations", Math. Surveys Monogr. 51, AMS, Providence 1998.

can be written so that the derivative $\partial v^\varepsilon/\partial t$ disappears. Then one may safely let $\varepsilon \to 0$. The result follows as before. $\square$

Proof of Theorem 38: Let (x_0, t_0) be a Lebesgue point for the weak supersolution v. Then

$$\lim_{TR^n \to 0} \frac{1}{TR^n} \int_{t_0-2T}^{t_0+T} \int_{B_{2R}} |v(x_0,t_0) - v(x,t)|^p \, dx \, dt \;=\; 0.$$

A fortiori

$$\lim_{TR^n \to 0} \frac{1}{TR^n} \int_{t_0-2T}^{t_0+T} \int_{B_{2R}} (v(x_0,t_0) - v(x,t))_+^p \, dx \, dt \;=\; 0. \tag{36}$$

We claim that

$$v(x_0,t_0) \leq \operatorname{ess} \liminf_{(x,t)\to(x_0,t_0)} v(x,t). \tag{37}$$

It is sufficient to establish that

$$\operatorname{ess} \limsup_{(x,t)\to(x_0,t_0)} (v(x_0,t_0) - v(x,t))_+ = 0,$$

since those points where $v(x,t) \geq v(x_0,t_0)$ can do no harm to inequality (37).

To this end, notice that the function $v(x_0,t_0) - v(x,t)$ is a weak subsolution and so is its positive part, the function

$$u(x,t) = (v(x_0,t_0) - v(x,t))_+$$

by Lemma 45. It is locally bounded according to Lemma 42. Thus the essliminf is $> -\infty$ in (37). Use Theorem 44 and let $TR^n \to 0$, keeping $R^p/T \leq$ Constant. In virtue of (35) it follows that

$$\operatorname{ess} \limsup_{(x,t)\to(x_0,t_0)} \left\{ u(x,t)^{n+p-\frac{2n}{p}} \right\} = 0$$

and the exponent can be erased. This proves the claim (37) at the given Lebesgue point.

Furthermore, the Lebesgue points have the property that

$$\begin{aligned} v(x_0,t_0) &\leq \operatorname{ess} \liminf_{(x,t)\to(x_0,t_0)} v(x,t) \\ &\leq \lim_{TR^n \to 0} \frac{1}{3TR^n|B_2|} \int_{t_0-2T}^{t_0+T} \int_{B_{2R}} v(x,t) \, dx \, dt \;=\; v(x_0,t_0). \end{aligned}$$

Since almost every point is a Lebesgue point, we have established that

$$v(x_0,t_0) = \operatorname{ess} \liminf_{(x,t)\to(x_0,t_0)} v(x,t)$$

almost everywhere. The right-hand member is a semicontinuous function. $\square$

6 The Equation With Measure Data

There is a close connexion between supersolutions and equations where the right-hand side is a Radon measure. The Barenblatt solution has the Dirac measure (multiplied by a suitable constant) as the right-hand side, and hence it is, indeed, a *solution* to an equation. The equation

$$\frac{\partial v}{\partial t} - \nabla \cdot (|\nabla v|^{p-2}\nabla v) = \mu$$

with a Radon measure μ has been much studied. For example, in [BD] a summability result is given for the spatial gradient ∇v of the solution. There the starting point was the given measure and the above equation. However, we can do the opposite and produce the measure. Indeed, every p-supercaloric function belonging to $L^{p-2}_{loc}(\Omega_T)$ induces a Radon measure $\mu \geq 0$. This follows from our summability theorem, combined with the Riesz Representation Theorem for linear functionals. However, if it so happens that v belongs to class $\mathfrak{M}$, then for some time t_0,

$$v(x,t) \geq (t-t_0)^{-\frac{1}{p-2}}\mathfrak{U}(x,t)$$

and it cannot induce any sigma finite measure, let alone a Radon measure.

Theorem 46 *Let v be a p-supercaloric function in $\Omega\times(0,T)$. If v is of class $\mathfrak{B}$ there exists a non-negative Radon measure μ such that*

$$\int_0^T\int_\Omega\Big(-v\frac{\partial\varphi}{\partial t} + \langle|\nabla v|^{p-2}\nabla v, \nabla\varphi\rangle\Big)dx\,dt = \int_{\Omega\times(0,T)}\varphi\,d\mu$$

for all $\varphi \in C_0^\infty(\Omega\times(0,T))$.

Proof: We already know that $v, \nabla v \in L^{p-1}_{loc}(\Omega \times (0,T))$. In order to use Riesz's Representation Theorem we define the linear functional

$$\Lambda_v : C_0^\infty(\Omega \times (0,T)) \longrightarrow \mathbb{R},$$
$$\Lambda_v(\varphi) = \int_0^T \int_\Omega \Big(-v \frac{\partial \varphi}{\partial t} + \langle |\nabla v|^{p-2} \nabla v, \nabla \varphi \rangle \Big) dx\, dt.$$

Now $\Lambda_v(\varphi) \geq 0$ for $\varphi \geq 0$ according to Theorem 2. Thus the functional is positive and the existence of the Radon measure follows from Riesz's theorem, cf. [EG, 1.8]. $\square$

Some further results can be found in [KLP]. The elliptic case has been thoroughly treated in [KL]. See also [Kuusi-Mingione].

7 Pointwise Behaviour

The viscosity supersolutions are defined at each point, not only almost everywhere. Actually, the results in this section imply that two viscosity supersolutions that coincide almost everywhere do so at each point.

7.1 The Stationary Equation

We begin with the stationary case. At each point a p-superharmonic function v satisfies

$$v(x) \leq \liminf_{y \to x} v(y) \leq \operatorname{ess} \liminf_{y \to x} v(y)$$

by lower semicontinuity. *Essential limes inferior* means that sets of Lebesgue measure zero be neglected in the calculation of the lower limit. The reverse inequalities also hold. To see this, we start by a lemma, which requires a pedantic formulation.

Lemma 47 *Suppose that v is p-superharmonic in the domain Ω. If $v(x) \leq \lambda$ at each point x in Ω and if $v(x) = \lambda$ at almost every point x in Ω, then $v(x) = \lambda$ at each point x in Ω.*

Proof: The proof is trivial for continuous functions and the idea is that v is everywhere equal to a p-harmonic function, which, of course, must coincide with the constant λ. We approximate v by the infimal convolutions v_ε. We can assume that the function v is bounded also from below in a given ball B_{2r}, strictly interior in Ω. We may even take $0 \leq v \leq \lambda$ by adding a constant. We approximate v by the infimal convolutions v_ε. Replace v_ε in B_r by the p-harmonic function h_ε having boundary values v_ε. Thus we have the function

$$w_\varepsilon = \begin{cases} h_\varepsilon \text{ in } B_r \\ v_\varepsilon \text{ in } B_{2r} \backslash B_r \end{cases}$$

As we have seen before, also w_ε is p-superharmonic. By comparison

$$w_\varepsilon \leq v_\varepsilon \leq v$$

pointwise in B_{2r}. As ε approaches zero via a decreasing sequence, say $1, 1/2, 1/3, \cdots$, the h_ε's converge to a p-harmonic function h, which is automatically continuous because the family is uniformly equicontinuous so that Ascoli's theorem applies. The equicontinuity is included in the Hölder estimate (6), because $0 \leq h_\varepsilon \leq \lambda$. Thus

$$h \leq v \leq \lambda$$

at *each* point in B_r. Since $\lambda - v_\varepsilon \geq \lambda - v \geq 0$, the Caccioppoli estimate

$$\int_{B_r} |\nabla h_\varepsilon|^p \, dx \leq \int_{B_r} |\nabla v_\varepsilon|^p \, dx$$
$$\leq p^p \int_{B_{2r}} (\lambda - v_\varepsilon)^p |\nabla \zeta|^p \, dx \leq C r^{-p} \int_{B_{2r}} (\lambda - v_\varepsilon)^p \, dx$$

is valid. The weak lower semicontinuity of the integral implies that

$$\int_{B_r} |\nabla h|^p \, dx \leq \lim_{\varepsilon \to 0} \int_{B_r} |\nabla h_\varepsilon|^p \, dx \leq C r^{-p} \int_{B_{2r}} (\lambda - v)^p \, dx = 0.$$

The conclusion is that h is constant almost everywhere, and hence everywhere by continuity. The constant must be λ, because it has boundary values λ in Sobolev's sense. We have proved that also $v(x) = \lambda$ at *each* point in the ball B_r. The result follows. $\square$

Lemma 48 *If v is p-superharmonic in Ω and if $v(x) > \lambda$ for a.e. x in Ω, then $v(x) \geq \lambda$ for every x in Ω.*

Proof: If $\lambda = -\infty$, there is nothing to prove. Applying the previous lemma to the p-superharmonic function defined by

$$\min\{v(x), \lambda\}$$

we obtain the result in the case $\lambda > -\infty$. $\Box$

Theorem 49 *At each point a p-superharmonic function v satisfies*

$$v(x) = \operatorname{ess} \liminf_{y \to x} v(y).$$

Proof: Fix an arbitrary point $x \in \Omega$. We must show only that

$$\lambda = \operatorname{ess} \liminf_{y \to x} v(y) \leq v(x),$$

since the opposite inequality was clear. Given any $\varepsilon > 0$, there is a δ such that $v(y) > \lambda - \varepsilon$ for a.e. $y \in B(x, \delta)$. By the lemma $v(y) \geq \lambda - \varepsilon$ for *each* such y. In particular, $v(x) \geq \lambda - \varepsilon$. Because ε was arbitrary, we have established that $v(x) \geq \lambda$. $\Box$

7.2 The Evolutionary Equation

We turn to the pointwise behaviour for the Evolutionary p-Laplacian Equation. At each point in its domain a lower semicontinuous function satisfies

$$v(x,t) \leq \liminf_{(y,\tau)\to(x,t)} v(y,\tau) \leq \operatorname{ess} \liminf_{(y,\tau)\to(x,t)} v(y,\tau) \leq \operatorname{ess} \liminf_{\substack{(y,\tau)\to(x,t) \\ \tau < t}} v(y,\tau).$$

We show that for a p-supercaloric function also the reverse inequalities hold, thus establishing Theorem 3 in the Introduction. In principle, the proof is similar to the stationary case, but now a delicate issue of regularization arises. We first consider a non-positive p-supercaloric function $v = v(x,t)$ which is equal to zero at *almost* each point and, again, we show that locally it coincides with the p-caloric function having the same boundary values, now in a space-time cylinder. Then one has to conclude that v was identically zero.

We seize the opportunity to describe a useful procedure of regularizing by taking the convolution[28]

$$u^\star(x,t) = \frac{1}{\sigma}\int_0^t e^{(s-t)/\sigma} u(x,s)\,ds, \quad \sigma > 0.$$

The notation hides the dependence on the parameter σ. For continuous and for bounded semicontinuous functions u the averaged function $u^\star$ is defined at each point. We will stay within this framework. Observe that

$$\sigma\frac{\partial u^\star}{\partial t} + u^\star = u.$$

Some of its properties are listed in the next lemma.

Lemma 50 *(i) If $u \in L^p(D_T)$, then*

$$\|u^\star\|_{L^p(D_T)} \le \|u\|_{L^p(D_T)}$$

and

$$\frac{\partial u^\star}{\partial t} = \frac{u - u^\star}{\sigma} \in L^p(D_T).$$

Moreover, $u^\star \to u$ in $L^p(D_T)$ as $\sigma \to 0$.

(ii) If, in addition, $\nabla u \in L^p(D_T)$, then $\nabla(u^\star) = (\nabla u)^\star$ componentwise,

$$\|\nabla u^\star\|_{L^p(D_T)} \le \|\nabla u\|_{L^p(D_T)},$$

and $\nabla u^\star \to \nabla u$ in $L^p(D_T)$ as $\sigma \to 0$.

(iii) Furthermore, if $u_k \to u$ in $L^p(D_T)$ then also

$$u_k^\star \to u^\star \quad \text{and} \quad \frac{\partial u_k^\star}{\partial t} \to \frac{\partial u^\star}{\partial t}$$

in $L^p(D_T)$.

(iv) If $\nabla u_k \to \nabla u$ in $L^p(D_T)$, then $\nabla u_k^\star \to \nabla u^\star$ in $L^p(D_T)$.

[28] The origin of this function is unknown to me. In connexion with the Laplace transform it would be the convolution of u and $\sigma^{-1}e^{-t/\sigma}$.

(v) Finally, if $\varphi \in C(\overline{D_T})$, then

$$\varphi^\star(x,t) + e^{-t/\sigma}\varphi(x,0) \to \varphi(x,t)$$

uniformly in D_T as $\sigma \to 0$.

Proof: We leave this as an exercise. (Some details are worked out on page 7 of [KL1].) □

The *averaged equation* for a weak supersolution u in D_T reads as follows:

$$\int_0^T \int_D \Big(\langle (|\nabla u|^{p-2}\nabla u)^\star, \nabla\varphi\rangle - u^\star \frac{\partial \varphi}{\partial t}\Big) dx\, dt + \int_D u^\star(x,T)\varphi(x,T)\, dx$$
$$\geq \int_D u(x,0)\left(\frac{1}{\sigma}\int_0^T \varphi(x,s)e^{-s/\sigma}\, ds\right) dx$$

valid for all test functions $\varphi \geq 0$ vanishing on the lateral boundary $\partial D \times [0,T]$ of the space-time cylinder. For solutions one has equality. Notice the typical difficulty with obtaining $(|\nabla u|^{p-2}\nabla u)^\star$ and not $|\nabla u^\star|^{p-2}\nabla u^\star$, except in the linear case. The averaged equation follows from the equation for the retarded supersolution $u(x, t-s)$, where $0 \leq s \leq T$:

$$\int_s^T \int_D \Big(\langle |\nabla u(x,t-s)|^{p-2}\nabla u(x,t-s), \nabla\varphi(x,t)\rangle - u(x,t-s)\frac{\partial\varphi}{\partial t}(x,t)\Big) dx\, dt$$
$$+ \int_D u(x,T-s)\varphi(x,T)\, dx \geq \int_D u(x,0)\varphi(x,s)\, dx.$$

Notice that $(x, t-s) \in \overline{D_T}$ when $0 \leq s \leq t \leq T$. Multiply by $\sigma^{-1}e^{-s/\sigma}$, integrate over $[0,T]$ with respect to s, and, finally, interchange the order of integration between s and t. This yields the averaged equation above.

The advantage of this procedure over more conventional convolutions is that no values outside the original space-time cylinder are evoked.

We begin with a simple situation.

Lemma 51 *Suppose that v is a p-supercaloric function in a domain containing the closure of $B_T = B \times (0,T)$. If*

(i) $v \leq 0$ at each point in B_T and

(ii) $v = 0$ at almost every point in B_T,

then $v = 0$ at each point in $B \times (0, T]$.

Proof: We may assume that v is bounded. Construct the infimal convolution v_ε with respect to a larger domain than B_T. Fix a small time $t' > 0$ and let h^ε be the p-caloric function with boundary values induced by v_ε on the parabolic boundary of the cylinder $B \times (t', T)$ and define the function

$$w_\varepsilon = \begin{cases} h^\varepsilon, & \text{in} \quad B \times (t', T] \\ v_\varepsilon & \text{otherwise.} \end{cases}$$

To be on the safe side concerning the validity at the terminal time T we may solve the boundary value problem in a slightly larger domain with terminal time $T' > T$. Also w_ε is a p-supercaloric function. By comparison

$$w_\varepsilon \leq v_\varepsilon \leq 0 \quad \text{pointwise in} \quad B_T.$$

We let ε go to zero through a monotone sequence, say $1, \frac{1}{2}, \frac{1}{3}, \cdots$. Then the limit

$$h = \lim_{\varepsilon \to 0} h^\varepsilon$$

exists pointwise and it follows from the uniform Hölder estimates (16) that this h is continuous without any correction made in a set of measure zero. It is important to preserve the information at *each* point. Thus h is a p-caloric function. The so obtained function

$$w = \begin{cases} h, & \text{in} \quad B \times (t', T') \\ v & \text{otherwise} \end{cases}$$

is a p-supercaloric function. For the verification of the semicontinuity and the comparison principle, which proves this, the fact that $h \leq v$ is essential.

We know that $w \leq v \leq 0$ everywhere in a domain containing $B \times (0, T)$. In particular,

$$h \leq v \leq 0 \quad \text{everywhere in} \quad B \times (0, T).$$

We claim that $h = 0$ at each point. The claim immediately implies that $v = 0$ at each point in $B \times (0, T)$. Concerning the statement at the terminal time T, we notice that $v \geq h$ and

$$v(x, T) \geq h(x, T) = \lim_{t \to T-} h(x, t) = 0,$$

since h is continuous. On the other hand $v(x,T) \leq 0$ by the lower semicontinuity. Thus also $v(x,T) = 0$.

Therefore it is sufficient to prove the claim. To conclude that h is identically zero we use the averaged equation for $w^\star$ and write

$$\int_0^T \int_B \Big(\langle (|\nabla w|^{p-2} \nabla w)^\star, \nabla \varphi \rangle + \varphi \frac{\partial w^\star}{\partial t} \Big) dx\, dt$$
$$\geq \int_B w(x,0) \left(\frac{1}{\sigma} \int_0^T \varphi(x,s) e^{-s/\sigma}\, ds \right) dx,$$

where the test function vanishes on the parabolic boundary (an integration by parts has been made with respect to time.) Select the test function $\varphi = (v_\varepsilon - w_\varepsilon)^\star$ and let ε approach zero. Taking into account that $\varphi = 0$ when $t < t'$, we arrive at

$$\int_{t'}^T \int_B \Big(\langle (|\nabla h|^{p-2} \nabla h)^\star, \nabla v^\star - \nabla h^\star \rangle + (v^\star - h^\star) \frac{\partial h^\star}{\partial t} \Big) dx\, dt$$
$$\geq \int_B v(x,0) \left(\frac{1}{\sigma} \int_{t'}^T (v^\star(x,s) - h^\star(x,s)) e^{-s/\sigma}\, ds \right) dx.$$

The last integral (which could be negative) approaches zero as the regularization parameter σ goes to zero, because $t' > 0$, so that the exponential decays. Integrating

$$(v^\star - h^\star) \frac{\partial h^\star}{\partial t} = -(v^\star - h^\star) \frac{\partial (v^\star - h^\star)}{\partial t} + (v^\star - h^\star) \frac{\partial v^\star}{\partial t}$$

we obtain

$$\int_{t'}^T \int_B (v^\star - h^\star) \frac{\partial h^\star}{\partial t}\, dx\, dt$$
$$= -\frac{1}{2} \int_B (v^\star(x,T) - h^\star(x,T))^2\, dx + \int_{t'}^T \int_D (v^\star - h^\star) \frac{\partial v^\star}{\partial t}\, dx\, dt.$$

because $v^\star(x,t') - h^\star(x,t') = 0$. The last integral is zero because $v^\star$ and $\frac{\partial v^\star}{\partial t}$ are zero almost everywhere according to property (i) in Lemma 50. Erasing this integral and letting the regularization parameter σ go to zero (so that the $\star$'s disappear) we finally obtain

$$\int_{t'}^T \int_B |\nabla h|^p\, dx\, dt + \frac{1}{2} \int_B h^2(x,T)\, dx \leq 0 \quad \text{i.e.} \quad = 0.$$

In fact[29] , the proof guarantees this only for almost all values of T in the range $t' < T < T'$. From this it is not difficult to conclude that h is identically zero. Thus our claim has been proved. □

Lemma 52 *Suppose that v is a p-supercaloric function in a domain containing $B_T = B \times (0, T)$. If $v(x,t) > \lambda$ for almost every $(x,t) \in B_T$, then $v(x,t) \geq \lambda$ for every $(x,t) \in B \times (0,T]$.*

Proof: The auxiliary function

$$u(x,t) = \min\{v(x,t), \lambda\} - \lambda$$

in place of v satisfies the assumptions in the previous lemma. Hence $u = 0$ everywhere in $B \times (0,T]$. This is equivalent to the assertion. □

Proof of Theorem 3: Denote

$$\lambda = \operatorname{ess} \liminf_{\substack{(x,t)\to(x_0,t_0) \\ t<t_0}} v(x,t).$$

According to the discussion in the beginning of this subsection, it is sufficient to prove that $\lambda \leq v(x_0,t_0)$. Thus we can assume that $\lambda > -\infty$.

First, we consider the case $\lambda < \infty$. Given $\varepsilon > 0$, we can find a $\delta > 0$ and a ball B with centre x_0 such that the closure of $B \times (t_0 - \delta, t_0)$ is comprised in the domain and

$$v(x,t) > \lambda - \varepsilon$$

for almost every $(x,t) \in B \times (t_0 - \delta, t_0)$. According to the previous lemma

$$v(x,t) \geq \lambda - \varepsilon$$

for *every* $(x,t) \in B \times (t_0 - \delta, t_0]$. In particular, we can take $(x,t) = (x_0,t_0)$. Hence $v(x_0,t_0) \geq \lambda - \varepsilon$. Since ε was arbitrary, we have proved that $\lambda \leq v(x_0,t_0)$, as desired.

[29]It is the validity of

$$\lim_{\sigma\to 0} \frac{1}{2}\int_B (v^\star(x,T) - h^\star(x,T))^2\, dx = \frac{1}{2}\int_B h^2(x,T)\, dx$$

that requires some caution. We know that $v^\star$ is zero almost everywhere but with respect to the $(n+1)$-dimensional measure.

Second, the case $\lambda = \infty$ is easily reached via the truncated functions $v_k = \min\{v(x,t), k\}$, $k = 1, 2, \ldots$. Indeed,

$$v(x_0, t_0) \geq v_k(x_0, t_0) \geq \min\{\infty, k\} = k,$$

in view of the previous case. This concludes the proof of Theorem 3. $\square$

8 Viscosity Supersolutions Are Weak Supersolutions

In this chapter[30] we give a simple proof, due to Julin and Juutinen, of the fact that the viscosity supersolutions are the same as those obtained in potential theory, cf. [JJ]. The proof in [JLM], which is more complicated, will be bypassed. (Thus we can avoid the uniqueness machinery for second order equations, the doubling of variables, and Jensen's auxiliary equations.) The proof is based on the fact that the *infimal convolutions have second derivatives in the sense of Alexandrov*, which can be used in the testing with so-called superjets. These occur in a reformulation of the definition of viscosity supersolutions. The idea is that a *one*-sided estimate makes it possible to use Fatou's lemma and finally pass to the limit in an integral.

In establishing the equivalence between the two concepts of supersolutions, the easy part is to show that p-superharmonic or p-supercaloric functions are viscosity supersolutions. The proof comes from the fact that an antithesis produces a touching test function which is p-*sub*harmonic or p-*sub*caloric in a neighbourhood, in which situation the comparison principle leads to a contradiction in the indirect proof. This was accomplished in the proof of Proposition 10 for the stationary case. The evolutionary case is similar, and we omit it here. —We now turn to the preparations for the difficult part of the equivalence proof.

Theorem 53 (Alexandrov) *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a convex function. Then f has second derivatives in the sense of Alexandrov: for a. e. point x there is a symmetric $n \times n$-matrix $\mathbb{A} = \mathbb{A}(x)$ such that the expansion*

$$f(y) = f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2}\langle y - x, \mathbb{A}(x)(y - x)\rangle + o(|y - x|^2)$$

[30] The previous chapters, do in fact, not require familiarity with the viscosity theory of second order equations, but now it is desirable that the reader knows the basics of this theory. Some chapters of Koike's book [Ko] are enough. A more advanced source is [CIL].

is valid as $y \to x$.

For a proof[31] we refer to [EG, Section 6.4, pp. 242–245]. The problem is not the first derivatives, since by Rademacher's Theorem they are Sobolev derivatives and $\nabla f \in L^{\infty}_{\text{loc}}$. The question is about the second ones. We will use the *notation* $D^2 f = \mathbb{A}$, although the Alexandrov derivatives are not always second Sobolev derivatives, because a singular Radon measure may be present. The proof in [EG] establishes that pointwise we have a. e. that

$$\mathbb{A} = \lim_{\varepsilon \to 0} (D^2(f \star \varrho_\varepsilon)) \tag{38}$$

where ϱ_ε is Friedrich's mollifier.

Alexandrov's theorem is applicable to the concave functions

$$v_\varepsilon(x) - \frac{|x|^2}{2\varepsilon}, \qquad v_\varepsilon(x,t) - \frac{|x|^2 + t^2}{2\varepsilon}$$

encountered in Section 2 and Section 3.2. They are in fact defined in the whole space (although the infima are taken over bounded sets). Then the theorem is also applicable to the v_ε, since the subtracted smooth functions have second derivatives.

8.1 The Stationary Case

The concept of viscosity solutions can be reformulated in terms of so-called jets. *Super*solutions require *sub*jets (and *sub*solutions *super*jets). We say that the pair $(\xi, \mathbb{X})$, where ξ is a vector in $\mathbf{R}^n$ and $\mathbb{X}$ is a symmetric $n \times n$-matrix, belongs to the *subjet* $J^{2,-}u(x)$ if

$$u(y) \geq u(x) + \langle \xi, y - x \rangle + \frac{1}{2}\langle y - x, \mathbb{X}(x)(y - x) \rangle + o(|y - x|^2)$$

as $y \to x$. See [Ko, 2.2, p. 17]. Notice the similarity with a Taylor polynomial. If it so happens that u has continuous second derivatives at the point x, then we must have $\xi = \nabla u(x)$, $\mathbb{X} = D^2 u(x) =$ the Hessian matrix. In other words,

$$J^{2,-}u(x) = \{(\nabla u(x), D^2 u(x))\}.$$

[31] Some details in [GZ, Lemma 7.11, p. 199] are helpful to understand the singular part of the Lebesgue decomposition, which is used in the proof in [EG].

The essential feature is that the Alexandrov derivatives always do as members of the jets.

For a wide class of second order equations the subjets can be used to give an equivalent characterization of the viscosity supersolutions. We need only the following necessary[32] condition.

Proposition 54 *Let $p \geq 2$. Suppose that $\Delta_p v \leq 0$ in the viscosity sense. If $(\xi, \mathbb{X}) \in J^{2,-}v(x)$, then*

$$|\xi|^{p-2}\mathsf{trace}(\mathbb{X}) + (p-2)|\xi|^{p-4}\langle \xi, \mathbb{X}\,\xi\rangle \;\leq\; 0. \tag{39}$$

Proof: A simple proof is given in [Ko, Proposition 2.6, pp. 18–19]. □

After these preparations we are in the position of proving that a bounded viscosity supersolution of the Stationary p-Laplace Equation, $p \geq 2$, is also a weak supersolution. This is the analogue of Theorem 16 in Section 2, but for viscosity supersolutions. It was based on Corollary 15. We will now prove Corollary 15 for viscosity supersolutions without evoking the reference [JLM]. To this end, assume that $0 \leq v(x) \leq L$ and that $\Delta_p v(x) \leq 0$ in the viscosity sense in Ω. The infimal convolution v_ε defined by formula (9) is, according to Proposition 14, also a viscosity supersolution in the shrunken domain Ω_ε. Given a non-negative test function ψ in $C_0^\infty(\Omega)$, we have to prove the following

$$\textsf{Claim :} \qquad\qquad \int_{\Omega_\varepsilon} \big\langle |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon, \nabla\psi\big\rangle\, dx \geq 0$$

when ε is so small that Ω_ε contains the support of ψ.

As we saw above, the second Alexandrov derivatives $D^2 v_\varepsilon(x)$ exist a.e. in $\mathbb{R}^n$ and therefore $\big(\nabla v_\varepsilon(x), D^2 v_\varepsilon(x)\big) \in J^{2,-}v_\varepsilon(x)$ at almost every point x. Hence, by the Proposition, the inequality

$$\begin{aligned}
&\Delta_p v_\varepsilon(x)\\
=\;& |\nabla v_\varepsilon(x)|^{p-4}\Big\{|\nabla v_\varepsilon(x)|^2\Delta v_\varepsilon(x) + (p-2)\big\langle \nabla v_\varepsilon(x), D^2 v_\varepsilon(x)\nabla v_\varepsilon(x)\big\rangle\Big\}\\
&\leq\; 0
\end{aligned} \tag{40}$$

is valid almost everywhere in Ω_ε. Here $\Delta v_\varepsilon = \mathsf{trace}(D^2 v_\varepsilon)$.

[32] Testing with subjets is also a sufficient condition when their "closures" are employed.

We need one further mollification. For

$$f_\varepsilon(x) = v_\varepsilon(x) - \frac{|x|^2}{2\varepsilon}$$

we define the convolution

$$f_{\varepsilon,j} = f_\varepsilon \star \varrho_{\varepsilon_j} \quad \text{where} \quad \varrho_{\varepsilon_j} = \begin{cases} \frac{C}{\varepsilon_j^n}\exp\bigl(-\frac{\varepsilon_j^2}{\varepsilon_j^2-|x|^2}\bigr), & \text{when } |x| < \varepsilon_j \\ 0, & \text{otherwise.} \end{cases}$$

The smooth functions $v_{\varepsilon,j} = v_\varepsilon \star \varrho_{\varepsilon_j}$ satisfy the identity

$$\int_{\Omega_\varepsilon} \bigl\langle |\nabla v_{\varepsilon,j}|^{p-2} v_{\varepsilon,j}, \nabla\psi \bigr\rangle\, dx = \int_{\Omega_\varepsilon} \psi\bigl(-\Delta_p v_{\varepsilon,j}\bigr)\, dx,$$

which identity we will extend to the function v_ε by passing to the limit. However, they are not viscosity supersolutions themselves! By the linearity of the convolution, we can from (38) conclude that

$$\lim_{j\to\infty} D^2 v_{\varepsilon,j} = D^2 v_\varepsilon$$

almost everywhere. Therefore we have

$$\lim_{j\to\infty} \Delta_p v_{\varepsilon,j}(x) = \Delta_p v_\varepsilon(x)$$

at a.e. point x in the support of ψ. Obviously, the convolution has preserved the concavity, and hence $D^2 f_{\varepsilon,j} \leq 0$. It follows that

$$D^2 v_{\varepsilon,j} \leq \frac{\mathbb{I}_n}{\varepsilon}, \qquad \Delta v_{\varepsilon,j} \leq \frac{n}{\varepsilon}$$

a.e.. Here $\mathbb{I}_n$ is the unit matrix. It is immediate that

$$|\nabla v_{\varepsilon,j}| \leq \|\nabla v_\varepsilon\|_\infty = C_\varepsilon.$$

These estimates yield the bound

$$-\Delta_p v_{\varepsilon,j} \geq -C_\varepsilon^{p-2}\frac{n+p-2}{\varepsilon} \tag{41}$$

valid almost everywhere in the support of ψ. This lower bound justifies the use of Fatous lemma below:

$$\begin{aligned}
\int_{\Omega_\varepsilon} \left\langle |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon, \nabla\psi\right\rangle dx &= \lim_{j\to\infty}\int_{\Omega_\varepsilon} \left\langle |\nabla v_{\varepsilon,j}|^{p-2}\nabla v_{\varepsilon,j}, \nabla\psi\right\rangle dx \\
&= \lim_{j\to\infty}\int_{\Omega_\varepsilon} \psi(-\Delta_p v_{\varepsilon,j})\, dx \\
&\geq \int_{\Omega_\varepsilon} \liminf_{j\to\infty}\Big(\psi(-\Delta_p v_{\varepsilon,j})\Big)\, dx \\
&= \int_{\Omega_\varepsilon} \psi(-\Delta_p v_\varepsilon)\, dx \quad \geq \quad 0.
\end{aligned}$$

In the very last step we used the inequality $-\Delta_p v_\varepsilon \geq 0$, which, as we recall, needed Alexandrov's theorem in its proof. This proves our claim. □

8.2 The Evolutionary Equation

Since the parabolic proof is very similar to the elliptic one, we only sketch the proof of the

$$\textsf{Claim :} \qquad \int_0^T\int_{\Omega_\varepsilon}\Big(-v_\varepsilon\frac{\partial\psi}{\partial t} + \left\langle|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon, \nabla\psi\right\rangle\Big)dx\,dt \geq 0$$

for all non-negative test functions $\psi \in C_0^\infty(\Omega)$. As in Section 3.2 the infimal convolution of the given bounded viscosity supersolution v, $0 \leq v(x,t) \leq L$, is defined as

$$v_\varepsilon(x,t) = \inf_{(y,\tau)\in\Omega_T}\left\{v(y,\tau) + \frac{|y-x|^2 + (\tau-t)^2}{2\varepsilon}\right\}$$

and the function

$$f_\varepsilon(x,t) = v_\varepsilon(x,t) - \frac{|x|^2+t^2}{2\varepsilon}$$

is concave in $n+1$ variables. Therefore it has second derivatives in the sense of Alexandrov. So has v_ε itself, since the quadratic term has no influence on this matter. Thus

$$
\begin{aligned}
&v_\varepsilon(y,\tau)\\
&\quad = v_\varepsilon(x,t) + \langle \nabla v_\varepsilon(x,t), y-x\rangle + \frac{1}{2}\langle y-x, D_x^2 v_\varepsilon(x,t)(y-x)\rangle\\
&\quad + \frac{\partial v_\varepsilon(x,t)}{\partial t}(t-\tau) + \Big\langle \nabla \frac{\partial v_\varepsilon(x,t)}{\partial t}, y-x\Big\rangle(\tau-t) + \frac{1}{2}\frac{\partial^2 v_\varepsilon(x,t)}{\partial t^2}(\tau-t)^2\\
&\quad + o\big(|y-x|^2 + |\tau-t|^2\big)
\end{aligned}
$$

as $(y,\tau)\to(x,t)$. Here $D_x^2 v_\varepsilon$ is not the complete Hessian but the $n\times n$-matrix consisting of the second space derivatives; the time derivatives are separately written. This implies that

$$
\begin{aligned}
v_\varepsilon(y,\tau) = v_\varepsilon(x,t) &+ \frac{\partial v_\varepsilon(x,t)}{\partial t}(t-\tau) + \langle \nabla v_\varepsilon(x,t), y-x\rangle\\
&+ \frac{1}{2}\langle y-x, D_x^2 v_\varepsilon(x,t)(y-x)\rangle + o\big(|y-x|^2 + |\tau-t|\big),
\end{aligned}
$$

where the error term is no longer quadratic in $\tau - t$.

The *parabolic subjet* $\mathcal{P}^{2,-}u(x,t)$ consists of all triples $(a,\xi,\mathbb{X})$, where $a = a(x,t)$ is a real number, $\xi = \xi(x,t)$ is a vector in $\mathbb{R}^n$ and $\mathbb{X} = \mathbb{X}(x,t)$, such that

$$
\begin{aligned}
u(y,\tau) \geq u(x,t) &+ a\,(\tau-t) + \langle \xi, y-x\rangle + \frac{1}{2}\langle y-x, \mathbb{X}(y-x)\rangle\\
&+ o\big(|y-x|^2 + |\tau-t|\big)
\end{aligned}
$$

as $(y,\tau)\to(x,t)$. See [CIL, equation(8.1), p.48]. The Alexandrov (and the Rademacher) derivatives will do in the parabolic subjet and the characterization of viscosity supersolutions in terms of jets yields now the pointwise inequality

$$
-\Delta_p v_\varepsilon + \frac{\partial v_\varepsilon}{\partial t} \geq 0 \tag{42}
$$

valid almost everywhere in the support of ψ.

Again we have to use a convolution. Because the second time derivatives will not be needed, we take the convolution $f_{\varepsilon,j} = f_\varepsilon \star \varrho_{\varepsilon_j}$ only with respect to the space variables: $\varrho_{\varepsilon_j} = \varrho_{\varepsilon_j}(x)$. (This does not matter.) We have

$$
\iint \Big(-v_{\varepsilon,j}\frac{\partial\psi}{\partial t} + \big\langle |\nabla v_{\varepsilon,j}|^{p-2} v_{\varepsilon,j}, \nabla\psi\big\rangle\Big)\,dx\,dt = \iint \psi\Big(\frac{\partial v_{\varepsilon,j}}{\partial t} - \Delta_p v_{\varepsilon,j}\Big)\,dx\,dt,
$$

where the integrals are taken over the support of ψ, and ε is small. One can clearly pass to the limit under the integral signs above, as $j \to \infty$, except that the integral of $-\Delta_p v_{\varepsilon,j}$ requires a justification. Actually, the estimate (41) is valid also in the parabolic case, whence Fatou's lemma can be used. We obtain

$$\begin{aligned} \iint \Big(-v_\varepsilon \frac{\partial \psi}{\partial t} + \big\langle |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon, \nabla \psi \big\rangle \Big)\, dx\, dt \\ = \lim_{j\to\infty} \iint \Big(-v_{\varepsilon,j} \frac{\partial \psi}{\partial t} + \big\langle |\nabla v_{\varepsilon,j}|^{p-2} v_{\varepsilon,j}, \nabla \psi \big\rangle \Big)\, dx\, dt \\ = \lim_{j\to\infty} \iint \psi \Big(\frac{\partial v_{\varepsilon,j}}{\partial t} - \Delta_p v_{\varepsilon,j} \Big)\, dx\, dt \\ \geq \iint \liminf_{j\to\infty} \psi \Big(\frac{\partial v_{\varepsilon,j}}{\partial t} - \Delta_p v_{\varepsilon,j} \Big)\, dx\, dt \\ = \iint \psi \Big(\frac{\partial v_\varepsilon}{\partial t} - \Delta_p v_\varepsilon \Big)\, dx\, dt \quad \geq \quad 0, \end{aligned}$$

where we used (42) in the last step. This proves our claim. $\Box$

$$\Big\langle \text{THE END} \Big\rangle$$

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