DEMAILLY APPROXIMATION AND HÖRMANDER THEORY WITH SINGULAR WEIGHTS

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ABSTRACT. This part is a sequel to [5] and was based on a private communication with Prof. Bo-Yong Chen [1]. We shall use Demailly's approximation theory of quasi-plurisubharmonic function [2] to prove a quasi–complete version of Demailly's theorem.

CONTENTS

1. A GENERALIZATION OF DEMAILLY'S THEOREM

Let us recall the following definition in [5].

Definition 1.1. *A Kähler metric ω on a complex manifold X is said to be quasi-complete if there exists a family of Kähler manifolds $\{(X_j, \omega_j, \chi_j)\}_{j=1}^{\infty}$ such that*

1) *Each X_j is an open set in X, $X_j \subset X_{j+1}$ and $X = \cup X_j$;*
2) *For each j we have $\omega_j \geq \omega$ on X_j and for every compact subset K of X,*

$$\lim_{j\to\infty} \sup_K |\omega_j - \omega|_\omega = 0;$$

3) *Each χ_j is smooth with compact support in X_j such that $0 \leq \chi_j \leq 1$ on X_j,*

$$\lim_{j\to\infty} \sup_{X_j} |\overline{\partial}\chi_j|_{\omega_j} = 0$$

and for every compact subset K in X we have $\chi_j \equiv 1$ on K for all $j \geq j(K)$.

The main theorem in [5] is the following:

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Theorem 1.1. *[Theorem 8.2 in [5]] Let $(L, e^{-\phi})$ be a holomorphic line bundle on an n-dimensional quasi-complete Kähler manifold (X, ω). Assume that*

$$i\Theta = i\partial\overline{\partial}\phi \geq i\,\bar{b} \wedge b$$

for a smooth $(0,1)$-form b on X. Then for every $\overline{\partial}$-closed L-valued (n,q)-form $(q \geq 1)$ c such that $c = b \wedge a$ on X and $||a|| < \infty$, we can find an L-valued $(n, q-1)$-form u on X such that

$$\overline{\partial}u = c$$

on X and $||u|| \leq ||a||$.

In this note, we shall use the above theorem together with Demailly's approximation theorem to prove the following theorem that is essentially due to Demailly.

Theorem 1.2. *[Demailly's theorem] Let $(L, e^{-\phi})$ (ϕ can be singular) be a holomorphic line bundle on an n-dimensional weakly pseudoconvex Kähler manifold (X, ω). Assume that*

$$i\Theta = i\partial\overline{\partial}\phi \geq i\,\bar{b} \wedge b$$

for a smooth $(0,1)$-form b on X. Then for every $\overline{\partial}$-closed L-valued (n,q)-form $(q \geq 1)$ c such that $c = b \wedge a$ on X and $||a|| < \infty$, we can find an L-valued $(n, q-1)$-form u on X such that

$$\overline{\partial}u = c$$

on X and $||u|| \leq ||a||$.

Remark: Our method can also be used to prove the following slightly generalized version of Demailly's theorem.

Theorem 1.3. *Let $(L, e^{-\phi})$ (ϕ can be singular) be a holomorphic line bundle on an n-dimensional quasi-complete Kähler manifold (X, ω). Assume that*

$$i\Theta = i\partial\overline{\partial}\phi \geq i\,\bar{b} \wedge b$$

for a smooth $(0,1)$-form b on X. Then for every $\overline{\partial}$-closed L-valued (n,q)-form $(q \geq 1)$ c such that $c = b \wedge a$ on X and $||a|| < \infty$, we can find an L-valued $(n, q-1)$-form u on X such that

$$\overline{\partial}u = c$$

on X and $||u|| \leq ||a||$.

Remark: It is known that a Kähler manifold (X, ω) is quasi-complete if X possesses a complete Kähler metric. In particular, by Proposition 2.1 below, the above theorem applies to (X, ω), where ω is a Kähler metric on Y, Y possesses a complete Kähler metric and

$$X = Y \setminus Z,$$

where Z is the zero set of a holomorphic section of some vector bundle over Y.

2. Demailly approximation, a weak version

Assume that (X, ω) is weakly pseudoconvex Kähler, we shall show that Demailly approximation theory implies the followings.

Fact 1: $X = \cup X^j$, each (X^j, ω) is weakly pseudoconvex Kähler, $X^j \Subset X^{j+1} \Subset X$;

Fact 2: There exists a decreasing sequence of metrics $\{\phi_\nu\}$ on X^j such that

$$\lim_{\nu\to\infty} \phi_\nu = \phi + \mathcal{E}_j, \ \ i\partial\overline{\partial}\phi_\nu \geq i\,\bar{b}\wedge b - 2^{-j}\omega \ \text{ on } \ X^j,$$

where $\mathcal{E}_j$ denotes a smooth function on X^j such that

$$0 < \mathcal{E}_j < 2^{-j}$$

on X^j and each ϕ_ν is smooth outside an analytic subset, say Z_ν, of X^j and $\phi_\nu = -\infty$ on Z_ν.

Fact 3: Each $(X^j \setminus Z_\nu, \omega)$ is quasi-complete.

Remark: **Fact 1** is trivial. Since weakly pseudoconvex Kähler implies quasi-complete, **Fact 3** follows from the following proposition.

Proposition 2.1. *Let (X, ω) be a quasi-complete Kähler manifold. Let ψ be a θ-psh function on X for some real smooth (1,1)-form θ. Put $S := \psi^{-1}(-\infty)$. Assume that $\psi \in C^\infty(X\backslash S)$. Then $(X \setminus S,\ \omega)$ is quasi-complete Kähler.*

Proof. We may assume that $\psi < -1$ on X. By a direct computation, we have

$$i\partial\overline{\partial}(-\log -\psi) \geq -|\theta| + i\partial\log(-\psi)\wedge\overline{\partial}\log(-\psi), \quad |\theta| := \max\{0, \theta\}.$$

Thus one may choose a decreasing sequence of sufficiently small positive numbers a_j such that

$$2^j a_j^2 \to 0, \ \ 5^j a_j^2 |\theta| < \omega,$$

on X_j (notice that one may assume that X_j is relatively compact). Take $\kappa \in C^\infty(\mathbb{R}, [0, 1])$ such that $\kappa \equiv 1$ on $(-\infty, 1/2)$ and $\kappa \equiv 0$ on $(1, \infty)$. Put

$$w_{j,k} := \omega_j + 2^{-k}\omega + 2^k a_k^2 \cdot i\partial\overline{\partial}(-\log -\psi)$$

and

$$\chi_{j,k} := \chi_j \cdot \kappa(a_k \log -\psi).$$

Then $(X_j \setminus S,\ w_{j,j},\ \chi_{j,j})$ fits our need. □

2.1. Proof of Fact 2.

Definition 2.1. *A function ψ on an n dimensional complex manifold X is said to have Bergman singularity if for every $x \in X$ there exists a positive integer N and local holomorphic coordinates, say $z = (z_1, \cdots, z_n)$, such that $x \in B := \{|z| < 1\}$,*

$$\psi = \frac{1}{N}\log K + \varphi$$

on B, where φ is smooth and K is a weighted Bergman kernel

$$K(z) = \sup_{f\in\mathcal{O}(B)} \frac{|f(z)|^2}{\int_B |f|^2 e^{-\rho}\,(i\partial\overline{\partial}|z|^2)^n}$$

for some plurisubharmonic function ρ on B.

Fact 2 follows from the following theorem.

Theorem 2.2 (weak version of Theorem 16.3, page 164 in [4])**.** *Let (X_0, ω) be a complex manifold with a smooth Hermitian metric ω. Let ϕ be a quasi-psh function on X_0 such that*

$$i\partial\overline{\partial}\phi \geq \gamma$$

on X_0 for some real smooth $(1,1)$-form γ. Fix $\varepsilon > 0$ and a relatively compact open subset, say X, of X_0. Then there exists a decreasing sequence of functions, $\{\phi_\nu\}$, with Bergman singularity on X such that

$$\lim_{\nu\to\infty} \phi_\nu = \phi + \mathcal{E}, \ \ i\partial\overline{\partial}\phi_\nu \geq \gamma - \varepsilon \cdot \omega,$$

where $\mathcal{E}$ is a smooth function on X such that $0 < \mathcal{E} < \varepsilon$.

Proof–from page 165–169 in [4]. Choose $0 < r < \delta < \varepsilon$ and cover $\bar{X}$ by finite open balls $\{B''_j\}$:

$$B''_j = \frac{1}{2}B'_j = \frac{1}{4}B_j, \ \ B_j := \{|z^{(j)}| < r\},$$

where $\{z^{(j)}\}$ are chosen such that

$$(1-\delta)\cdot i\partial\overline{\partial}|z^{(j)}|^2 \leq \omega \leq (1+\delta)\cdot i\partial\overline{\partial}|z^{(j)}|^2$$

on B_j and there exist Hermitian forms Q_j with

$$0 \leq \gamma + i\partial\overline{\partial}Q_j(z^{(j)}) \leq \delta\cdot\omega$$

on B_j. Then when δ and r are small, we have for every $z_0 \in B'_j \cap B'_k$,

$$\{|z^{(j)} - z_0| < r/4\} \subset B_\omega(z_0, r/3) \subset \{|z^{(k)} - z_0| < r/2\} \subset B_k,$$

where $B_\omega(z_0, r/3)$ denotes the ω ball centered at z_0 with radius $r/3$. Denote by $K_{\nu,j}$ the Bergman kernel on B_j with respect to the weight

$$2\nu(\phi + Q_j(z^{(j)})).$$

Put

$$\psi_{\nu,j} := \frac{1}{2\nu}\log K_{\nu,j} - Q_j(z^{(j)}).$$

Then

$$i\partial\overline{\partial}\psi_{\nu,j} \geq \gamma - \delta\cdot\omega$$

on B_j and the Ohsawa–Takegoshi extension theorem gives (when r is smooth enough)

$$\psi_{\nu,j} \geq \phi$$

on B_j. Moreover, the sub-mean inequality gives, for every $z \in B'_j$,

$$\psi_{\nu,j}(z) \leq \sup_{|\zeta - z|\leq t}(\phi(\zeta) + Q_j(\zeta)) - Q_j(z) + \frac{1}{2v}\log\frac{n!}{\pi^n t^{2n}}, \quad \forall\, 0 < t < r/2.$$

The Ohsawa–Takegoshi extension from the diagonal to $B_j \times B_j$ implies that

$$\psi_{2^\nu,j}$$

is decreasing with respect to ν on B_j when r_j is small. Demailly further proved that

(2.1) $$|\psi_{\nu,j} - \psi_{\nu,k}| \le \frac{1}{\nu} \log \frac{17 \cdot 6^n}{r^{(n+1)}} + 2\delta r^2$$

on $B_j' \cap B_k'$. Choose ν_0 such that for all $\nu \ge \nu_0$ we have

$$\frac{1}{v} \log \frac{17 \cdot 6^n}{r^{(n+1)}} < 2\delta r^2,$$

then it is enough to take

$$\phi_v(z) := \max_{j, B_j' \ni z} \{\psi_{2^\nu, j}(z) + 12\delta(r^2/4 - |z^{(j)}|^2)\}$$

and

$$\mathcal{E}(z) := \max_{j, B_j' \ni z} \{12\delta(r^2/4 - |z^{(j)}|^2)\},$$

where $\max$ denotes the regularized maximum function. It is clear that

$$0 < \mathcal{E}(z) < 3\delta r^2 < \varepsilon$$

and

$$i\partial\overline{\partial}\phi_v \ge \gamma - \delta \cdot \omega - \frac{12\delta}{1-\delta}\omega \ge \gamma - \varepsilon\omega$$

when δ is smooth enough. □

Proof of (2.1). Fix $z_0 \in B_j' \cap B_k'$, the extremal property of the Bergman kernel gives

$$f \in \mathcal{O}(B_j)$$

with $||f|| = 1$ such that

$$K_{\nu,j}(z_0) = |f(z_0)|^2.$$

Let χ be a smooth function on $[0, \infty)$ such that

$$\chi = \begin{cases} 1 & [0,1] \\ 0 & [2,\infty], \end{cases}$$

with $|\chi| \le 1, \;\; |\chi'| \le 2$. Let us define θ on B_k such that

$$\theta(z^{(k)}) = \chi\left(\frac{32 \cdot |z^{(k)} - z_0|^2}{r^2}\right),$$

then the support of θ lies in $B_j \cap B_k$. Let us solve

$$\overline{\partial}(u) = \overline{\partial}(\theta f e^{2\nu g})$$

on B_j wrt $\Phi := 2\nu(\phi + Q_k) + 2n \log |z^{(k)} - z_0| + |z^{(k)}|^2$, where

$$g = Q_k(z - z_0, z_0) - Q_j(z - z_0, z_0).$$

Notice that when r is small enough we have

$$|Q_k(z - z_0) - Q_j(z - z_0)| \le 2\delta r^2$$

for every $z \in B_j \cap B_k$. Thus Hörmander's theorem gives

$$\int_{B_k} |u|^2 e^{-\Phi} \leq \int_{B_k} |\overline{\partial}\theta|^2_{i\partial\overline{\partial}|z^{(k)}|^2} |f|^2 e^{2\nu(g+\bar{g})-\Phi}.$$

Notice that

$$Q_k - Q_j - g - \bar{g} = Q_k(z_0) - Q_j(z_0) + Q_k(z-z_0) - Q_j(z-z_0),$$

thus we get (when r is small enough) $u(z_0) = 0$ and

$$||u||^2 e^{-r^2} \leq \int_{B_k} |u|^2 e^{-\Phi} \leq \frac{256 \cdot 6^{2n}}{r^{2(n+1)}} e^{-2\nu(Q_k(z_0)-Q_j(z_0))} e^{4\nu\delta r^2}$$

thus

$$||u|| \leq \frac{16 \cdot 6^n}{r^{(n+1)}} e^{r^2/2} e^{-\nu(Q_k(z_0)-Q_j(z_0))} e^{2\nu\delta r^2}.$$

Together with

$$||\theta f e^{2vg}|| \leq e^{-\nu(Q_k(z_0)-Q_j(z_0))} e^{2\nu\delta r^2}.$$

Put

$$f' = \theta f e^{2vg} - u,$$

we get

$$f'(z_0) = f(z_0), \ \ ||f|| \leq \left(\frac{16 \cdot 6^n}{r^{(n+1)}} e^{r^2/2} + 1\right) e^{-\nu(Q_k(z_0)-Q_j(z_0))} e^{2\nu\delta r^2}$$

When r is small we have

$$\frac{16 \cdot 6^n}{r^{(n+1)}} e^{r^2/2} + 1 \leq \frac{17 \cdot 6^n}{r^{(n+1)}}.$$

Thus the extremal property of the Bergman kernel gives

$$\psi_{\nu,k}(z_0) \geq \psi_{\nu,j}(z_0) - \frac{1}{\nu} \log \frac{17 \cdot 6^n}{r^{(n+1)}} - 2\delta r^2.$$

Exchanging j and k gives (2.1). □

3. Proof of Demailly's theorem

Since each $(X^j \setminus Z_\nu, \omega)$ is quasi-complete, by our definition, we can find the associated approximation family (X_l, ω_l, χ_l). Denote by $\square_\nu$ the $\overline{\partial}$-Laplacian with respect to ϕ_ν. Since

$$i\partial\overline{\partial}\phi_\nu \geq i\,\bar{b} \wedge b - 2^{-j}\omega \geq i\,\bar{b} \wedge b - 2^{-j}\omega_l,$$

we can solve

$$(\square_v + 2^{-j} \cdot q) v_{j,\nu,l} = c$$

on X_l such that

$$||\overline{\partial} v_{j,\nu,l}||^2_{l,\nu} + ||\overline{\partial}^* v_{j,\nu,l}||^2_{l,\nu} + 2^{-j} \cdot q \cdot ||v_{j,\nu,l}||^2_{l,\nu} \leq ||a||^2,$$

where $||\cdot||_{l,\nu}$ the (ω_l, ϕ_ν) norm on X_l. The proof of Theorem 6.5 in [5] gives

$$||\chi_l \overline{\partial}^* \overline{\partial} v_{j,\nu,l}||_{l,v} \leq \varepsilon_l \cdot ||a||.$$

Let l go to infinity, we get weak limits $u_{j,\nu}$ and $v_{j,\nu}$ such that

$$\overline{\partial} u_{j,\nu} + v_{j,\nu} = c \tag{3.1}$$

on $X^j \setminus Z_\nu$ and

$$||u_{j,\nu}||_\nu \leq ||a||, \ \ ||v_{j,\nu}||_\nu \leq \sqrt{2^{-j}q} \cdot ||a||.$$

By Lemma 7.3 (see page 382 of [3]), in the sense of distribution, (3.1) is true on X^j. Thus first let v go to infinity and then let j go to infinity, we get weak limits u such that $\overline{\partial} u = c$ on X and $||u|| \leq ||a||$.

4. Proof of Theorem 1.3

Let (X_l, ω_l, χ_l) be the approximation family for (X, ω). Assume each X_l is relatively compact in X. Then we can use Demailly approximation to each X_l and obtain approximation sequence ϕ_ν^l with singularity Z_ν^l. As in the proof of Proposition 2.1, one may assume $\phi_\nu^l < -1$ and consider

$$\omega_{l,\nu,k} := \omega_l + 2^{-k}\omega + 2^k a_k^2 \cdot i\partial\overline{\partial}(-\log -\phi_\nu^l)$$

and

$$\chi_{l,\nu,k} := \chi_l \cdot \kappa(a_k \log -\phi_\nu^l).$$

Solve

$$(\Box_{l,\nu,k} + 2^{-l} \cdot q) v_{l,\nu,k} = c$$

on $(X_l \setminus Z_\nu^l, \omega_{l,\nu,k})$ such that

$$||\overline{\partial} v_{l,\nu,k}||_{l,\nu,k}^2 + ||\overline{\partial}^* v_{l,\nu,k}||_{l,\nu,k}^2 + 2^{-l} \cdot q \cdot ||v_{l,\nu,k}||_{l,\nu,k}^2 \leq ||a||^2.$$

The proof of Theorem 6.5 in [5] gives

$$||\chi_{l,\nu,k} \overline{\partial}^* \overline{\partial} v_{l,\nu,k}||_{l,\nu,k} \leq \varepsilon_{l,k} \cdot ||a||.$$

First let k go to infinity, we get

$$||\chi_l \overline{\partial}^* \overline{\partial} v_{l,\nu}||_{l,\nu} \leq \varepsilon_l \cdot ||a||.$$

By Lemma 7.3 (see page 382 of [3]), in the sense of distribution, we get

$$(\Box_{l,\nu} + 2^{-l} \cdot q) v_{l,\nu} = c$$

on an open set where $\chi_l \equiv 1$. Let ν, l go to infinity, our result follows.

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