HÖRMANDER $\overline{\partial}$ THEORY ON QUASI-COMPLETE KÄHLER MANIFOLD

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ABSTRACT. These notes were written for a PhD course at NTNU. Most of the results are well known. In the Hörmander L^2 theory part, we introduce the concept "quasi-completeness", which we hope could simplify and slightly generalize the theory.

CONTENTS

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1. Jordan normal form of a matrix

Let A be an (n,n) complex matrix. The starting point of the whole story is

Lemma 1.1. *$\lambda\in\mathbb{C}$ is an eigenvalue of A if and only if $\det(A-\lambda)=0$.*

Proof. It is clear that $Ax=\lambda x$ iff $(A-\lambda)x=0$ iff column vectors, say $\{(A-\lambda)e_j\}_{1\leq j\leq n}$, of $(A-\lambda)$ are linearly dependent iff (by Gauss elimination)

$$\det(A-\lambda):=\frac{(A-\lambda)e_1\wedge\cdots\wedge(A-\lambda)e_n}{e_1\wedge\cdots\wedge e_n}=0.$$

□

Now let $\{\lambda_j\}_{1\leq j\leq l}$ be the eigenvalue set of A. By the above lemma, we must have

$$\det(A-\lambda)=(\lambda_1-\lambda)^{n_1}\cdots(\lambda_l-\lambda)^{n_l}.$$

Put

$$V_j^p:=\ker(A-\lambda_j)^p,$$

then we can choose $1\leq m_j\leq n_j$ such that

$$V_j^1\subsetneq V_j^2\subsetneq\cdots\subsetneq V_j^{m_j}=V_j^{m_j+1}=\cdots.$$

It is clear that

$$V_j^{m_j}\cap V_k^{m_k}=\{0\},\ \ j\neq k.$$

If fact if $(A-\lambda_j)^{m_j}(e)=(A-\lambda_k)^{m_k}(e)=0$ then we must have $(\lambda_j-\lambda_k)^{m_j+m_k}e=0$, which gives $e=0$. Thus we have

$$\oplus_{j=1}^l V_j^{m_j}\subset\mathbb{C}^n.$$

The Jordan theorem is the following

Theorem 1.2. $\mathbb{C}^n=\oplus_{j=1}^l V_j^{m_j}$.

Proof. It is enough to show that $\dim V_j^{m_j}=n_j$. By the above lemma, we know that $\dim V_j^{m_j}\geq 1$, thus the theorem is true in case all $n_j=1$. Assume that $n_1>1$ and

$$Ae_1=\lambda_1e_1,\ \ Ae_j=\sum_{k=1}^n c_{kj}e_k.$$

Consider $A'=(c_{kj})_{2\leq k,j\leq n}$, then we have

$$\det(\lambda-A')=(\lambda_1-\lambda)^{n_1-1}\cdots(\lambda_l-\lambda)^{n_l}.$$

thus $\det(\lambda_1-A')=0$. By the above lemma we can find a $\mathbb{C}$ linear combination of $e_2,\cdots,e_n$ (assume that it is e_2) such that

$$A'e_2=\lambda_1e_2.$$

Thus we have

$$(A-\lambda_1)^2 e_2 = (A-\lambda_1)(Ae_2 - A'e_2) = c_{12}(A-\lambda_1)e_1 = 0,$$

which implies that e_1, e_2 lie in $V_1^{m_1}$. Thus $\dim V_1^{m_1} \geq 2$. Continue the above process (replace A be A'), we can finally prove that $\dim V_1^{m_1} = n_1$. □

Exercise 1: Complete the above proof and show that the above proof gives a basis, say $\{e_j\}$, of $\mathbb{C}^n$ under which A is upper triangular, i.e.

$$Ae_j = \sum_{k\leq j} c_{kj} e_k.$$

Think of $A - \lambda_j$ as a $\mathbb{C}$-linear transform, say T, on $V_j^{m_j}$. Then we have

$$T^{m_j-1} \neq 0, \ \ T^{m_j} \equiv 0.$$

Definition 1.1. *A $\mathbb{C}$-linear map T from $\mathbb{C}^n$ to itself is said to be k-nilpotent ($k \geq 0, \ T^0 := 1$) if $T^{k+1} = 0$ but $T^k \neq 0$.*

If T is k-nilpotent then

$$0 \subsetneq \ker T \subsetneq \cdots \subsetneq \ker T^k \subsetneq \ker T^{k+1} = \ker T^{k+2} = \mathbb{C}^n.$$

Moreover, we can inductively choose subspaces S^j ($0 \leq j \leq k$) of $\mathbb{C}^n$ such that

$$S^0 \oplus \ker T^k = \mathbb{C}^n,$$

$$S^1 \oplus TS^0 \oplus \ker T^{k-1} = \ker T^k,$$

and finally

$$S^k \oplus TS^{k-1} \oplus \cdots \oplus T^k S^0 = \ker T.$$

(Thanks to Tai for pointing out a mistake in an early version of this notes).

Exercise 2: Check the following theorem:

Theorem 1.3 (Lefschetz isomorphism)**.** *We have*

$$\mathbb{C}^n = \oplus_{j=0}^{2k} V^j,$$

where

$$V^j := \oplus_{l\leq j-l\leq k} T^l S^{j-2l}.$$

Moreover, for every $0 \leq j \leq k$,

$$T^j : V^{k-j} \to V^{k+j},$$

is an isomorphism.

The above theorem says that $\mathbb{C}^n$ is generated by T and the $\oplus S^j$. We call $\oplus S^j$ a *primitive space* for T. The primitive space is not unique. But in the next section, we shall show that if we fix the graded structure with Lefschetz isomorphism then the primitive space is unique.

Remark 1: The main example that we will use is the following: each V^j is the space of *degree j* differential forms and T is given by the wedge product of a degree two symplectic form.

Remark 2: Once we have fixed a basis, say A_j, for each S^j, then

$$\cup_{i+j\leq k} T^i A_j$$

defines a basis of $\mathbb{C}^n$, with respect to which we get the *Jordan normal form* of T. The Jordan normal forms for all $(A-\lambda_j)|_{V_j^{m_j}}$ give the Jordan normal form of a matrix A.

2. LINEAR LEFSCHETZ THEORY ASSOCIATED TO A SYMPLECTIC STRUCTURE

We shall show that there is a natural Lefschetz isomorphism associated to a symplectic structure. Then we define the symplectic star operator and the associated sl_2-triple.

Let V be an N-dimensional real vector space. Let ω be a bilinear form on V. We call ω a *symplectic form* if ω is non-degenerate and ω is anti-symmetric (i.e. $\omega(u,v)=-\omega(v,u),\ \forall\, u,v\in V$, and we know that $\omega\in\wedge^2 V^*$).

Proposition 2.1. *Assume that there is a symplectic form ω on V. Then $N=2n$ for some integer n and there exists a basis, say $\{e_1^*, f_1^*; \cdots; e_n^*, f_n^*\}$, of V^* such that*

$$\omega=\sum_{j=1}^n e_j^*\wedge f_j^*.$$

Proof. Since ω is non-degenerate, we know that $N\geq 2$. If $N=2$ and $\omega(e,f)=1$ then

$$\omega=e^*\wedge f^*,$$

where $\{e^*, f^*\}$ denotes the dual basis of $\{e,f\}$. Assume that $N\geq 3$, consider

$$V':=\{u\in V: \omega(u,e)=\omega(u,f)=0\}.$$

Then for every $u\in V$, we have

$$u':=u-\omega(u,f)e+\omega(u,e)f\in V',$$

and

$$ae+bf\in V' \text{ iff } a=b=0,$$

thus

$$V=V'\oplus \mathrm{Span}\{e,f\}.$$

Since ω is non-degenerate, we know for every $v\in V'$ there exists $u\in V$ with $\omega(u,v)\neq 0$. Thus

$$\omega(u',v)=\omega(u,v)\neq 0,$$

which implies that $\omega|_{V'}$ is a symplectic form on V'. Induction on N gives the theorem. $\square$

One may use ω to define a bilinear form, say ω^{-1}, on V^* such that

$$\omega^{-1}(f_j^*, e_k^*)=-\omega^{-1}(e_k^*, f_j^*)=\delta_{jk},\ \omega^{-1}(f_j^*, f_k^*)=\omega^{-1}(e_j^*, e_k^*)=0.$$

Exercise: Check that

$$\omega^{-1}(u\rfloor\omega, v\rfloor\omega)=\omega(v,u).$$

Thus the definition of ω^{-1} does not depend on the choice of basis in the above proposition.

We shall use the same notation ω^{-1} for the following bilinear form on $\wedge^p V^*$:

$$\omega^{-1}(\mu,\nu) := \det(\omega^{-1}(\alpha_i,\beta_j)),\ \mu=\alpha_1\wedge\cdots\wedge\alpha_p,\ \nu=\beta_1\wedge\cdots\wedge\beta_p. \tag{2.1}$$

Definition 2.1 (By Guillemin [12]). *The symplectic star operator $*_s : \wedge^p V^* \to \wedge^{2n-p}V^*$ is defined by*

$$\mu\wedge *_s\nu = \omega^{-1}(\mu,\nu)\frac{\omega^n}{n!}. \tag{2.2}$$

We shall show how to use the Lefschetz isomorphism to decode the structure of $*_s$.

Theorem 2.2 (Hard Lefschetz theorem–pointwise version). *For each $0\le k\le n$,*

$$L^{n-k} : u\mapsto \omega^{n-k}\wedge u,\ u\in\wedge^k V^*,$$

defines an isomorphism between $\wedge^k V^*$ and $\wedge^{2n-k}V^*$.

Proof. Notice that the theorem is true if $n=1$ or $k=0,n$. Now assume that it is true for $n\le l$, $l\ge 1$. We need to prove that it is true for $n=l+1$, $1\le k\le l$. Put

$$\omega' = \sum_{j=1}^{l} e_j^*\wedge f_j^*.$$

Then we have

$$\omega^{l+1-k} = (\omega')^{l+1-k} + (l+1-k)(\omega')^{l-k}\wedge e_{l+1}^*\wedge f_{l+1}^*.$$

Let us write $u\in\wedge^k(V^*)$ as

$$u = u^0 + u^1\wedge e_{l+1}^* + u^2\wedge f_{l+1}^* + u^3\wedge e_{l+1}^*\wedge f_{l+1}^*,$$

where each u^j contains no e_{l+1}^* or f_{l+1}^* term. Then $\omega^{l+1-k}\wedge u=0$ is equivalent to

$$(\omega')^{l+1-k}\wedge u^0 = (\omega')^{l+1-k}\wedge u^1 = (\omega')^{l+1-k}\wedge u^2 = (\omega')^{l+1-k}\wedge u^3 + (l+1-k)(\omega')^{l-k}\wedge u^0 = 0,$$

which implies $u^1=u^2=0$ by our theorem for $n=l$. Moreover, $u^3=0$ since

$$(\omega')^{l+2-k}\wedge u^3 = (\omega')^{l+2-k}\wedge u^3 + (l+1-k)(\omega')^{l-k+1}\wedge u^0 = 0.$$

Thus $(\omega')^{l-k}\wedge u^0=0$, which implies $u^0=0$. Now we know that $u\mapsto\omega^{l+1-k}\wedge u$ is an injection, thus an isomorphism since $\dim\wedge^k V^* = \dim\wedge^{2n-k}V^*$. □

The notion of primitive form is an analogy of primitive space in section 1:

Definition 2.2. *We call $u\in\wedge^k V^*$ a primitive form if $k\le n$ and $\omega^{n-k+1}\wedge u=0$.*

The following Lefschetz decomposition theorem follows directly from Theorem 2.2.

Theorem 2.3 (Lefschetz decomposition formula). *Every $u\in\wedge^k V^*$ has a unique decomposition as follows:*

$$u=\sum L_r\wedge u^r,\quad L_r := \frac{L^r}{r!}, \tag{2.3}$$

where each u^r is a primitive $(k-2r)$-form.

Proof. Put $V^k = \wedge^k V^*$. We can assume that $k \leq n$ since we have the isomorphism $L^k : V^{n-k} \to V^{n+k}$. Notice that the theorem is trivial if $k = 0, 1$. Assume that $2 \leq k \leq n$. The isomorphism

$$L^{n-k+2} : V^{k-2} \to V^{2n-k+2},$$

gives $\hat{u} \in V^{k-2}$ such that $L^{n-k+2}\hat{u} = L^{n-k+1}u$. Put $u^0 = u - L\hat{u}$, we know that u^0 is primitive and $u = u^0 + L\hat{u}$. Consider $\hat{u}$ instead of u, we have $\hat{u} = u^1 + L\tilde{u}$, where u^1 is primitive. By induction, we know that u can be written as

$$u = \sum L_r u^r,$$

where each $u^r \in V^{k-2r}$ is primitive. For the uniqueness part, assume that

$$0 = \sum_{r=0}^{j} L_r u^r,$$

where each $u^r \in V^{k-2r}$ is primitive. Then we have

$$0 = L_{n-k+j}\left(\sum_{r=0}^{j} L_r u^r\right) = L_{n-k+j} L_j u^j,$$

which gives $u^j = 0$. By induction on j we know that all $u^r = 0$. □

By the above theorem, it is enough to study the symplectic star operator on

$$\omega_r \wedge u, \quad \omega_r := \frac{\omega^r}{r!},$$

where u is primitive. The main result is the following:

Theorem 2.4. *If u is a primitive k-form then $*_s(\omega_r \wedge u) = (-1)^{1+2+\cdots+k}\omega_{n-k-r} \wedge u$.*

The above theorem implies

$$*_s^2 = 1.$$

The proof of the above theorem depends on a symplectic analogy of the lemma proved by Berndtsson (see Lemma 3.6.10 in [1]).

Definition 2.3. *$u \in \wedge^k V^*$ is said to be an elementary form if there exists a basis, say*

$$\{e_1^*, f_1^*; \cdots ; e_n^*, f_n^*\},$$

of V^* such that

$$\omega = \sum_{j=1}^{n} e_j^* \wedge f_j^*, \ u = e_1^* \wedge \cdots \wedge e_k^*.$$

Lemma 2.5 (Berndtsson lemma)**.** *The space of primitive forms is equal to the linear space spanned by elementary forms.*

Proof. Since

$$\omega_{n-k+1} = \sum_{j_1<\cdots<j_{n-k+1}} e^*_{j_1} \wedge f^*_{j_1} \wedge \cdots \wedge e^*_{j_{n-k+1}} \wedge f^*_{j_{n-k+1}},$$

we know that $\omega_{n-k+1} \wedge u = 0$ if u is an elementary k-form. Thus every elementary form is primitive. Let us prove the other side by induction on n. Notice that the lemma is true if $n = 1$. Assume that it is true for $n \leq l$, $l \geq 1$. We shall prove that it is also true for $n = l + 1$. With the notation in the proof of Theorem 2.2, $\omega^{l-k+2} \wedge u = 0$ is equivalent to

$$(\omega')^{l+2-k} \wedge u^0 = (\omega')^{l+2-k} \wedge u^1 = (\omega')^{l+2-k} \wedge u^2 = (\omega')^{l+2-k} \wedge u^3 + (l+2-k)(\omega')^{l-k+1} \wedge u^0 = 0,$$

which is equivalent to the ω'-primitivity of u^1, u^2, u^3 and $(l+2-k)u^0 + \omega' \wedge u^3$. Now it suffices to show that (consider $(l + 2 - k)u$)

$$u' := u^3 \wedge ((l + 2 - k)e^*_{l+1} \wedge f^*_{l+1} - \omega')$$

is a linear combination of elementary forms. Since u^3 is ω'-primitive, by the induction hypothesis, we can assume that

$$u^3 = e^*_1 \wedge \cdots \wedge e^*_{k-2}.$$

Thus

$$u' = \sum_{j=k-1}^{l} e^*_1 \wedge \cdots \wedge e^*_{k-2} \wedge (e^*_{l+1} \wedge f^*_{l+1} - e^*_j \wedge f^*_j).$$

Now it suffices to show that if $n = 2$ then $e^*_1 \wedge f^*_1 - e^*_2 \wedge f^*_2$ is a linear combination of elementary forms. Notice that

$$e^*_1 \wedge f^*_1 - e^*_2 \wedge f^*_2 = (e^*_1 + e^*_2) \wedge (f^*_1 - f^*_2) + e^*_1 \wedge f^*_2 + f^*_1 \wedge e^*_2.$$

It is clear that $e^*_1 \wedge f^*_2$ and $f^*_1 \wedge e^*_2$ are elementary. $(e^*_1 + e^*_2) \wedge (f^*_1 - f^*_2)$ is also elementary since we can write

$$\omega = (e^*_1 + e^*_2) \wedge f^*_1 + e^*_2 \wedge (f^*_2 - f^*_1).$$

The proof is complete. □

We shall also use the following lemma from [12].

Lemma 2.6 (Guillemin Lemma)**.** *Assume that* $(V, \omega) = (V_1, \omega^{(1)}) \oplus (V_2, \omega^{(2)})$. *Then*

$$*_s(u \wedge v) = (-1)^{k_1 k_2} *^1_s u \wedge *^2_s v, \ u \in \wedge^{k_1} V^*_1, \ v \in \wedge^{k_2} V^*_2,$$

where $*^1_s$ *and* $*^2_s$ *are symplectic star operators on* V_1 *and* V_2 *respectively.*

Proof. For every $a \in \wedge^{k_1} V^*_1, \ b \in \wedge^{k_2} V^*_2$, we have

$$a \wedge b \wedge (-1)^{k_1 k_2} *^1_s u \wedge *^2_s v = a \wedge *^1_s u \wedge b \wedge \wedge *^2_s v = \omega^{-1}(a \wedge b, u \wedge v)\omega_n,$$

which gives the lemma. □

Now we are able to prove Theorem 2.4.

Proof of Theorem 2.4. By the Berndtsson lemma, we can assume that

$$u = e_1^* \wedge \cdots \wedge e_k^*.$$

Consider $V = \mathrm{Span}\{e_j^*, f_j^*\}_{1\leq j\leq k} \oplus \mathrm{Span}\{e_{k+1}^*, f_{k+1}^*\} \oplus \cdots \oplus \mathrm{Span}\{e_n^*, f_n^*\}$ and write

$$*_s = *_s^{\leq k} \oplus *_s^{k+1} \oplus \cdots \oplus *_s^n.$$

Since

$$*_s^j(1) = e_j^* \wedge f_j^*, \ *_s^j(e_j^* \wedge f_j^*) = 1, \ \forall\, k+1 \leq j \leq n,$$

by the Guillemin lemma, we have

$$*_s(e_{k+1}^* \wedge f_{k+1}^* \wedge \cdots \wedge e_{k+r}^* \wedge f_{k+r}^* \wedge u) = e_{k+r+1}^* \wedge f_{k+r+1}^* \wedge \cdots \wedge e_n^* \wedge f_n^* \wedge *_s^{\leq k} u,$$

which implies

$$*_s(\omega_r \wedge u) = \omega_{n-k-r} \wedge *_s^{\leq k} u.$$

Since $*_s^{\leq k} = *_s^1 \oplus \cdots \oplus *_s^k$ and (try!)

$$*_s^j e_j^* = -e_j^*, \ \forall\, 1 \leq j \leq k,$$

the Guillemin lemma gives

$$*_s^{\leq k} u = (-1)^{k-1}(-e_1^*) \wedge *_s^{\leq (k-1)}(e_2^* \wedge \cdots \wedge e_k^*) = \cdots = (-1)^{k+\cdots+1} u,$$

the proof is complete. □

Definition 2.4. *We call* $\{L, \Lambda, B\}$ *the* sl_2*-triple on* $\oplus_{0\leq k\leq 2n} \wedge^k V^*$*, where*

$$Lu := \omega \wedge u, \ \Lambda := *_s^{-1} L *_s, \ B := [L, \Lambda].$$

We have

$$\omega^{-1}(Lu, v) = \omega^{-1}(u, \Lambda v).$$

Hence Λ is the adjoint of L. Put

$$L_r := L^r/r!, \ L_0 := 1, \ L_{-1} := 0.$$

We have:

Proposition 2.7. *If* u *is a primitive* k*-form then*

$$\Lambda(L_r u) = (n-k-r+1) L_{r-1} u, \ B(L_r u) = (k+2r-n) L_r u,$$

for every $0 \leq r \leq n-k+1$.

Proof. Put $c = (-1)^{k+\cdots+1}$, then

$$L *_s (L_r u) = cL(L_{n-k-r} u) = (n-k-r+1) c L_{n-k-r+1} u = (n-k-r+1) *_s (L_{r-1} u),$$

which gives the first identity. The second follows directly from the first. □

3. Complex structure and Hodge star operator

3.1. **Compatible complex structure.** Let (V, ω) be a symplectic space. Let us consider another structure on V, which can be used to define an inner product structure on V.

Definition 3.1. *We call a linear map $J : V \to V$ a complex structure on V if $J(Ju) = -u$ for every $u \in V$.*

Definition 3.2. *A complex structure J on (V, ω) is said to be compatible with ω if*

$$(u, v) := \omega(u, Jv),$$

defines a inner product structure on V (i.e.

$$\omega(u, Jv) = \omega(v, Ju), \ \forall\, u, v \in V,$$

and $\omega(u, Ju) > 0$ if u is not zero). We call $(\cdot, \cdot)$ the (ω, J)-metric on V.

If J is a complex structure on V then

$$J(v)(u) := v(Ju), \ \forall\, u \in V, \ v \in V^*,$$

defines a complex structure on V^*. The dual formulation of compatibility is the following:

Proposition 3.1. *If a complex structure J on (V, ω) is compatible with ω then*

$$(\alpha, \beta) := \omega^{-1}(\alpha, J\beta),$$

defines an inner product structure on V^*. We call $(\cdot, \cdot)$ the (ω, J)-metric on V^*.

Proof. The theorem follows from

$$\omega^{-1}(u \rfloor \omega, J(v \rfloor \omega)) = -\omega^{-1}(u \rfloor \omega, (Jv) \rfloor \omega) = \omega(u, Jv),$$

where the first formula follows from

$$(J(u \rfloor \omega))(v) = (u \rfloor \omega)(Jv) = \omega(u, Jv) = -\omega(Ju, v) = -((Ju) \rfloor \omega))(v).$$

□

Definition 3.3. *We call*

$$J(v_1 \wedge \cdots \wedge v_k) := J(v_1) \wedge \cdots \wedge J(v_k),$$

the Weil operator on $\oplus_{0 \leq k \leq 2n} \wedge^k V^*$.

Since the eigenvalues of J are $\pm i$, its eigenvectors lie in $\mathbb{C} \otimes V^*$. Put

$$E_i := \{u \in \mathbb{C} \otimes V^* : J(u) = iu\}, \ \ E_{-i} := \{u \in \mathbb{C} \otimes V^* : J(u) = -iu\},$$

we know that

$$E_i = \{u - iJu : u \in V^*\}, \ \ E_{-i} = \{u + iJu : u \in V^*\}.$$

and $\mathbb{C} \otimes V^* = E_i \oplus E_{-i}$. Put

$$\wedge^{p,q} V^* := (\wedge^p E_i) \wedge (\wedge^q E_{-i}).$$

Then we have

$$\mathbb{C} \otimes (\wedge^k V^*) = \wedge^k (\mathbb{C} \otimes V^*) = \oplus_{p+q=k} \wedge^{p,q} V^*,$$

and

$$Ju = i^{p-q}u, \ \forall\, u \in \wedge^{p,q}V^*.$$

Proposition 3.2. *$\omega \in \wedge^{1,1}V^*$ iff $\omega(u, Jv) = \omega(v, Ju), \ \forall\, u, v \in V$.*

Proof. Let $\omega = \omega^{2,0} + \omega^{1,1} + \overline{\omega^{2,0}}$ be the bidegree decomposition of ω, then

$$J\omega = -\omega^{2,0} + \omega^{1,1} - \overline{\omega^{2,0}}.$$

Thus we have $\omega \in \wedge^{1,1}V^*$ iff $J\omega = \omega$ iff

$$\omega(Ju, Jv) = \omega(u, v), \ \forall\, u, v \in V,$$

iff $\omega(u, Jv) = \omega(v, Ju), \ \forall\, u, v \in V$. □

Remark: We can also extend ω^{-1} to a bilinear form on $\mathbb{C} \otimes V^*$ as follows:

$$\omega^{-1}(c_1 u, c_2 v) := c_1 c_2\, \omega^{-1}(u, v), \ \ \forall\, u, v \in V, \ \ \forall\, c_1, c_2 \in \mathbb{C}.$$

Then it is clear that if $\omega \in \wedge^{1,1}V^*$ then

$$\omega^{-1}(\wedge^{1,0}V^*, \wedge^{1,0}V^*) = \omega^{-1}(\wedge^{0,1}V^*, \wedge^{0,1}V^*) = 0.$$

Definition 3.4. *Assume that J is compatible with ω. Then the (ω, J)-metric on V^* extends to a hermitian metric on $\mathbb{C} \otimes V^*$ as follows*

$$(u, v) := \omega^{-1}(u, J\bar{v}), \ \ \forall\, u, v \in \mathbb{C} \otimes V^*.$$

We call it the (ω, J)-metric on $\mathbb{C} \otimes V^*$.

Remark: It is clear that

$$\wedge^{0,1}V^* \perp \wedge^{1,0}\, V^*$$

with respect to the (ω, J)-metric. Let $\{\xi^j\}_{1 \le j \le n}$ be an orthonormal basis of $\wedge^{1,0}V^*$, then

$$(\xi^j, \xi^k) = (\bar{\xi}^k, \bar{\xi}^j) = \delta_{jk}$$

implies that $\{\bar{\xi}^j\}_{1 \le j \le n}$ is an orthonormal basis of $\wedge^{1,0}V^*$.

Exercise 1: Check that we can write

$$\omega = i \sum_{j=1}^n \xi^j \wedge \bar{\xi}^j,$$

moreover, if we write

$$\xi^j = a^j + ib^j, \ \ a^j, b^j \in V^*$$

(e g $dz^j = dx^j + idy^j$) then

$$a^j = \frac{\xi^j + \bar{\xi}^j}{2}, \ \ b^j = \frac{\xi^j - \bar{\xi}^j}{2i}, \ \ J(a^j) = -b^j,$$

and the associated (ω, J)-metric on V can be written as

$$g = 2 \sum_{j=1}^n (a^j \otimes a^j + b^j \otimes b^j).$$

In particular, $\{\sqrt{2}a^j, \sqrt{2}b^j\}$ defines an orthonormal basis of V^*.

Exercise 2: We can further extend the (ω, J)-metric on $\mathbb{C}\otimes V^*$ to each $\wedge^k(\mathbb{C}\otimes V^*)$ as follows

$$(u,v) := \omega^{-1}(u, J\bar{v}), \quad u, v \in \wedge^k(\mathbb{C}\otimes V^*),$$

where ω^{-1} denotes the $\mathbb{C}$-linear extension of the bilinear form ω^{-1} on $\wedge^k \otimes V^*$, J denotes the Weil operator. Check that $\wedge^{p,q}V^* \perp \wedge^{k,l} V^*$ if $(p,q)\neq(k,l)$ and

$$(u_1\wedge\cdots\wedge u_p, v_1\wedge\cdots\wedge v_p) = \det((u_j, v_k)), \quad u_j, v_k \in \mathbb{C}\otimes V^*,$$

moreover if $\{\xi^j\}_{1\leq j\leq n}$ be an orthonormal basis of $\wedge^{1,0}V^*$ then

$$\{\xi^{J\bar{K}} := \xi^{j_1}\wedge\cdots\wedge\xi^{j_p}\wedge\bar{\xi}^{k_1}\wedge\cdots\wedge\bar{\xi}^{k_q}\}_{j_1<\cdots<j_p,\, k_1<\cdots<k_p}$$

defines an orthonormal basis of $\wedge^{p,q}V^*$, in particular

$$(\omega_n, \omega_n) = 1. \tag{3.1}$$

Definition 3.5. *We call the hermitian inner product, say $(\cdot,\cdot)$, defined in the above exercise the (ω, J)-metric on $\oplus\wedge^{p,q} V^*$ and we shall always write the associated norm as $|u| := \sqrt{(u,u)}$.*

Now we can define a $\mathbb{C}$-linear map, say $\star$, on $\oplus\wedge^{p,q} V^*$ such that

$$u\wedge\star\bar{v} = (u,v)\,\omega_n.$$

Exercise 3: Check that

$$\star(\wedge^{p,q}V^*) = \wedge^{n-q,n-p}V^*$$

and $\star\bar{v} = \overline{\star v}$.

Definition 3.6. *We call $\star$ defined above the Hodge star operator on $\oplus\wedge^{p,q} V^*$.*

Remark: Compare it with the symplectic star operator, we get

$$\star = \star_s\circ J = J\circ\star_s,$$

where J denotes the Weil-operartor. In particular, we have

$$\star^2 = J^2 = (-1)^{p-q} = (-1)^{p+q},$$

on $\wedge^{p,q}V^*$, which implies that

$$(u,v)\,\omega_n = u\wedge\star\bar{v} = (-1)^{\deg u}\overline{\star v\wedge\bar{u}} = \overline{\star v\wedge\star\star\bar{u}} = \overline{(\star v, \star u)}\,\omega_n = (\star u, \star v)\,\omega_n.$$

Thus *the Hodge star operator preserve the metric*. In the next section, we shall show that the above Hodge star operator is just the $\mathbb{C}$-linear extension of the usual Hodge star operator on $(V, (\cdot,\cdot))$ with respect to the canonical orientation ω_n.

3.2. **Compare with the classical Hodge star operator.** In general, if we have an inner product structure, say $(\cdot,\cdot)$, on an N dimensional real linear space V. Then the induced inner product structure on V^* is defined by the Riesz reprentation of V^* in V, more precisely, if $\alpha \in V^*$ then we can find a unique $R(\alpha) \in V$ such that

$$\alpha(u) = (u, R(\alpha)),$$

we call $R(\alpha)$ the *Riesz representation* of α in V. Then the inner product on V^* is defined by

Definition 3.7. $(\alpha, \beta) := (R(\beta), R(\alpha))$.

Exercise 4: Prove that

$$(\alpha, \alpha) := \sup_{u \in V} \frac{|\alpha(u)|^2}{(u,u)}.$$

The above inner product on V^* also defines an inner product structure on $\oplus \wedge^k V^*$ as follows:

Definition 3.8. *If $u \in \wedge^k V^*$, $v \in \wedge^j V^*$ and $j \neq k$ then $(u, v) := 0$; moreover the inner product structure on each $\wedge^k V^*$ is defined by*

$$(u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k) := \det((u_j, v_k)).$$

Definition 3.9. *An orientation on $(V, (\cdot,\cdot))$ is an N-form, say $dV \in \wedge^N V^*$, such that*

$$(dV, dV) = 1.$$

A general N-form $d\mu$ is said to be semi-positive (with respect to dV) if

$$d\mu = f dV, \quad f \geq 0.$$

Remark: It is clear that there are two orientations, $\pm\, dV$, on (V, g). (3.1) implies that ω_n defines a *canonical orientation* with respect to the (ω, J) metric.

Definition 3.10. *The classical Hodge star operator $\star$ on $\oplus \wedge^k V^*$ with respect to the orientation dV is defined by*

$$u \wedge \star v = (u, v)\, dV.$$

Now it is clear that thle Hodge star operator defined in the above section is equal to the $\mathbb{C}$-*linear extension* of the classical Hodge star operator $\star$ with respect to the orientation ω_n.

3.3. **Basic notions in complex geometry.**

Definition 3.11. *An N-dimensional smooth manifold is a Hausdorff space X such that X is locally homeomorphic to domains in $\mathbb{R}^N$ with smooth transition maps and X can be written as*

$$X = \cup_{j \geq 1} X_j,$$

where $\{X_j\}$ is an increasing sequence of relatively compact open sets in X.

Definition 3.12. *Replace $\mathbb{R}^N$ by $\mathbb{C}^n$ and "smooth" by holomorphic, we get an n-dimensional complex manifold.*

Example: $\mathbb{C}P^n$ (complex projective space) and the complex torus $T_L := \mathbb{C}^n / L$, where L is a lattice in $\mathbb{C}^n$ are examples of complex manifolds.

Definition 3.13. *A rank r holomorphic vector bundle E over a complex manifold X means a holomorphic map, say*

$$\pi : E \to X,$$

such that for every $x \in X$ there exists an open neighborhood, say U_x, on X together with a biholomorphic map

$$\sigma_x : \pi^{-1}(U_x) \to U_x \times \mathbb{C}^r$$

such that

$$\sigma_x(E_z) = \{z\} \times \mathbb{C}^r, \ \ \forall\, z \in U_x, \ \ E_z := \pi^{-1}(z),$$

and if $U_x \cap U_y \neq \emptyset$ then for every $z \in U_x \cap U_y$, $\sigma_x \circ \sigma_y^{-1}$ maps $\{z\} \times \mathbb{C}^r$ to itself $\mathbb{C}$-linearly .

Remark: Replace "holomorphic" by "smooth", one gets the notion of *complex vector bundles*, similarly one may define real vector bundles over a smooth manifold. A rank one holomorphic vector bundle is also called a *holomorphic line bundle*.

Exercise 5: If we write

$$\sigma_x \circ \sigma_y^{-1}(z, v) = (z, \sigma_{yx}(z)v), \ \ z \in U_x \cap U_y, \ \ v \in \mathbb{C}^r,$$

then we know that each $\sigma_{xy}(z)$ is a holomorphic matrix-valued function such that

$$\sigma_{xy}^{-1}(z) = \sigma_{yx}(z), \quad z \in U_x \cap U_y$$

and

$$\sigma_{xy}(z)\sigma_{yt}(z) = \sigma_{xt}(z), \ \ z \in U_x \cap U_y \cap U_t.$$

Definition 3.14. *A holomorphic section of E on an open set U in X means a holomorphic map $e : U \mapsto E$ such that $e(z) \in E_z$ for every $z \in U$. We write $H^0(U, E)$ the space of all holomorphic sections of E on U.*

Exercise 6: Fix a basis, say $\{e_j\}$, of $\mathbb{C}^r$, then each

$$z \mapsto \sigma_x^{-1}(z, e_j),$$

defines a holomorphic section of E over U_x. We shall still denote it by e_j and call $\{e_j\}$ a holomorphic frame of E over U_x. Check that every holomorphic section of E over U_x can be written as

$$\xi(z) = \sum \xi^j(z) e_j(z),$$

where ξ^j are holomorphic functions on U_x.

Example: If X is a complex manifold with local coordinates

$$z^j = x^j + iy^j,$$

then its tangent bundle T_X is a smooth vector bundle over X with local frame

$$\partial/\partial x^j, \partial/\partial y^j$$

If we write

$$\mathbb{C} \otimes T_X = T \oplus \bar{T},$$

where T are defined by the local frame

$$\partial/\partial z^j := (\partial/\partial x^j - i\partial/\partial y^j)/2,$$

and $\bar{T}$ are defined by the local frame

$$\partial/\partial \bar{z}^j := (\partial/\partial x^j + i\partial/\partial y^j)/2,$$

Then we know that T is a holomorphic vector bundle over X. Its dual bundle T^* is defined by the local frame

$$dz^j := dx^j + idy^j.$$

The dual bundle $\bar{T}^*$ of $\bar{T}$ is defined by the local frame

$$d\bar{z}^j := dx^j - idy^j.$$

Definition 3.15. *We call T the holomorphic tangent bundle on X and T^* the holomorphic cotangent bundle. Let E be a holomorphic vector bundle on X. We call smooth section of $E \otimes (\wedge^p T \wedge \wedge^q \bar{T})$ a smooth E-valued (p,q)-form; smooth section of $E \otimes \wedge^k(\mathbb{C} \otimes T_X)$ a smooth E-valued k-form. We shall denote by V^k (resp. $V^{p,q}$) the space of smooth E-valued k-forms (resp. (p,q)-forms) with compact support on X.*

Remark: If $\{e_\alpha\}$ is a local holomorphic frame of E then we can write a smooth E-valued (p,q)-form, say u, as

$$u = \sum u^\alpha \otimes e_\alpha,$$

where u^α are locally defined smooth (p,q)-forms on X.

Definition 3.16. *The $\overline{\partial}$-operator on $V^{p,q}$ is defined by*

$$\overline{\partial} u := \sum \overline{\partial} u^\alpha \otimes e_\alpha,$$

where

$$\overline{\partial} u^\alpha := \sum d\bar{z}^j \wedge \partial/\partial \bar{z}^j(u^\alpha).$$

Exercise 7: Check that $\overline{\partial}$ does not depend on the choice of $\{e_\alpha\}$ and $\{z^j\}$.

3.4. **Hermitian metrics and Lefschetz theory in complex geometry.** Let E be a complex vector bundle over a smooth manifold X. A *hermitian metric structure* on E is defined as a *smooth* family, say

$$h_E = \{h_{E_x}\},$$

of hermitian metrics h_{E_x} on E_x. Here smooth family means that

$$h_E(e_\alpha, e_\beta) : x \mapsto h_{E_x}(e_\alpha(x), e_\beta(x))$$

are smooth function of x if $\{e_\alpha\}$ is a smooth local frame on E.

Definition 3.17. *A Hermitian metric on a complex manifold X means a hermitian metric, say h_T, on its holomorphic tangent bundle T.*

Exercise 8: Put

$$h_{j\bar{k}}(z) := h_T(\partial/\partial z^j, \partial/\partial z^k),$$

then we know that (try!)

$$\omega := i \sum h_{j\bar{k}}(z) dz^j \wedge d\bar{z}^k$$

is a *globally defined non-degenerate hermitian* $(1,1)$*-form* on X (hermitian means each $h_{j\bar{k}}(z)$ is a positive definite hermitian matrix) and $h_T \mapsto \omega$ is a one to one correspondence (late we shall identify h_T with ω).

Definition 3.18. *We call* ω *the hermitian form of the Hermitian metric on* X*. If* $d\omega = 0$ *then we say that* (X, ω) *is a Kähler manifold and the associated hermitian metric is a Kähler metric.*

The Lefschetz isomorphism associated to each $\omega(x)$, $x \in X$ gives

Theorem 3.3 (Hard Lefschetz theorem)**.** *For each* $0 \leq k \leq n$*,*

$$u \mapsto \omega^{n-k} \wedge u, \ \ u \in V^k, \tag{3.2}$$

defines an isomorphism between V^k *and* V^{2n-k}*.*

Definition 3.19. *We call an* E*-valued* k*-form, say* u*, on* X *a primitive form if* $k \leq n$ *and* $\omega^{n-k+1} \wedge u \equiv 0$*.*

We also have the associated Lefschetz decomposition:

Theorem 3.4 (Lefschetz decomposition formula)**.** *Every* E*-valued* k*-form* u *on* X *has a unique decomposition as follows:*

$$u = \sum \omega_r \wedge u^r, \ \omega_r := \frac{\omega^r}{r!}, \tag{3.3}$$

where each u^r *is an* E*-valued primitive* $(k-2r)$*-form.*

Let $\{e_\alpha\}$ be a local holomorphic frame of E, then ($\star$ denotes the Hodge star operator)

$$||u||^2 := \int_X \sum h_E(e_\alpha, e_\beta) u^\alpha \wedge \star \bar{u}^\beta, \ u := \sum u^\alpha \otimes e_\alpha,$$

defines a Hermitian inner product structure on V^k, we call it the (ω, J, h_E)-metric on V^k.

Definition 3.20. *The Hodge star operator on* V^k *is defined by*

$$\star u = \sum (\star u^\alpha) \otimes e_\alpha, \ u := \sum u^\alpha \otimes e_\alpha.$$

Definition 3.21. *We call* $\{\cdot, \cdot\}$ *defined by*

$$\left\{\sum u^\alpha \otimes e_\alpha, \sum u^\beta \otimes e_\beta\right\} := \sum h_E(e_\alpha, e_\beta) u^\alpha \wedge \bar{u}^\beta,$$

the sesquilinear product on $\oplus V^k$*.*

Exercise 9: Check that with respect to the above definition, we have

$$||u||^2=\int_X\{u,\star u\}=\int_X\sum h_E(e_\alpha,e_\beta)u^\alpha\wedge\star\bar{u}^\beta,$$

in particular, if E is a line bundle with local holomorphic frame e and we write $h_E(e,e)(z)=e^{-\phi(z)},\ u=\hat{u}(z)\otimes e$. then

$$||u||^2=\int_X e^{-\phi}\hat{u}(z)\wedge\star\overline{\hat{u}(z)}=\int_X|\hat{u}(z)|^2_\omega\, e^{-\phi}\omega_n,$$

is just the "weighted" L^2-norm, the only difference is that φ is not globally defined in general. *Later, we shall just write* $||u||^2$ *as* $\int_X|u|^2_\omega e^{-\phi}\omega_n$ *or* $\int_X|u|^2e^{-\phi}$.

The following result is a direct consequence of Theorem 2.4 and $\star=J\circ\star_s$.

Theorem 3.5 (Hodge-Riemann bilinear relation–pointwise version)**.** *If u is an E-valued primitive (p,q)-form then its (ω,J,h_E)-norm satisfies*

$$||u||^2=\int_X\{u,\omega_{n-k}\wedge Iu\},\ Iu:=(-1)^{1+\cdots+k}i^{p-q}u\geq 0,\ k:=p+q. \tag{3.4}$$

Remark: In particular, if E is a line bundle with smooth metric $e^{-\phi}$ then

$$\{u,\omega_{n-k}\wedge Iu\}=(-1)^{\frac{k(k+1)}{2}}(-i)^{p-q}e^{-\phi}u\wedge\bar{u}\wedge\omega_{n-k}=|u|^2e^{-\phi}\omega_n.$$

Since we always have $|u|^2e^{-\phi}\geq 0$, thus the Hodge Riemann bilinear relation just means that

$$(-1)^{\frac{k(k+1)}{2}}(-i)^{p-q}e^{-\phi}u\wedge\bar{u}\wedge\omega_{n-k}$$

is a semipositive (n,n)-form. Assume further that u is an (n,q)-form. By HLT, we can write

$$u=\gamma_u\wedge\omega_q, \tag{3.5}$$

where γ_u is an E-valued $(n-q,0)$-form. In particular, γ_u is primitive, thus

$$\star\gamma_u=\omega_q\wedge I\gamma_u=c\omega_q\wedge\gamma_u=cu,\ \ c:=(-1)^{\frac{(n-q)(n-q+1)}{2}}i^{n-q}.$$

Since $|c|=1$ and $\star$ preserves the metric, we have

$$|\gamma_u|=|u|,\ \ (\eta,\xi)e^{-\phi}\,\omega_n=(\gamma_\eta,\gamma_\xi)e^{-\phi}\,\omega_n=c_{n-q}e^{-\phi}\gamma_\eta\wedge\overline{\gamma_\xi}\wedge\omega_q,\ \ c_{n-q}:=\bar{c}, \tag{3.6}$$

for every smooth forms η,ξ of bidegree (n,q). Check that (try) $c=i^{(n-q)^2}$.

4. CHERN CONNECTION AND $\partial\overline{\partial}$-BOCHNER FORMULA

We shall follow [3], page 25–28. Let η,ξ be two smooth L-valued (n,q)-forms $(q\geq 1)$ on an n-dimensional compact Kähler manifold, where L is a holomorphic line bundle with smooth metric $e^{-\phi}$. Recall that we have defined a map

$$\eta\mapsto\gamma_\eta,$$

such that

$$(\eta,\xi)e^{-\phi}\,\omega_n=c_{n-q}e^{-\phi}\gamma_\eta\wedge\overline{\gamma_\xi}\wedge\omega_q=c_{n-q}e^{-\phi}\eta\wedge\overline{\gamma_\xi}. \tag{4.1}$$

The formal adjoint of the $\overline{\partial}$ operator, $\overline{\partial}^*$, must, for any smooth u of bidegree $(n, q-1)$, satisfy

$$(\overline{\partial} u, \xi) = (u, \overline{\partial}^* \xi).$$

By (4.1), the left hand side equals

$$\int_X c_{n-q} e^{-\phi} \overline{\partial} u \wedge \overline{\gamma_\xi}.$$

We want to use Stokes' formula to move $\overline{\partial}$ to an operator on γ_ξ. Notice that $e^{-\phi} u \wedge \overline{\gamma_\xi}$ is a *globally defined* $(n, n-1)$-form and for bidegree reason we have

$$d(e^{-\phi} u \wedge \overline{\gamma_\xi}) = \overline{\partial}(e^{-\phi} u \wedge \overline{\gamma_\xi}),$$

thus Stokes' formula gives

$$0 = \int_X \overline{\partial}(u \wedge e^{-\phi} \overline{\gamma_\xi}) = \int_X \overline{\partial} u \wedge e^{-\phi} \overline{\gamma_\xi} + (-1)^{\deg u} u \wedge \overline{\partial} e^{-\phi}(\overline{\gamma_\xi}).$$

Put

$$\partial_\phi \gamma_\xi = e^{\phi} \partial(e^{-\phi} \gamma_\xi), \tag{4.2}$$

then we have

$$(\overline{\partial} u, \xi) = \int_X (-1)^{n+q} c_{n-q} e^{-\phi} u \wedge \overline{\partial_\phi \gamma_\xi}.$$

On the other hand

$$(u, \overline{\partial}^* \xi) = \int_X c_{n-q+1} e^{-\phi} u \wedge \overline{\gamma_{\overline{\partial}^*_\phi \xi}}.$$

Since $c_{n-q+1} = (-1)^{n-q} i c_{n-q}$, we see that $\overline{\partial}^*$ satisfies

$$\gamma_{\overline{\partial}^*_\phi \xi} = i \partial_\phi \gamma_\xi.$$

Definition 4.1. *We call*

$$D := \overline{\partial} + \partial_\phi$$

the Chern operator (connection) associated to $(L, e^{-\phi})$.

Remark: Notice that the $(0,1)$-part of the Chern connection is just the $\overline{\partial}$-operator on L valued forms and the $(1,0)$-part is further determined by the weight ϕ, more precisely, we have the following *weighted product rule*

$$\overline{\partial}(e^{-\phi} u \wedge v) = e^{-\phi} \overline{\partial} u \wedge v + (-1)^{\deg u} e^{-\phi} u \wedge \partial_\phi u.$$

In general, if (E, h_E) is a hermitian vector bundle then we can define the *Chern connection*, say D, on E-valued forms, such that the $(0,1)$-part of D is just $\overline{\partial}$ and

$$d\{u, v\} = \{Du, v\} + (-1)^{\deg u} \{u, Dv\}.$$

Exercise 1: Check that if we write

$$D = \sum dz^j \wedge D_j + \sum d\bar{z}^j \wedge \overline{\partial}_j,$$

where z^j are holomorphic local coordinates on the base manifold, then $\overline{\partial}_j = \partial/\partial\bar{z}^j$ and D_j is determined by

$$\partial/\partial z^j h_E(e_\alpha, e_\beta) = h_E(D_j e_\alpha, e_\beta),$$

where $\{e_\alpha\}$ is a local holomorphic frame on E. The above formula gives a precise formula for $D_j e_\alpha$. In the line bundle case, $h_E(e, e) = e^{-\phi}$ gives

$$D_j e = -\phi_j e, \ \ D_j(fe) = (f_j - \phi_j f)e,$$

for an arbitray smooth function f. Try to find similar formulas for the vector bundle case.

Definition 4.2. *If D is the Chern connection then we call $\Theta := D^2$ the Chern curvature.*

Remark: In the line bundle case

$$D = \overline{\partial} + \partial_\phi,$$

notice that $\overline{\partial}^2$ and $\partial_\phi = e^\phi \partial(e^{-\phi}\cdot)$ implies $\partial_\phi^2 = 0$. Thus

$$\Theta = \overline{\partial}\partial_\phi + \partial_\phi\overline{\partial}.$$

Exercise 2: Check that

$$\Theta u = \partial\overline{\partial}\phi \wedge u,$$

moreover, in the vector bundle case, show that we can write

$$\Theta = \sum dz^j \wedge d\bar{z}^k \wedge [D_j, \overline{\partial}_k], \ \ [D_j, \overline{\partial}_k] := D_j\overline{\partial}_k - \overline{\partial}_k D_j.$$

In the line bundle case, we have

$$[D_j, \overline{\partial}_k] = [\partial_j - \phi_j, \overline{\partial}_k] = \phi_{j\bar{k}},$$

Try to find similar formulas for the vector bundle case.

Definition 4.3. *Siu's $\partial\overline{\partial}$–Bochner trick is to compute $i\partial\overline{\partial}T$, where*

$$T := c_{n-q} e^{-\phi} \gamma_u \wedge \overline{\gamma_u} \wedge \omega_{q-1}$$

is an $(n-1, n-1)$-form associated to an L-valued smooth (n, q)-form u, $q \geq 1$.

Theorem 4.1 ($\partial\overline{\partial}$–Bochner formula)**.** *$i\partial\overline{\partial}T$ can be written as*

$$\left(-2\mathrm{Re}\langle\overline{\partial\overline{\partial}^*}u, u\rangle + |\overline{\partial}^* u|^2 - |\overline{\partial}u|^2 + |\overline{\partial}\gamma_u|^2\right) e^{-\phi}\omega_n + i\partial\overline{\partial}\phi \wedge T \tag{4.3}$$

and

$$||\overline{\partial}^* u||^2 + ||\overline{\partial}u||^2 = ||\overline{\partial}\gamma_u||^2 + \int_X i\partial\overline{\partial}\phi \wedge T. \tag{4.4}$$

Proof. Note first that the second formula follows immediately from the first one, since the integral of the left hand side of (4.3) vanishes by Stokes' formula. Let us write

$$T = c_{n-q}\{\gamma_u, \gamma_u\} \wedge \omega_{q-1}.$$

Notice that $d\omega = 0$, thus the definition of ∂_ϕ (the Chern connection) gives

$$\overline{\partial}T = c_{n-q}\{\overline{\partial}\gamma_u, \gamma_u\} \wedge \omega_{q-1} + (-1)^{n-q} c_{n-q}\{\gamma_u, \partial_\phi\gamma_u\} \wedge \omega_{q-1},$$

and

$$i\partial\overline{\partial}T =$$

$$ic_{n-q}\{\partial_\phi\overline{\partial}\gamma_u, \gamma_u\} \wedge \omega_{q-1} + i(-1)^{n-q+1}c_{n-q}\{\overline{\partial}\gamma_u, \overline{\partial}\gamma_u\} \wedge \omega_{q-1}$$

$$+i(-1)^{n-q}c_{n-q}\{\partial_\phi\gamma_u, \partial_\phi\gamma_u\} \wedge \omega_{q-1} + i(-1)^{n-q}(-1)^{n-q}c_{n-q}\{\gamma_u, \overline{\partial}\partial_\phi\gamma_u\} \wedge \omega_{q-1}.$$

Now we use the commutation rule (Chern curvature formula)

$$\overline{\partial}\partial_\phi + \partial_\phi\overline{\partial} = \partial\overline{\partial}\phi$$

in the first term. The first and last terms then combine to give the first and last terms in (9.35). Moreover, by the formula for $\overline{\partial}^*$, the third term equals the second term in (9.35) (here we use $c_{n-q+1} = (-1)^{n-q}ic_{n-q}$). The trickiest term is the second, see the following lemma. □

Lemma 4.2. $i(-1)^{n-q+1}c_{n-q}\{\overline{\partial}\gamma_u, \overline{\partial}\gamma_u\} \wedge \omega_{q-1} = (-|\overline{\partial}u|^2 + |\overline{\partial}\gamma_u|^2)e^{-\phi}\omega_n.$

Proof. The main idea is to use HLT to the $(n-q,1)$-form $\overline{\partial}_\gamma u$. Let us write

$$\overline{\partial}\gamma_u = a + \omega \wedge b,$$

where a is primitive of degree $(n-q,1)$ and b is primitive of degree $(n-q-1,0)$. Since $a \wedge \omega_q = 0$, we have

$$\{\overline{\partial}\gamma_u, \overline{\partial}\gamma_u\} \wedge \omega_{q-1} = \{a, a\} \wedge \omega_{q-1} + \{\omega \wedge b, \omega \wedge b\} \wedge \omega_{q-1}.$$

Moreover, since HLT is an orthogonal decomposition, we get

$$|\overline{\partial}\gamma_u|^2 = |a|^2 + |\omega \wedge b|^2,$$

now it is enough to use

$$|\overline{\partial}u| = |\omega_q \wedge \overline{\partial}\gamma_u| = |\omega_q \wedge \omega \wedge b|,$$

and compare the coefficients. We leave the detail as an exercise. □

Remark: In case $q = n = 1$ then γ_u is locally given by a smooth function, thus

$$i\partial\overline{\partial}\phi \wedge T = c_0 e^{-\phi}|\gamma_u|^2 i\partial\overline{\partial}\phi.$$

Let us define an function B such that

$$i\partial\overline{\partial}\phi = B\omega,$$

then we have

$$i\partial\overline{\partial}\phi \wedge T = B|u|^2e^{-\phi}\omega_n.$$

Notice that in case $X = \mathbb{C}$ and $\omega = idz \wedge d\bar{z}$ then

$$B = \phi_{z\bar{z}}.$$

In particular, $B > 0$ if ϕ is strictly subharmonic. In general, we shall use the following formula to define an *operator* B:

$$i\partial\overline{\partial}\phi \wedge T = \{Bu, \star u\}.$$

then we have

Theorem 4.3. $B = [i\Theta, \Lambda]$ *is a pointwise self-adjoint operator. All eigenvalues of* B *are positive if* $i\Theta = i\partial\overline{\partial}\phi > 0$ *(in which case we say that* $B > 0$*).*

Proof. Since $u = \omega_q \wedge \gamma_u$ and γ_u is primitive, we have

$$\Lambda u = \omega_{q-1} \wedge \gamma_u.$$

Thus (note that $\Theta u = i\partial\overline{\partial}\phi \wedge u = 0$)

$$[i\Theta, \Lambda]u = i\Theta\Lambda u = i\partial\overline{\partial}\phi \wedge \omega_{q-1} \wedge \gamma_u = Bu.$$

Fix $x \in X$, let $\{\xi^j\}$ be an orthonormal basis of $\wedge^{1,0}T^*_xX$. Assume that

$$\gamma_u(x) = \sum_{j_1<\cdots<j_{n-q}} c_{j_1\cdots j_{n-q}}\, \xi^{j_1} \wedge \cdots \wedge \xi^{j_{n-q}}, \quad i\partial\overline{\partial}\phi(x) = i\sum \lambda_j\, \xi^j \wedge \bar{\xi}^j.$$

A direct computation gives

$$i\partial\overline{\partial}\phi \wedge \omega_{q-1} \wedge (\xi^{j_1} \wedge \cdots \wedge \xi^{j_{n-q}}) = \left(\sum_{j\notin\{j_1,\cdots,j_{n-q}\}} \lambda_j\right) \omega_q \wedge (\xi^{j_1} \wedge \cdots \wedge \xi^{j_{n-q}}).$$

Thus the theorem follows. □

Exercise 3: Check that in general, we have

$$[i\Theta, \Lambda](\xi^{J\bar{K}}) = \left(\sum_{j\in J} \lambda_j + \sum_{j\in K} \lambda_j - \sum_{j=1}^{n} \lambda_j\right)(\xi^{J\bar{K}}).$$

In particular, if

$$\omega = i\Theta = i\partial\overline{\partial}\phi$$

then $B = [i\Theta, \Lambda] = p + q - n$ on the space of (p, q)-forms. *Hint: Use the fact that*

$$[i\Theta, \Lambda](a \wedge b) = [i\Theta_1, \Lambda_1]a \wedge b + a \wedge [i\Theta_2, \Lambda_2]b,$$

where $\Theta_1 = \sum_{j\leq m} \lambda_j \xi^j \wedge \bar{\xi}^j$, $\Lambda_1 := (\sum_{j\leq m} i\xi^j \wedge \bar{\xi}^j)^*$, $\Theta_2 = \Theta - \Theta_1$, $\Lambda_2 = \Lambda - \Lambda_1$ *and*

$$a = \xi^{J_1\overline{K_1}},\ b = \xi^{J_2\overline{K_2}}, \quad J_1, K_1 \in \{1, \cdots, m\}, \quad J_2, K_2 \in \{m+1, \cdots, n\}.$$

Theorem 4.4 (L^2-estimate for the Laplacians)**.** *Let $(L, e^{-\phi})$ be a holomorphic line bundle on an n-dimensional Kähler manifold (X, ω). Assume that L is positive (i.e. $i\Theta = i\partial\overline{\partial}\phi > 0$). Then for every smooth L-valued (n, q)-form ($q \geq 1$) c on X, we can find a smooth L-valued (n, q)-form v on X such that*

$$\square v := (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})v = c,$$

and

$$||\overline{\partial}v||^2 + ||\overline{\partial}^*v||^2 \leq (B^{-1}c, c).$$

Proof. Denote by $D^{n,q}$ the space of smooth L-valued (n, q)-forms with *compact support* in X. The $\partial\overline{\partial}$–Bochner formula gives

$$|(c, u)|^2 \leq (B^{-1}c, c)(Bu, u) \leq (B^{-1}c, c)\left(||\overline{\partial}u||^2 + ||\overline{\partial}^*u||^2\right),$$

for every $u \in D^{n,q}$. Consider the following inner product on$D^{n,q}$

$$(u, v)_\square := (\overline{\partial}u, \overline{\partial}v) + (\overline{\partial}^*u, \overline{\partial}^*v),$$

we know that

$$u \mapsto (u, c)$$

defines a bounded linear functional on $(D^{n,q}, ||\cdot||_\Box)$, which extends to a bounded linear functional on its Hilbert completion, say H, of $D^{n,q}$. Thus the Riesz representation theorem gives $v \in H$ such that

$$(u, c) = (u, v)_\Box, \quad ||v||_\Box^2 \leq (B^{-1}c, c).$$

Since $||u||_\Box^2 \geq (Bu, u)$, we know that H is a subspace of the L^2 space. Thus we have

$$(u, c) = (u, v)_\Box = (u, \Box v),$$

in the sense of current, which implies that

$$\Box v = c,$$

moreover, smoothness of c gives smoothness of v. □

Theorem 4.5 ($\overline{\partial}$-L^2-estimate on compact Kähler manifold)**.** *With the assumptions in the above theorem, assume further that X is compact and $\overline{\partial} c = 0$. Then there exists a smooth L-valued $(n, q-1)$-form a on X such that*

$$\overline{\partial} a = c$$

on X and

$$||a||^2 \leq (B^{-1}c, c).$$

In case $i\Theta \geq \varepsilon\omega$ we have $B \geq q\varepsilon$.

Proof. $\overline{\partial} c = 0$ implies that $\overline{\partial}\overline{\partial}^*\overline{\partial} v = 0$, thus

$$0 = (\overline{\partial}\overline{\partial}^*\overline{\partial} v, \overline{\partial} v) = ||\overline{\partial}^*\overline{\partial} v||^2,$$

where we use the compact-ness in the the second identity. Now it is enough to take $a = \overline{\partial}^* v$. □

Remark: The above theorem implies that if L is positive and X is compact then every smooth $\overline{\partial}$-closed form is $\overline{\partial}$-exact, i.e. the following $\overline{\partial}$–Dolbeault cohomology group

$$H_L^{p,q} := C^\infty_{p,q}(L, \ker \overline{\partial}) / C^\infty_{p,q}(L, \mathrm{Im}\, \overline{\partial})$$

vanishes if $p = n$ and $q \geq 1$. If fact the above proof only use

$$\Box \geq [i\Theta, \Lambda] := B > 0$$

Later we shall use Kähler identities to prove that $\Box \geq [i\Theta, \Lambda]$ always holds. Thus if we choose $\omega = i\Theta$ then $B = p + q - n$ on the space of smooth L-valued (p, q)-forms and we get the following *Kodaira vanishing theorem*:

Theorem 4.6. *If L is positive and X is compact then $H_L^{p,q} = 0$ if $p + q > n$.*

Exercise 4: Check that if $p = n, q = 1$ then

$$(B^{-1}c, c) = ||c||_{i\Theta}^2,$$

does not depend on the choice of ω.

5. COMPLETE KÄHLER CASE

Now let us consider the non–compact case. Recall that by Theorem 4.4, we can always find smooth v such that

$$(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})v = c, \quad ||\overline{\partial}v||^2 + ||\overline{\partial}^*v||^2 \leq (B^{-1}c, c).$$

The main problem is the following:

Under which conditions $\overline{\partial}c = 0$ *implies* $\overline{\partial}^*\overline{\partial}v = 0$ *?*

Notice that $\overline{\partial}c = 0$ iff $\overline{\partial}\overline{\partial}^*\overline{\partial}v = 0$, which implies

$$(\overline{\partial}\overline{\partial}^*\overline{\partial}v, \overline{\partial}v) = 0.$$

but if X is not compact, we cannot move $\overline{\partial}$ to get

$$(\overline{\partial}\overline{\partial}^*\overline{\partial}v, \overline{\partial}v) = ||\overline{\partial}^*\overline{\partial}v||^2.$$

The main idea in [8] is to consider a family of smooth forms, say f_ε, with *compact support* such that f_ε converges to $\overline{\partial}v$ in a nice way such that

$$0 = (\overline{\partial}\overline{\partial}^*\overline{\partial}v, f_\varepsilon) = (\overline{\partial}^*\overline{\partial}v, \overline{\partial}^* f_\varepsilon)$$

gives $\overline{\partial}\overline{\partial}^*\overline{\partial}v = 0$. The approach that we will introduce is the following:

Andreotti–Vesentini–Hörmander trick — geometry behind a good family of cut–off functions: The idea is very simple: suppose we have a family of smooth functions $\chi_j \in C_0^\infty(X)$ such that

$$\chi_j|_{K_j} = 1, \ K_j \subset K_{j+1}, \ \cup K_j = X,$$

and

$$|\chi_j| \leq 1, \ \ |\overline{\partial}\chi_j|_\omega \leq 1/j.$$

Then we have

$$0 = (\chi_j^2\overline{\partial}\overline{\partial}^*\overline{\partial}v, \overline{\partial}v) = ||\chi_j\overline{\partial}^*\overline{\partial}v||^2 - 2(\chi_j\overline{\partial}\chi_j \wedge \overline{\partial}^*\overline{\partial}v, \overline{\partial}v),$$

Since $||\overline{\partial}v||^2 \leq (B^{-1}c, c)$, together with Cauchy–Schwarz inequality, the above identity gives

$$||\chi_j\overline{\partial}^*\overline{\partial}v||^4 \leq (4/j^2) \cdot ||\chi_j\overline{\partial}^*\overline{\partial}v||^2 \cdot (B^{-1}c, c).$$

Thus

$$||\chi_j\overline{\partial}^*\overline{\partial}v||^2 \leq (4/j^2) \cdot (B^{-1}c, c),$$

let j go to infinity we get $\overline{\partial}^*\overline{\partial}v = 0$.

5.1. **Complete Kähler manifold.**

Definition 5.1. *A Riemannian manifold (X, g) is said to be complete if there is a real smooth function ρ on X such that for every $c \in \mathbb{R}$,*

$$\rho_c := \{\rho < c\}$$

is relatively compact in X and

$$|d\rho|_g \leq 1$$

on X. A Käher manifold is said to be complete if the underlying Riemannian manifold is so.

Remark: In case X is compact one may take $\rho = 1$. Thus every compact manifold is complete. In general,

$$\chi_j := \chi(\rho/j),$$

gives a good family of cut–off functions, where χ is smooth on $\mathbb{R}$, such that $\chi = 1$ on $(-\infty, 1]$, $\chi = 0$ on $[3, \infty)$, $0 \leq \chi \leq 1$ and $|\chi'| \leq 1$.

Relation between pseudoconvexity and plurisubharmonicity.

Definition 5.2. *A complex manifold X is said to be weakly pseudoconvex if there is a smooth real function ψ on X such that for every $c \in \mathbb{R}$,*

$$\psi_c := \{\psi < c\}$$

is relatively compact in X and ψ is plurisubharmonic on X, i.e.

$$i\partial\overline{\partial}\psi \geq 0$$

on X. X is said to be Stein if moreover $i\partial\overline{\partial}\psi > 0$ everywhere on X.

Exercise 5: Prove that if X is Kähler and weakly pseudoconvex then there is a complete Kähler metric on X (in particular every Stein manifold is complete Kähler).

The Andreotti–Vesentini–Hörmander trick implies the following theorem:

Theorem 5.1 ($\overline{\partial}$-L^2-estimate on complete Kähler manifold)**.** *Let $(L, e^{-\phi})$ be a holomorphic line bundle on an n-dimensional complete Kähler manifold (X, ω). Assume that L is positive (i.e. $i\Theta = i\partial\overline{\partial}\phi > 0$). Then for every $\overline{\partial}$-closed smooth L-valued (n, q)-form ($q \geq 1$) c on X with $(B^{-1}c, c) < \infty$, we can find a smooth L-valued $(n, q-1)$-form a on X such that*

$$\overline{\partial}a = c$$

on X and $||a||^2 \leq (B^{-1}c, c)$.

6. HÖRMANDER THEORY, DEGREE $(n, 1)$-CASE

6.1. L^2**-estimate with respect to a non-complete Kähler metric.** This part is around an unpublished result of the author and Bo-Yong Chen (based on [8]).

Definition 6.1. *A Kähler metric ω on a complex manifold X is said to be quasi-complete if there exists a family of Kähler manifolds $\{(X_j, \omega_j, \chi_j)\}_{j=1}^{\infty}$ such that*

1) *Each X_j is an open set in X, $X_j \subset X_{j+1}$ and $X = \cup X_j$;*

2) *For each j we have $\omega_j \geq \omega$ on X_j and for every compact subset K of X,*

$$\lim_{j\to\infty} \sup_K |\omega_j - \omega|_\omega = 0;$$

3) *Each χ_j is smooth with compact support in X_j such that $0 \leq \chi_j \leq 1$ on X_j,*

$$\lim_{j\to\infty} \sup_{X_j} |\overline{\partial}\chi_j|_{\omega_j} = 0$$

and for every compact subset K in X we have $\chi_j \equiv 1$ on K for all $j \geq j(K)$.

Exercise 1: Recall that we have defined $B := [i\Theta, \Lambda_\omega]$. Assume that $i\Theta > 0$, try to show that

$$(B^{-1}c, c)_\omega = (c, c)_{i\Theta}$$

for every smooth L-valued $(n, 1)$-form c on X (in particular, the above inner product does not depend on the choice of ω).

We shall use the following lemma to prove the Hörmander L^2-estimate on *non-complete* Kähler manifold.

Lemma 6.1. *Every Kähler metric ω on X is quasi-complete if there exists a complete Kähler metric, say $\hat{\omega}$, on X.*

Proof. Enough to take $X_j = X$, $\omega_j = \omega + (1/j)\hat{\omega}$ and $\chi_j = \chi(\rho/j^2)$. □

Remark: In particular every Kähler metric on a weakly pseudoconvex manifold is quasi–complete. The main result that we will prove is the following:

Theorem 6.2 ($\overline{\partial}$-L^2-estimate on quasi-complete Kähler manifold)**.** *Let $(L, e^{-\phi})$ be a holomorphic line bundle on an n-dimensional quasi-complete Kähler manifold (X, ω). Assume that L is positive (i.e. $i\Theta = i\partial\overline{\partial}\phi > 0$). Then for every $\overline{\partial}$-closed L-valued $(n, 1)$-form c on X with $||c||^2_{i\Theta} < \infty$, we can find an L-valued $(n, 0)$-form a on X (i.e. a section of $K_X + L$) such that*

$$\overline{\partial}a = c$$

on X and $||a|| \leq ||c||_{i\Theta}$.

Proof. Denote by $\square_j$ and B_j the operators with respect to (X_j, ω_j), solving the Laplace equation gives v_j on X_j such that

$$\square_j v_j = c, \quad ||\overline{\partial}v_j||^2_{\omega_j} + ||\overline{\partial}^* v_j||^2_{\omega_j} \leq (B_j^{-1}c, c)_{\omega_j} = ||c||^2_{i\Theta, X_j} \leq ||c||^2_{i\Theta}.$$

Claim: $\overline{\partial}^*\overline{\partial}v_j$ goes to zero in the sense of distribution on every relatively compact open subset in X. In fact $\overline{\partial}c = 0$ implies that $\square_j\overline{\partial}v_j = 0$, thus each $\overline{\partial}v_j$ is smooth and we have

$$0 = (\chi_j^2\overline{\partial}\overline{\partial}^*\overline{\partial}v_j, \overline{\partial}v_j)_{\omega_j} = ||\chi_j\overline{\partial}^*\overline{\partial}v_j||^2_{\omega_j} - 2(\chi_j\overline{\partial}\chi_j \wedge \overline{\partial}^*\overline{\partial}v_j, \overline{\partial}v_j)_{\omega_j},$$

which gives

$$||\chi_j\overline{\partial}^*\overline{\partial}v_j||^2_{\omega_j} \leq 4 \sup_{X_j} |\overline{\partial}\chi_j|^2_{\omega_j}(B^{-1}c, c),$$

which implies that (since $\{\omega_j\}$ is locally bounded form above)

$$\lim_{j\to\infty} ||\overline{\partial}^*\overline{\partial} v_j||_{U,\omega} = 0,$$

for every relatively compact open subset U in X. Thus our claim is true.

Since ω_j converges to ω locally uniformly, taking a weak limit of $\{\overline{\partial}^* v_j|_U\}$ we get an L^2-form a_U on U such that

$$\overline{\partial} a_U = c, \quad ||a_U|| \leq ||c||_{i\Theta}.$$

Now weak limit of $\{a_U\}$ gives an L^2-form a on X such that $\overline{\partial} a = c$ and $||a|| \leq ||c||_{i\Theta}$. □

Remark 1: The above theorem implies that

$$H^{n,1}_{L^2}(L) := L^2_{n,1}(\ker \overline{\partial}, i\Theta)/L^2_{n,1}(\mathrm{Im}\, \overline{\partial}, i\Theta)$$

is trivial if X is quasi-complete and $i\Theta > 0$.

Remark 2: In case X is a pseudoconvex domain in $\mathbb{C}^n$, ϕ is a smooth strictly plurisubharmonic function on X, we have

$$||a||^2 = \int_X i^{n^2} a \wedge \bar{a}\, e^{-\phi}.$$

If we write $a = \hat{a}\, dz$, $dV = i^{n^2} dz \wedge \overline{dz}$ (i.e. dV is 2^n times the Lebesgue measure $d\lambda$) then

$$||a||^2 = \int_X |\hat{a}|^2 e^{-\phi}\, dV, \quad ||c||^2_{i\partial\overline{\partial}\phi} = \int_X \sum c_{\bar{j}} \overline{c_{\bar{k}}} \phi^{\bar{j}k} e^{-\phi}\, dV, \quad c := \sum c_{\bar{j}}\, d\bar{z}^j \wedge dz.$$

Thus we get the following *Hörmander theorem*:

Theorem 6.3. *Let ϕ be a smooth strictly plurisubharmonic function on a pseudoconvex domain X in $\mathbb{C}^n$. Then for every $\overline{\partial}$-closed $(0,1)$-form $\hat{c} := \sum c_{\bar{j}}\, d\bar{z}^j$ on X there exists a function $\hat{a}$ on X such that $\overline{\partial}\hat{a} = \hat{c}$ on X and*

$$\int_X |\hat{a}|^2 e^{-\phi}\, d\lambda \leq \int_X \sum c_{\bar{j}} \overline{c_{\bar{k}}} \phi^{\bar{j}k} e^{-\phi}\, d\lambda,$$

provided the right hand side is finite.

Exercise 2: If ϕ is not strictly plurisubharmonic, then one may consider

$$\psi := \phi + |z|^2.$$

In case X is bounded, try to prove the following theorem:

Theorem 6.4. *Let ϕ be a smooth plurisubharmonic function on a pseudoconvex domain X in $\mathbb{C}^n$. Then for every $\overline{\partial}$-closed $(0,1)$-form $\hat{c} := \sum c_{\bar{j}}\, d\bar{z}^j$ there exists a function $\hat{a}$ on X such that $\overline{\partial}\hat{a} = \hat{c}$ on X and*

$$\int_X |\hat{a}|^2 e^{-\phi}\, d\lambda \leq e^{\sup_X |z|^2} \int_X \sum |c_{\bar{j}}|^2 e^{-\phi}\, d\lambda,$$

provided the right hand side is finite.

We will also need the following generalized version of Theorem 6.2.

Theorem 6.5 (Semi-positive version)**.** *Let $(L, e^{-\phi})$ be a holomorphic line bundle on an n-dimensional quasi-complete Kähler manifold (X, ω). Assume that L is semi-positive (i.e. $i\Theta = i\partial\overline{\partial}\phi \geq 0$). Then for every $\overline{\partial}$-closed L-valued $(n,1)$-form c on X with*

$$I(c) := \lim_{\varepsilon \to 0} ||c||_{i\Theta+\varepsilon\omega} < \infty,$$

we can find an L-valued $(n,0)$-form a on X such that

$$\overline{\partial} a = c$$

on X and $||a|| \leq I(c)$.

Proof. Consider $\Box_j + \varepsilon$ instead of $\Box_j$, we find $v_{j,\varepsilon}$ on X_j such that

$$(\Box_j + \varepsilon) v_{j,\varepsilon} = c, \quad ||\overline{\partial} v_{j,\varepsilon}||^2_{\omega_j} + ||\overline{\partial}^* v_{j,\varepsilon}||^2_{\omega_j} + \varepsilon ||v_{j,\varepsilon}||^2_{\omega_j} \leq ((B_j+\varepsilon)^{-1} c, c)_{\omega_j} \leq I(c)^2.$$

Now $\overline{\partial} c = 0$ gives

$$\overline{\partial}\overline{\partial}^* \overline{\partial} v_{j,\varepsilon} + \varepsilon \overline{\partial} v_{j,\varepsilon} = 0,$$

thus $(\Box_j + \varepsilon)\overline{\partial} v_{j,\varepsilon} = 0$ and each $\overline{\partial} v_{j,\varepsilon}$ is smooth. Taking inner product with $\chi_j^2 \overline{\partial} v_{j,\varepsilon}$ gives

$$0 = \varepsilon ||\chi_j \overline{\partial} v_{j,\varepsilon}||^2_{\omega_j} + ||\chi_j \overline{\partial}^* \overline{\partial} v_{j,\varepsilon}||^2_{\omega_j} - 2(\chi_j \overline{\partial}\chi_j \wedge \overline{\partial}^* \overline{\partial} v_{j,\varepsilon}, \overline{\partial} v_{j,\varepsilon})_{\omega_j}.$$

Assume that $\sup_{X_j} |\overline{\partial}\chi_j|_{\omega_j} = \varepsilon_j$, we get

$$||\chi_j \overline{\partial}^* \overline{\partial} v_{j,\varepsilon}||^2_{\omega_j} \leq 2\varepsilon_j ||\chi_j \overline{\partial}^* \overline{\partial} v_{j,\varepsilon}||_{\omega_j} I(c),$$

which gives

$$||\chi_j \overline{\partial}^* \overline{\partial} v_{j,\varepsilon}||_{\omega_j} \leq 2\varepsilon_j I(c).$$

The theorem follows if we first let ε go to zero then let j go to infinity. □

Exercise 3: Assume that

$$c = \alpha \wedge u,$$

where α is a smooth $(0,1)$-form such that

$$i\bar{\alpha} \wedge \alpha \leq A^2 \cdot i\Theta$$

on X, where $A > 0$ is a constant. Then

$$I(c) \leq A \cdot ||u||.$$

We will use this estimate later.

6.2. **Approximation theorem.** Let us recall the following definition first.

Definition 6.2. *A complex manifold X is said to be weakly pseudoconvex if there is a smooth real function ρ on X such that for every $c \in \mathbb{R}$,*

$$X_c := \{\rho < c\}$$

is relatively compact in X and ρ is plurisubharmonic on X, i.e.

$$i\partial\overline{\partial}\rho \geq 0$$

on X (i.e. ρ is psh exhaustion). X is said to be Stein if moreover $i\partial\overline{\partial}\rho > 0$ everywhere on X.

Theorem 6.6. *Let $(L, e^{-\phi})$ be a holomorphic line bundle on an n-dimensional weakly pseudo-convex Kähler manifold (X, ω, ρ). Assume that $i\Theta = i\partial\overline{\partial}\phi \geq 0$. Let u be an L-valued L^2 holomorphic n-form on X_0. Then there exists a family of L-valued holomorphic n-forms $\{u_\varepsilon\}$, each u_ε is holomorphic on X_ε and*

$$\lim_{\varepsilon \to 0} ||u_\varepsilon - u||_{X_0} = 0.$$

Proof. Put

$$\psi_\varepsilon := -\log -(\rho - 2\varepsilon), \ \varepsilon > 0. \tag{6.1}$$

Then we know that each ψ_ε is smooth and bounded on X_ε. Moreover,

$$i\partial\overline{\partial}\psi_\varepsilon \geq i\partial\psi_\varepsilon \wedge \overline{\partial}\psi_\varepsilon, \tag{6.2}$$

on X_ε. Let λ be a smooth function on $\mathbb{R}$, such that $\lambda \equiv 1$ on $(-\infty, 1]$, $\lambda \equiv 0$ on $[2, \infty]$ and $|\lambda'| \leq 2$ on $\mathbb{R}$. Put

$$\lambda_\varepsilon := \lambda\left(\frac{\rho + 3\varepsilon}{\varepsilon}\right). \tag{6.3}$$

Then on X_ε we have

$$i\partial\lambda_\varepsilon \wedge \overline{\partial}\lambda_\varepsilon \leq 64 \cdot i\partial\psi_\varepsilon \wedge \overline{\partial}\psi_\varepsilon \leq 64 \cdot i\partial\overline{\partial}\psi_\varepsilon, \tag{6.4}$$

and each λ_ε is a smooth function with compact support in X_0. Thus

$$\int_{X_0} |\lambda_\varepsilon u|^2 e^{-(\phi + \psi_\varepsilon/2)} < \infty. \tag{6.5}$$

We claim that it is enough to take u_ε as the Bergman projection of $\lambda_\varepsilon u$, i.e.

$$u_\varepsilon := P(\lambda_\varepsilon u)$$

on X_ε with respect to the weight function $\phi + \psi_\varepsilon/2$. Put

$$a_\varepsilon := \lambda_\varepsilon u - u_\varepsilon. \tag{6.6}$$

Then each a_ε is the L^2-minimal solution (with respect to the weight function $\phi + \psi_\varepsilon/2$) of

$$\overline{\partial}(\cdot) = \overline{\partial}(\lambda_\varepsilon u) = \overline{\partial}\lambda_\varepsilon \wedge u, \tag{6.7}$$

on X_ε. Thus each $a_\varepsilon e^{\psi_\varepsilon/2}$ is the L^2-minimal solution (with respect to the weight function $\phi+\psi_\varepsilon$, try!) of the following equation on X_ε

$$\overline{\partial}(\cdot)=\overline{\partial}(a_\varepsilon e^{\psi_\varepsilon/2})=e^{\psi_\varepsilon/2}\left(\overline{\partial}\lambda_\varepsilon\wedge u+\frac{1}{2}\overline{\partial}\psi_\varepsilon\wedge a_\varepsilon\right):=f_\varepsilon. \tag{6.8}$$

Thus Theorem 6.5 gives

$$\int_{X_\varepsilon}|a_\varepsilon|^2e^{-\phi}=\int_{X_\varepsilon}|a_\varepsilon e^{\psi_\varepsilon/2}|^2e^{-\phi-\psi_\varepsilon}\leq\lim_{\delta\to 0}\int_{X_\varepsilon}|f_\varepsilon|^2_{\delta\omega+i\partial\overline{\partial}(\phi+\psi_\varepsilon)}e^{-\phi-\psi_\varepsilon}. \tag{6.9}$$

Notice that (6.2), (6.4) and Exercise 3 together imply that

$$\lim_{\delta\to 0}|f_\varepsilon|^2_{\delta\omega+i\partial\overline{\partial}(\phi+\psi_\varepsilon)}\leq e^{\psi_\varepsilon}\cdot 2\cdot(64|u|^2+\frac{1}{4}|a_\varepsilon|^2). \tag{6.10}$$

Thus we have

$$\int_{X_\varepsilon}|a_\varepsilon|^2e^{-\phi}\leq 256\int_{X_{-\varepsilon}\setminus X_{-2\varepsilon}}|u|^2e^{-\phi}\to 0,\ \text{as }\varepsilon\to 0, \tag{6.11}$$

which gives

$$\lim_{\varepsilon\to 0}||u_\varepsilon-u||_{X_0}=0. \tag{6.12}$$

Thus the theorem follows. □

Remark: For each $\varepsilon>0$ consider

$$\mu_\varepsilon:=\lambda\left(\frac{3\rho}{\varepsilon}\right). \tag{6.13}$$

Then we know that each μ_ε is a smooth function with compact support on X_ε such that $\mu_\varepsilon\equiv 1$ on X_0. Let us choose $k_\varepsilon>0$ such that

$$k_\varepsilon\chi''(\rho)\geq\frac{9}{\varepsilon^2}\cdot\left|\lambda'\left(\frac{3\rho}{\varepsilon}\right)\right|^2 \tag{6.14}$$

on X, where χ is smooth convex increasing function such that $\chi\equiv 0$ on $(-\infty,0)$. Then we have

$$k\cdot i\partial\overline{\partial}(\chi\circ\rho)\geq i\partial\mu_\varepsilon\wedge\overline{\partial}\mu_\varepsilon,\ \forall\, k\geq k_\varepsilon, \tag{6.15}$$

on X. For each $k\geq k_\varepsilon$, let $a_{k,\varepsilon}$ be the L^2-minimal solution of

$$\overline{\partial}(\cdot)=\overline{\partial}(\mu_\varepsilon u_\varepsilon):=v_\varepsilon, \tag{6.16}$$

on X with respect to the weight function $\phi+k\chi\circ\rho$. Notice that (6.15) implies that

$$\lim_{\delta\to 0}\int_X|v_\varepsilon|^2_{\delta\omega+i\partial\overline{\partial}(\phi+k\chi\circ\rho)}e^{-\phi-k\chi\circ\rho}\leq\int_{X_\varepsilon\setminus X_0}|u_\varepsilon|^2e^{-\phi-k\chi\circ\rho}\to 0\ (k\to\infty). \tag{6.17}$$

Thus by Hörmander's L^2-estimates, we have

$$\int_X|a_{k,\varepsilon}|^2e^{-\phi-k\chi\circ\rho}\leq\lim_{\delta\to 0}\int_{X_s}|v_\varepsilon|^2_{\delta\omega+i\partial\overline{\partial}(\phi+k\chi\circ\rho)}e^{-\phi-k\chi\circ\rho}\to 0\ (k\to\infty). \tag{6.18}$$

Put

(6.19) $$u_{k,\varepsilon} := \mu_\varepsilon u_\varepsilon - a_{k,\varepsilon}.$$

Then each $u_{k,\varepsilon}$ ($k \geq k_\varepsilon$) is holomorphic on X and

(6.20) $$||u_{k,\varepsilon} - u||_{X_0} \to 0 \ (\varepsilon \to 0, \ k \to \infty).$$

Thus we can make each u_ε to be holomorphic on X in the above approximation theorem.

6.3. **Vanishing theorem and Levi problem on Stein manifold.** Recall that a Stein manifold is a complex manifold, say X, with a smooth strictly psh exhaustion function, say ρ. Since for every smooth E-valued (n,q)-form c on X, we can choose a convex increasing function, say χ, such that $||e^{-\chi(\rho)}c||_{i\partial\overline{\partial}\rho} < \infty$, we know that if c is $\overline{\partial}$-closed and $q \geq 1$ then c must be $\overline{\partial}$-exact.

Theorem 6.7 (Exercise)**.** *Let E be a holomorphic vector bundle over a Stein manifold X. Then the Dolbeault cohomology group $H^{n,q}(X,E)$ is trivial if $q \geq 1$. Since*

$$H^{n,q}(X,E) \sim H^q(X, K_X \otimes E),$$

Consider $E \otimes K_X^{-1}$ instead of E, we know that the q-th Cech cohomology group $H^q(X,E)$ is trivial as long as $q \geq 1$.

Another application of the Hörmander theory is a nice solution of the *Levi problem*:

Is a Stein manifold (X,ρ) holomorphically convex ? Here holomorphically convex means that: for every sequence of points $\{x_j\}$ in X such that $\rho(x_j)$ goes to infinity we can find a holomorphic function f on X such that $|f(x_j)|$ goes to infinity.

Theorem 6.8. *Stein manifold is holomorphically convex.*

Proof. Apply the Hörmander theorem to the case that $L = -K_X$. Then an L-valued $(n,0)$-form is just a function and an L-valued $(n,1)$-form is a $(0,1)$-form. Fix an arbitrary smooth metric $e^{-\phi}$ on L, at each point x_j choose local holomorphic coordinate system $z_j = \{z_j^k\}_{1\leq k\leq n}$ centered at x_j such that $\{|z_j| < 3\} \cap \{x_j\} = x_j$. Consider

$$c = \sum j\overline{\partial}\lambda(|z_j|) \wedge dz_j^1 \wedge \cdots \wedge dz_j^n \otimes (dz_j^1 \wedge \cdots \wedge dz_j^n)^{-1}$$

where $\lambda(x) \equiv 1$ on $|x| \leq 1$ and $\lambda \equiv 0$ on $|x| \geq 3$. Consider an extra weight function

$$\psi_\varepsilon := \chi(\rho) + \sum n\lambda(|z_j|)\log(|z_j|^2 + \varepsilon),$$

where χ is choosing such that

$$i\partial\overline{\partial}\psi_\varepsilon + i\partial\overline{\partial}\phi \geq \omega := i\partial\overline{\partial}\rho$$

on X (the key point is that χ does not depend on ε) and

$$||e^{-\psi_\varepsilon/2}c||_\omega \leq 1.$$

Then Hörmander's theorem gives u_ε such that $\overline{\partial}u_\varepsilon = c$ and

$$||e^{-\psi_\varepsilon/2}u_\varepsilon||_\omega \leq 1$$

Let u_ε goes to zero. Then we get u such that $\overline{\partial} u = 0$ and $u(x_j) = 0$. Thus

$$f := -u + \sum j\lambda(|z_j|) \wedge dz_j^1 \wedge \cdots \wedge dz_j^n \otimes (dz_j^1 \wedge \cdots \wedge dz_j^n)^{-1}$$

fits our needs. □

Remark: The original definition of Stein manifold is the following:

A complex manifold X is said to be Stein if it is holomorphically convex and for every $x \neq y \in X$, there exists a holomorphic function f on X such that $f(x) \neq f(y)$.

It is known that if X is Stein in the above sense then X is Stein (see Demailly's book).

Exercise 4: Prove that if X is Stein then X is also Stein in the above sense.

6.4. **Kodaira embedding theorem.**

Definition 6.3. *Let L be a holomorphic line bundle on a compact complex manifold X, we say that L is positive if there exists a smooth metric $e^{-\phi}$ on L such that its Chern curvature satisfies*

$$i\Theta = i\partial\overline{\partial}\phi > 0$$

on X. We say that L is ample if there exists $m \in \mathbb{N}$ such that a basis, say $\{e_j\}$, of the space of holomorphic sections of mL on X defines a holomorphic embedding of X into $\mathbb{P}^N$, $N + 1 = \dim H^0(X, mL)$ as follows

$$x \mapsto [e_0(x), \cdots, e_N(x)] \in \mathbb{P}^N.$$

We say that mL is very ample.

Remark: A result of Chow says that every submanifold of $\mathbb{P}^n$ is given by the common zero set of a finite number of homogeneous polynomials in $\mathbb{C}^{n+1}$, i.e. every submanifold of $\mathbb{P}^n$ is algebraic. The following lemma implies that every algebraic manifold is Kähler. But not every Kähler manifold is algebraic (there exists a non-algebraic two dimensional torus).

Proposition 6.9. *Ample implies positive.*

Proof. Show that the $\mathcal{O}(1)$ bundle on $\mathbb{P}^N$ is positive. If $\{e_j\}$ above defines a holomorphic embedding, say

$$\Phi : X \to \mathbb{P}^N.$$

Then the pull back of the $\mathcal{O}(1)$ bundle is equal to mL. Thus mL has a positively curved metric, say $e^{-\phi}$. Then $e^{-\phi/m}$ defines a positively curved metric on L. □

Theorem 6.10 (Kodaira embedding theorem)**.** *Positive implies ample.*

Proof. Assume that L is positive. Denote by Φ_m the map (depends on the choice of basis!) to $\mathbb{P}^N$ defined by a basis of $H^0(X, mL)$. First let us show that the map Φ_m is well defined, i.e. for every $x \in X$, we can find $u \in H^0(X, mL)$ such that $u(x) \neq 0$ (we say that mL is base point free). The proof is very similar to the Levi problem. In fact, let z be the local holomorphic coordinate system centered at x such that $\{|z| < 3\}$ is a well defined open set in X and on which L has a holomorphic frame, say σ. Let $e^{-\phi}$ be the positively curved metric on L. Let $e^{-\psi}$ be a smooth metric on K_X. Since X is compact and $i\partial\overline{\partial}\phi$ is positive on X, we can choose a sufficiently large m such that

$$i\partial\overline{\partial}(m\phi - \psi + n\lambda(|z|)\log(|z|^2+\varepsilon)) > \omega := i\partial\overline{\partial}\phi$$

for every ε (the same m for all $\varepsilon \leq 1$ and all x, think why!). Consider

$$c = \overline{\partial}\lambda(|z|) \wedge dz^1 \wedge \cdots dz^n \otimes (dz^1 \wedge \cdots dz^n)^{-1} \otimes \sigma^m,$$

Same as before, solving $\overline{\partial}$ and let ε goes to zero, we get u such that $\overline{\partial}a = c$ and $a(x) = 0$. Thus

$$u := \lambda(|z|)dz^1 \wedge \cdots dz^n \otimes (dz^1 \wedge \cdots dz^n)^{-1} \otimes \sigma^m - a$$

fits our needs. Similar argument implies that when m is sufficiently large Φ_m is injective with injective differential. Thus Φ_m defines a holomorphic embedding when m is sufficiently large. The proof is complete. □

Exercise 5: Add the details in the above proof.

7. Degree (n,n)-case

Recall that Siu's $\partial\overline{\partial}$–Bochner trick is to compute $i\partial\overline{\partial}T$, where

$$T := c_{n-q}e^{-\phi}\gamma_u \wedge \overline{\gamma_u} \wedge \omega_{q-1}$$

is an $(n-1, n-1)$-form associated to an L-valued smooth (n,q)-form u with compact support, $q \geq 1$. The main formula is: assume that $d\omega = 0$ then

$$i\partial\overline{\partial}T = \left(-2\mathrm{Re}\langle\overline{\partial}\overline{\partial}^* u, u\rangle + |\overline{\partial}^* u|^2 - |\overline{\partial}u|^2 + |\overline{\partial}\gamma_u|^2\right) e^{-\phi}\omega_n + i\partial\overline{\partial}\phi \wedge T.$$

In case $q = n$ we have

$$T = e^{-\phi}|\gamma_u|^2\omega_{n-1}$$

and $\overline{\partial}u = 0$, thus

$$i\partial\overline{\partial}T = \left(-2\mathrm{Re}\langle\overline{\partial}\overline{\partial}^* u, u\rangle + |\overline{\partial}^* u|^2 + |\overline{\partial}\gamma_u|^2\right) e^{-\phi}\omega_n + i\partial\overline{\partial}\phi \wedge T$$

Exercise 6: Check that the above formula is also true in case $d\omega^{n-1} = 0$.

Integrate the above formula, we get

$$||\overline{\partial}^* u||^2 = ||\overline{\partial}\gamma_u||^2 + \int_X i\partial\overline{\partial}\phi \wedge T.$$

Notice that

$$i\partial\overline{\partial}\phi \wedge T = e^{-\phi}|\gamma_u|^2\omega_{n-1} \wedge i\partial\overline{\partial}\phi.$$

Exercise 7: Assume that the eigenvalue of $i\partial\overline{\partial}\phi$ with respect to ω is $\lambda_1 \leq \cdots \leq \lambda_n$. Then

$$\omega_{n-1} \wedge i\partial\overline{\partial}\phi = (\lambda_1 + \cdots + \lambda_n)\omega_n.$$

Definition 7.1. *We call $\lambda_1 + \cdots + \lambda_n$ the trace of $i\partial\overline{\partial}\phi$ with respect to ω, and denote it by $\mathrm{Tr}_\omega(i\partial\overline{\partial}\phi)$:*

$$\omega_{n-1} \wedge i\partial\overline{\partial}\phi = \mathrm{Tr}_\omega(i\partial\overline{\partial}\phi)\, \omega_n.$$

In case $L = -K_X$, and ϕ is defined by

$$e^{-\phi(z)} i^{n^2} dz \wedge \overline{dz} = \omega_n.$$

Then we call $i\partial\overline{\partial}\phi$ the Ricci form associated to ω and $\mathrm{Tr}_\omega(i\partial\overline{\partial}\phi)$ the Scalar curvature of ω.

Exercise 8: Prove the following theorem.

Theorem 7.1 ($\overline{\partial}$-L^2-estimate for (n,n)-forms)**.** *Let $(L, e^{-\phi})$ be a holomorphic line bundle on an n-dimensional Kähler manifold (X, ω). Assume that $\mathrm{Tr}_\omega(i\Theta) > 0$. Then for every L-valued (n,n)-form c on X with*

$$I(c) := \int_X \mathrm{Tr}_\omega(i\Theta)^{-1} |c|^2_\omega e^{-\phi} \omega_n < \infty,$$

we can find an L-valued $(n, n-1)$-form a on X such that $\overline{\partial} a = c$ on X and $||a||^2 \leq I(c)$.

Remark 1: The above theorem is also true for non-Kähler manifolds with d-closed ω^{n-1}.

Remark 2: In case $\mathrm{Tr}_\omega(i\Theta) \geq 0$, just replace $I(c) < \infty$ by

$$I(c) := \lim_{\varepsilon \to 0} \int_X \mathrm{Tr}_\omega(i\Theta + \varepsilon\omega)^{-1} |c|^2_\omega e^{-\phi} \omega_n < \infty$$

then the theorem still holds (try!).

Remark 3: By the above theorem, if the scalar curvature $S(\omega) > 0$ on X (X is compact Kähler) then

$$H^{n,n}(X, -K_X) = H^{0,0}(X, K_X) = \{0\}.$$

If $S(\omega) < 0$ then

$$H^{n,n}(X, K_X) = H^{0,0}(X, -K_X) = \{0\}.$$

7.1. **Approximation theorem.**

Theorem 7.2. *Let $(L, e^{-\phi})$ be a holomorphic line bundle on an n-dimensional weakly pseudo-convex Kähler manifold (X, ω, ρ). Assume that $i\Theta = i\partial\overline{\partial}\phi \geq 0$. Let u be a $\overline{\partial}$-closed L-valued L^2 $(n, n-1)$-form on X_0. Then there exists a family of L-valued $(n, n-1)$-forms $\{u_\varepsilon\}$, each u_ε is $\overline{\partial}$-closed on X and*

$$\lim_{\varepsilon \to 0} ||u_\varepsilon - u||_{X_0} = 0.$$

Proof. The proof is very similar to degree $(n, 0)$ case. With the same notation, we need to estimate

$$\mathrm{Tr}_\omega(\delta\omega + i\partial\overline{\partial}(\phi + \psi_\varepsilon))^{-1} \cdot |\overline{\partial}\psi_\varepsilon \wedge a_\varepsilon|^2_\omega.$$

What we have is

$$i\partial\psi_\varepsilon \wedge \overline{\partial}\psi_\varepsilon \leq i\partial\overline{\partial}\psi_\varepsilon,$$

which implies that

$$|\overline{\partial}\psi_\varepsilon|^2_\omega \omega_n \leq \omega_{n-1} \wedge i\partial\overline{\partial}\psi_\varepsilon.$$

By the definition of Tr_ω, we have

$$\mathrm{Tr}_\omega(\delta\omega + i\partial\overline{\partial}(\phi+\psi_\varepsilon))^{-1}\omega_{n-1} \wedge (\delta\omega + i\partial\overline{\partial}(\phi+\psi_\varepsilon)) = \omega_n,$$

which gives

$$\mathrm{Tr}_\omega(\delta\omega + i\partial\overline{\partial}(\phi+\psi_\varepsilon))^{-1}\omega_{n-1} \wedge i\partial\overline{\partial}\psi_\varepsilon \leq \omega_n.$$

Thus we have

$$\mathrm{Tr}_\omega(\delta\omega + i\partial\overline{\partial}(\phi+\psi_\varepsilon))^{-1}|\overline{\partial}\psi_\varepsilon|^2_\omega \omega_n \leq \omega_n,$$

which gives

$$\mathrm{Tr}_\omega(\delta\omega + i\partial\overline{\partial}(\phi+\psi_\varepsilon))^{-1} \cdot |\overline{\partial}\psi_\varepsilon \wedge a_\varepsilon|^2_\omega \leq |a_\varepsilon|^2_\omega.$$

□

Exercise 9: Add the details in the above proof.

8. General (n,q)-version

8.1. **Approximation theorem.** Our main aim is to prove the (n,q)-version of the approximation theorem. We need the following linear algebra lemma (generalization of our estimate in degree $(n,1)$ and (n,n) cases).

Lemma 8.1. *Fix $x \in X$, the following pointwise inner product (with respect to ω) satisfies*

$$\lim_{\varepsilon\to 0}((B+\varepsilon)^{-1}(b\wedge a), b\wedge a)(x) \leq (a,a)(x), \tag{8.1}$$

where b is of degree-$(0,1)$, a is of degree $(n,q-1)$, $B=[i\Theta,\Lambda_\omega]$ and we assume that

$$i\Theta(x) \geq i\,\overline{b(x)} \wedge b(x). \tag{8.2}$$

Proof. Assume that $\{e_j\}$ is an orthonormal basis of $\wedge^{1,0}T^*_x X$ with respect to $\omega(x)$ and $\overline{b(x)} = e_1$. It is enough to prove the case (try!) when

$$\Theta(x) = e_1 \wedge \overline{e_1}.$$

Assume that

$$a(x) = \overline{e_1} \wedge a_1 + a_2,$$

where a_1, a_2 contain no $\overline{e_1}$ terms, then

$$(B+\varepsilon)^{-1}(b\wedge a) = (1+\varepsilon)^{-1}\overline{e_1}\wedge a_2,$$

which gives

$$\lim_{\varepsilon\to 0}((B+\varepsilon\omega)^{-1}(b\wedge a), b\wedge a)(x) = (a_2,a_2)(x) \leq (a,a)(x).$$

Thus the lemma follows. □

Exercise 10: Use the above lemma and the proof of Theorem 6.2 to prove the following two theorems.

Theorem 8.2. *Let $(L, e^{-\phi})$ be a holomorphic line bundle on an n-dimensional quasi-complete Kähler manifold (X, ω). Assume that*

$$i\Theta = i\partial\overline{\partial}\phi \geq i\,\bar{b} \wedge b$$

for a smooth $(0,1)$-form b on X. Then for every $\overline{\partial}$-closed L-valued (n,q)-form $(q \geq 1)$ c such that $c = b \wedge a$ on X and $||a|| < \infty$, we can find an L-valued $(n, q-1)$-form u on X such that

$$\overline{\partial} u = c$$

on X and $||u|| \leq ||a||$.

Theorem 8.3. *Let $(L, e^{-\phi})$ be a holomorphic line bundle on an n-dimensional weakly pseudoconvex Kähler manifold (X, ω, ρ). Assume that $i\Theta = i\partial\overline{\partial}\phi \geq 0$. Let u be a $\overline{\partial}$-closed L-valued L^2 (n,q)-form on X_0. Then there exists a family of L-valued (n,q)-forms $\{u_\varepsilon\}$, each u_ε is $\overline{\partial}$-closed on X and $\lim_{\varepsilon \to 0} ||u_\varepsilon - u||_{X_0} = 0$.*

8.2. **Hörmander theory, general degree (n,q) case.** The proof of Lemma 8.1 also gives:

Lemma 8.4. *Let (X, ω) be a Kähler manifold. Fix $m \in \mathbb{N}$. Let b_j (resp. a_j) be $(1,0)$ (resp. $(n, q-1)$) forms on X, $1 \leq j \leq m$. Let $i\Theta$ be a semi-positive $(1,1)$ form on X. Assume that*

$$\sum_{j=1}^{m} A_{j\bar{k}}(x)\, \xi_j \bar{\xi}_k \cdot i\Theta(x) \geq i\, \overline{b_\xi} \wedge b_\xi, \quad b_\xi := \sum_{j=1}^{m} b_j(x)\xi_j, \;\; \forall\, \xi \in \mathbb{C}^m, \;\; x \in X, \tag{8.3}$$

where each $(A_{j\bar{k}}(x))$ is a semi-positive definite Hermitian matrix. Put $B = [i\Theta, \Lambda_\omega]$, then

$$\lim_{\varepsilon \to 0} \left((B+\varepsilon)^{-1} \sum_{j=1}^{m} b_j \wedge a_j, \sum_{j=1}^{m} b_j \wedge a_j \right)(x) \leq \sum_{j,k=1}^{m} A_{j\bar{k}}(x)(a_j, a_k)(x), \tag{8.4}$$

for every $x \in X$.

Proof. By a linear transform, it suffices to prove the case that $(A_{j\bar{k}}(x))$ is the identity matrix. For simplicity's sake, we will only prove the case that $i\Theta = \omega$, $\{b_j(x)\} = \{e_j\}_{1 \leq j \leq n}$ is an orthonormal basis of $\wedge^{0,1} T_x^* X$ (the general case follows by a similar argument). Then (8.4) is equivalent to that

$$\Big| \sum_{j=1}^{n} e_j \wedge a_j \Big|^2 \leq q \cdot \sum_{j=1}^{n} |a_j|^2. \tag{8.5}$$

Using orthogonal decomposition, one may assume that each a_j does not contain the e_j term, then (8.5) follows from Hilbert space inequality (due to the convexity of $|| \cdot ||$)

$$\Big| \frac{v_1 + \cdots + v_q}{q} \Big|^2 \leq \frac{|v_1|^2 + \cdots + |v_q|^2}{q}.$$

□

Remark: One may also generalize the above lemma to the "q-semipositivity" case (see the proof of Lemma 3.10 in [17]). Similar as before, the above lemma gives the following generalization of Theorem 8.2.

Theorem 8.5. *Let $(L, e^{-\phi})$ be a holomorphic line bundle on an n-dimensional quasi-complete Kähler manifold (X, ω). Let*

$$c = \sum_{j=1}^{m} b_j \wedge a_j$$

be a $\overline{\partial}$-closed L-valued (n, q)-form $(q \geq 1)$, where $b_1, \cdots, b_m$ are smooth $(0,1)$-forms on X. Assume that (8.3) holds for $\Theta = \partial\overline{\partial}\phi$. Then we can find an L-valued $(n, q-1)$-form u on X such that

$$\overline{\partial} u = c$$

on X and

$$||u||^2 \leq \sum_{j,k=1}^{m} (A_{j\bar{k}} a_j, a_k)$$

provided the right hand side is finite.

Remark: The above theorem is inspired by Berndtsson's Nakano positivity of the direct image bundle [2], in which case,

$$c = \sum_{j=1}^{m} \overline{\partial}\psi_{t_j} \wedge a_j,$$

where ψ is a smooth function on $B \times X$ (B denotes the unit ball in $\mathbb{C}^m$) such that

$$\psi(t, x) + \phi(x)$$

is (locally) psh as a function of (t, x). Then we know that for each $t \in B$, (8.3) holds for $b_j = \overline{\partial}\psi_{t_j}$, $\Theta = \partial_z\overline{\partial}_z(\psi + \phi)$ and

$$A_{j\bar{k}} = \psi_{t_j \bar{t}_k}.$$

Thus the above theorem implies that if u is the L^2 minimal solution of $\overline{\partial} u = c$ then

$$||u||^2 \leq \sum_{j,k=1}^{m} (\psi_{t_j \bar{t}_k} a_j, a_k).$$

9. Newlander–Nirenberg theorem

In this short note we shall recall the classical Newlander-Nirenberg theorem and its vector bundle version. We shall also recall an L^2-Hörmander-proof given by Demailly.

9.1. Classical Newlander-Nirenberg theorem.

Let M be a smooth manifold. Let us recall the following definition:

Definition 9.1. *We call a smooth bundle mapping J from TM to itself an almost complex structure on M if $J^2 = -1$.*

Now similar as before, we have the following decompositon

$$\wedge^k (T_x^*M \otimes_{\mathbb{R}} \mathbb{C}) = \oplus_{p+q=k}(\wedge^p T_x^* M_{\mathbb{C}} \ \wedge \ \wedge^q \overline{T_x^* M_{\mathbb{C}}}), \ \forall x \in M, \ 1 \leq k \leq \dim_{\mathbb{R}} M. \tag{9.1}$$

Thus every complex valued k-form can be written as the sum of (p,q)-forms. Now let u be a smooth (p,q)-form on M, locally one may write (not unique)

$$u = \sum u^{j_1} \wedge \cdots \wedge u^{j_p} \wedge \bar{u}^{k_1} \wedge \cdots \wedge \bar{u}^{k_q}, \tag{9.2}$$

where each u^{j_t} is a section of $T^*M_{\mathbb{C}}$ and each $\bar{u}^{k_s}$ is a section of $\overline{T^*M_{\mathbb{C}}}$. By the Leibniz rule, du can be written as

$$du^{j_1} \wedge \cdots \wedge u^{j_p} \wedge \bar{u}^{k_1} \wedge \cdots \wedge \bar{u}^{k_q} + \cdots + (-1)^{p+q-1} u^{j_1} \wedge \cdots \wedge u^{j_p} \wedge \bar{u}^{k_1} \wedge \cdots \wedge d\bar{u}^{k_q}. \tag{9.3}$$

Thus we have

$$du = du^{p+1,q} + du^{p,q+1} + du^{p-1,q+2} + du^{p+2,q-1}, \tag{9.4}$$

where each $du^{t,s}$ denotes the (t,s)-part of du.

Definition 9.2. $\overline{\partial} u := du^{p,q+1}$, $\partial u := du^{p+1,q}$. *We say that J is integrable if $d = \partial + \overline{\partial}$.*

Proposition 9.1. *J is integrable if and only if $\overline{\partial}^2 = 0$.*

Proof. It suffices to show that $\overline{\partial}^2 = 0$ implies that J is integrable. By definition, we have $\partial = \overline{\overline{\partial}}$. Thus $\overline{\partial}^2 = 0$ is equivalent to $\partial^2 = 0$. By (9.3), it suffice to show that if u is an $(1,0)$ or $(0,1)$ form, then $du = \overline{\partial} u + \partial u$. For example, if u is an $(1,0)$-form, then one may write $u = \sum a_j \partial b_j$. Thus

$$du = \sum da_j \wedge \partial b_j + a_j d\partial b_j.$$

Now it suffices to show that each $d\partial b_j$ has no $(0,2)$-part. Since each b_j is a function, we have $db_j = \partial b_j + \overline{\partial} b_j$. Thus

$$d\partial b_j = d(d - \overline{\partial}) b_j = -d\overline{\partial} b_j,$$

and the $(0,2)$-part of $d\partial b_j$ is just $-\overline{\partial}^2 b_j$, which vanishes by our assumption. Same argument works for $(0,1)$-form. □

We shall present Hörmander's L^2-proof of the following theorem of Newlander-Nirenberg:

Theorem 9.2 (Newlander-Nirenberg)**.** *If J is integrable then for every $x \in M$ there exist n smooth complex valued functions, say $z_1, \cdots, z_n$, near x such that $\overline{\partial} z_j = 0$ near x for each j and $\{\partial z_j(x)\}_{1 \leq j \leq n}$ defines a basis of $T_x^* M_{\mathbb{C}}$.*

Let us show how to use this theorem first. Since $\{\partial z_j(x)\}_{1 \leq j \leq n}$ defines a basis of $T_x^* M_{\mathbb{C}}$, we know that $\{\overline{\partial} \bar{z}_j(x)\}_{1 \leq j \leq n}$ defines a basis of $\overline{T_x^* M_{\mathbb{C}}}$. Since $\overline{\partial} z_j = 0$ near x, we know that $\partial \bar{z}_j = 0$ near x also. Thus $\{d\mathrm{Im} z_j, d\mathrm{Re} z_j\}_{1 \leq j \leq n}$ defines a $\mathbb{R}$-basis of $T_x^* M$. Now we can use $z := \{\mathrm{Re} z_j, \mathrm{Im} z_j\}_{1 \leq j \leq n}$ to define a coordinate covering of M. Let $z : U \to \mathbb{R}^{2n}$ and $w : V \to \mathbb{R}^{2n}$ be two such coordinate charts. Assume that $U \cap V$ is a non-empty open subset of M. Then $w \circ z^{-1}$ is a smooth mapping from $z(U \cap V)$ to $w(U \cap V)$. Now let us look at z and w as complex valued functions, then both $z(U \cap V)$ and $w(U \cap V)$ can be seen as open subsets of $\mathbb{C}^n$. We shall show that $w \circ z^{-1}$ is holomorphic.

Definition 9.3. *Let U be an open subset in $\mathbb{C}^n$. Let f be a smooth mapping from U to $\mathbb{C}^m$. f is said to be holomorphic on U if*

$$\partial f^k/\partial \bar{z}_j \equiv 0, \text{ on } U, \ \forall\, 1 \leq k \leq m, \ \forall\, 1 \leq j \leq n. \tag{9.5}$$

Now let us prove $f := w \circ z^{-1}$ is holomorphic on $z(U \cap V)$ in the sense of the above definition. Since $w_k = f^k(z)$ and $\{\mathrm{Re}z_j, \mathrm{Im}z_j\}_{1 \leq j \leq n}$ defines a smooth local coordinate chart on M, we have

$$dw_k = \sum \partial f^k/\partial \mathrm{Re}z_j \; d\mathrm{Re}z_j + \sum \partial f^k/\partial \mathrm{Im}z_j \; d\mathrm{Im}z_j. \tag{9.6}$$

Since $\overline{\partial} w_k = 0$, thus by definition of $\overline{\partial} w_k$ above, we have

$$0 = \sum \partial f^k/\partial \mathrm{Re}z_j \; \overline{\partial}\mathrm{Re}z_j + \sum \partial f^k/\partial \mathrm{Im}z_j \; \overline{\partial}\mathrm{Im}z_j. \tag{9.7}$$

Since $\overline{\partial} z_j = 0$ and

$$\mathrm{Re}z_j = \frac{z_j + \bar{z}_j}{2}, \ \ \mathrm{Im}z_j = \frac{z_j - \bar{z}_j}{2i}, \tag{9.8}$$

we have

$$\overline{\partial}\mathrm{Re}z_j = \frac{1}{2}\overline{\partial}\bar{z}_j; \ \ \overline{\partial}\mathrm{Im}z_j = \frac{-1}{2i}\overline{\partial}\bar{z}_j. \tag{9.9}$$

Thus we have

$$0 = \sum \left(\partial f^k/\partial \mathrm{Re}z_j - \frac{1}{i} \sum \partial f^k/\partial \mathrm{Im}z_j \right) \overline{\partial}\bar{z}_j. \tag{9.10}$$

Since $\{\overline{\partial}\bar{z}_j\}_{1 \leq j \leq n}$ are linearly independent, we have

$$\partial f^k/\partial \bar{z}_j = \frac{1}{2} \left(\partial f^k/\partial \mathrm{Re}z_j + i \sum \partial f^k/\partial \mathrm{Im}z_j \right) \equiv 0, \tag{9.11}$$

on $z(U \cap V)$. Thus $f = w \circ z^{-1}$ is holomorphic on $z(U \cap V)$.

The Newlander-Nirenberg theorem tells us that a smooth manifold with an integrable almost complex structure is in fact a complex manifold. Now let us recall the Hörmander proof [13] of the Newlander-Nirenberg theorem. The basic idea of Hörmander is: by using a precise L^2-estimate, a "good" J-plurisubharmonic functions gives J-holomorphic sections. Moreover, if J is integrable then locally there exist "good" J-plurisubharmonic functions.

Step 1: Construct "good" J-plurisubharmonic functions (see [9]).

Let us fix a point x in M. Let us take $2n$ smooth functions, say $x_1, \cdots, x_{2n}$, near x such that

$$x_1(x) = \cdots = x_{2n}(x) = 0,$$

and $\{dx_1(x), \cdots, dx_{2n}(x)\}$ defines a base of T_x^*M. Thus there exists a small open neighborhood, say U_x, of x such that

$$(x_1, \cdots, x_{2n}) : \ U_x \to \mathbb{R}^{2n},$$

defines a smooth coordinate chart on U_x. Now let us fix a base, say $\{\sigma_1, \cdots, \sigma_n\}$, of $T_x^*M_{\mathbb{C}}$. By definition, we have

$$\sigma_j = \sum a_j^k dx_k(x), \ \ a_j^k \in \mathbb{C}, \ 1 \leq j \leq n, \ 1 \leq k \leq 2n. \tag{9.12}$$

Since each σ_j has no $(0,1)$-part, we know that

$$\sum a_j^k \overline{\partial} x_k(x) = 0. \tag{9.13}$$

Put

$$f_j = \sum a_j^k x_k, \ \ 1 \leq j \leq n, \tag{9.14}$$

then f_j are smooth complex valued functions on U_x and $\overline{\partial} f_j$ are zero at x, i.e.

$$\overline{\partial} f_j(x) = 0. \tag{9.15}$$

By choosing sufficiently small U_x, one may assume that $\{\mathrm{Re} f_j, \mathrm{Im} f_j\}_{1\leq j\leq n}$ defines a coordinate chart on U_x. Put

$$\psi = |f|^2 := \sum f_j \bar{f}_j, \tag{9.16}$$

then ψ is a smooth function on U_x. Thus choose $\delta > 0$ sufficienely small, we know that

$$\Omega := \{\psi < \delta\} \tag{9.17}$$

is a relatively compact open subset of U_x and the gradient of ψ has no zero point on the boundary, $\{\psi = 1\}$, of Ω. Put

$$\omega = i\partial\overline{\partial}\psi, \ \hat{\omega} = i\partial\overline{\partial}\left(-\log(\delta - \psi)\right). \tag{9.18}$$

If J is integrable then both ω and $\hat{\omega}$ are real d-closed $(1,1)$-forms on Ω. Since

$$f_j(x) = \overline{\partial} f_j(x) = 0, \tag{9.19}$$

we have

$$i\partial\overline{\partial}\psi(x) = \sum i\partial f_j(x) \wedge \overline{\partial} \bar{f}_j(x) = i \sum \sigma_j \wedge \bar{\sigma}_j > 0. \tag{9.20}$$

Thus if δ is small enough then $i\partial\overline{\partial}\psi > 0$ on Ω. We call ω a J-Kähler form on Ω. In order to constuct a singular J-plurisubharmonic function, we have to use the following lemma of Demailly [9]:

Lemma 9.3 (Demailly)**.** *If J is integrable then there exist n smooth complex valued functions, say $g_1, \cdots, g_n$, on a neighborhood of x such that*

$$g_j(x) = 0, \ \partial g_j(x) = \sigma_j, \ \overline{\partial} g_j = O(\psi),$$

for every $1 \leq j \leq n$.

Proof. Since $\{\overline{\partial} \bar{f}_k\}_{1\leq k\leq n}$ defines a base of $(0,1)$-form on U_x and $\overline{\partial} f_j(x) = 0$, we have

$$\overline{\partial} f_j = \sum p_j^{kl} f_l \overline{\partial} \bar{f}_k + \sum q_j^{kl} \bar{f}_l \overline{\partial} \bar{f}_k, \tag{9.21}$$

where p_j^{kl}, q_j^{kl} are smooth function on U_x. Since J is integrable, we know that

$$0 = \overline{\partial}^2 f_j(x) = \sum q_j^{kl}(x) \overline{\partial} \bar{f}_l(x) \wedge \overline{\partial} \bar{f}_k(x), \tag{9.22}$$

which implies that

$$q_j^{kl}(x) = q_j^{lk}(x). \tag{9.23}$$

Let us consider

$$g_j = f_j + \sum a^{kl} \bar{f}_k f_l + \sum b^{kl} \bar{f}_k \bar{f}_l, \tag{9.24}$$

where a^{kl}, b^{kl} are complex constants and $b^{kl} = b^{lk}$. Thus

$$\overline{\partial} g_j = \overline{\partial} f_j + \sum a^{kl} f_l \overline{\partial} \bar{f}_k + \sum a^{kl} \bar{f}_k \overline{\partial} f_l + 2 \sum b^{kl} \bar{f}_l \overline{\partial} \bar{f}_k. \tag{9.25}$$

It suffices to choose a^{kl}, b^{kl} with $b^{kl} = b^{lk}$ such that

$$2b^{kl} + q_j^{kl}(x) = 0, \;\; a^{kl} + p_j^{kl}(x) = 0. \tag{9.26}$$

the existence of a^{kl}, b^{kl} follows from (9.23). □

By Demailly's lemma, one may assume that (consider g_j instead of f_j)

$$\overline{\partial} f_j = O(\psi) \tag{9.27}$$

Then we have

$$\partial \psi = \sum \bar{f}_j \partial f_j + O(\psi^{3/2}), \tag{9.28}$$

thus

$$i\partial\psi \wedge \overline{\partial}\psi \leq \psi \sum i\partial f_j \wedge \overline{\partial} \bar{f}_j + O(\psi^2). \tag{9.29}$$

Moreover, $\overline{\partial} f_j = O(\psi)$ implies that

$$i\partial\overline{\partial} f_j = O(\psi^{1/2}), \tag{9.30}$$

thus

$$i\partial\overline{\partial}\psi = \sum i\partial f_j \wedge \overline{\partial} \bar{f}_j + O(\psi). \tag{9.31}$$

Hence

$$i\partial\psi \wedge \overline{\partial}\psi \leq \psi i\partial\overline{\partial}\psi + O(\psi^2). \tag{9.32}$$

Thus (choose a smaller δ if necessary) there exists a sufficiently big positive number N such that

$$i\partial\psi \wedge \overline{\partial}\psi \leq (\psi + N\psi^2) i\partial\overline{\partial}\psi, \tag{9.33}$$

on Ω, which implies that

$$i\partial\overline{\partial} \log(\psi + \varepsilon) \geq (\varepsilon - N\psi^2) \frac{i\partial\overline{\partial}\psi}{(\psi+\varepsilon)^2} \geq -N i\partial\overline{\partial}\psi, \tag{9.34}$$

for every $\varepsilon > 0$.

Step 2: $\partial\overline{\partial}$-Bochner formula for integrable J.

Let γ be an arbitrary smooth $(n-q, 0)$-form on Ω and let ϕ be a real valued smooth function on Ω. Put

$$\omega_q = \omega^q / q!, \;\; u = \gamma \wedge \omega_q,$$

and

$$T = i^{(n-q)^2} \gamma \wedge \bar{\gamma} \wedge e^{-\phi} \wedge \omega_{q-1}.$$

Let us define

$$\overline{\partial}^* = - * \partial_\phi *,$$

where

$$\partial_\phi := \partial - \partial\phi \wedge .$$

Since J is integrable, we have

$$\partial_\phi^2 = \overline{\partial}^2 = 0.$$

Thus $i\partial\overline{\partial}T$ can be written as

$$-2\mathrm{Re}\langle \partial\overline{\partial}^* u, u\rangle e^{-\phi}\omega_n + |\overline{\partial}^* u|^2 e^{-\phi}\omega_n + i\partial\overline{\partial}\phi \wedge T - S, \tag{9.35}$$

where

$$S = i^{(n-q+1)^2}\overline{\partial}\gamma \wedge \overline{\overline{\partial}\gamma} \wedge e^{-\phi} \wedge \omega_{q-1}.$$

The following lemma follows from the primitive decomposition of $\overline{\partial}\gamma$ (see Lemma 4.2 in Berndtsson's lecture notes [3] or section 8 of the notes).

Lemma 9.4. $S = (|\overline{\partial}u|^2 - |\partial_\phi^* u|^2)e^{-\phi}\omega_n$.

Thus we have the following $\partial\overline{\partial}$-Bochner formula for (n, q)-forms:

$$i\partial\overline{\partial}T = \left(-2\mathrm{Re}\langle \overline{\partial\overline{\partial}^*} u, u\rangle + |\overline{\partial}^* u|^2 - |\overline{\partial}u|^2 + |\partial_\phi^* u|^2\right) e^{-\phi}\omega_n + i\partial\overline{\partial}\phi \wedge T. \tag{9.36}$$

Step 3: Hörmander's L^2-estimate.

In order to construct $\overline{\partial}$-closed functions that we need, we have to solve the $\overline{\partial}$-equation for $(0, 1)$-form. But we only have $\partial\overline{\partial}$-Bochner formula for (n, q)-forms, thus it is necessary to solve $\overline{\partial}$-equation for $(n, 1)$-form and construst a "good" $\overline{\partial}$-closed holomorphic $(n, 0)$-form, then we can use it to solve the $\overline{\partial}$-equation for $(0, 1)$-form.

We shall use Chen's method [8] to solve the $\overline{\partial}$-equation. More precisely, put

$$v = \overline{\partial}(\partial f_1 \wedge \cdots \wedge \partial f_n), \tag{9.37}$$

and

$$\psi_\varepsilon = nN\psi + n\log(\psi + \varepsilon) - \log(\delta - \psi), \tag{9.38}$$

By (9.30), (9.18) and (9.34), we have

$$i\partial\overline{\partial}\psi_\varepsilon \geq \hat{\omega}, \tag{9.39}$$

and

$$\int_\Omega |v|^2_{\hat{\omega}} e^{-\psi_\varepsilon}\hat{\omega}_n \leq \int_\Omega |v|^2_{\hat{\omega}} e^{-\psi_0}\hat{\omega}_n := I(v) < \infty, \ \forall\, \varepsilon > 0. \tag{9.40}$$

Put

$$Q(\alpha, \beta) := \int_\Omega \langle \overline{\partial}^*\alpha, \overline{\partial}^*\beta\rangle_{\hat{\omega}} e^{-\psi_\varepsilon}\hat{\omega}_n + \int_\Omega \langle \overline{\partial}\alpha, \overline{\partial}\beta\rangle_{\hat{\omega}} e^{-\psi_\varepsilon}\hat{\omega}_n. \tag{9.41}$$

By (9.36), we have

$$Q(\alpha,\alpha)\geq\int_{\Omega}|\alpha|^2_{\hat{\omega}}e^{-\psi_\varepsilon}\hat{\omega}_n, \tag{9.42}$$

for every smooth $(n,1)$-form α with compact support in Ω. Denote by H the completion under Hermitian form Q of the space of smooth $(n,1)$-forms with compact support in Ω. Thus

$$\alpha\to\int_{\Omega}\langle\alpha,v\rangle_{\hat{\omega}}e^{-\psi_\varepsilon}\hat{\omega}_n, \tag{9.43}$$

extends to a Q-bounded linear functional on H. By the Riesz representation theorem and (9.42), we know that there exists $a\in H$ such that

$$Q(\alpha,a)=\int_{\Omega}\langle\alpha,v\rangle_{\hat{\omega}}e^{-\psi_\varepsilon}\hat{\omega}_n,\ \forall\,\alpha\in H, \tag{9.44}$$

and

$$Q(a,a)\leq\int_{\Omega}|v|^2_{\hat{\omega}}e^{-\psi_\varepsilon}\hat{\omega}_n\leq I(v). \tag{9.45}$$

Notice that (9.44) implies that

$$\left(\overline{\partial}\overline{\partial}^*+\overline{\partial}^*\overline{\partial}\right)a=v, \tag{9.46}$$

in the sense of current. Since $\overline{\partial}\overline{\partial}^*+\overline{\partial}^*\overline{\partial}$ is elliptic and v is smooth, we know that there exists a smooth representative, say $\mathbf{a}$, of the current a such that

$$\left(\overline{\partial}\overline{\partial}^*+\overline{\partial}^*\overline{\partial}\right)\mathbf{a}=v, \tag{9.47}$$

on D (in fact, by using the Fourier transform, we get a Gårding inequality for elliptic operator with constant coefficients, then one can get the Gårding inequality for general elliptic operator by comparing with the case of constant coefficients. In our case, by (9.45) and convolution of a with a smooth function, say the Gaussian kernel, and using the Arzel$\grave{a}$- Ascoli theorem, we get that the current solution a locally has a smooth representative. By using the partition of unity, we finally get a global smooth representative, say $\mathbf{a}$, of a) and

$$Q(\mathbf{a},\mathbf{a})\leq I(v). \tag{9.48}$$

Since J is integrable, (9.47) implies that

$$\overline{\partial}\overline{\partial}^*\overline{\partial}\mathbf{a}\equiv 0 \tag{9.49}$$

on Ω. Let χ be a smooth function on $\mathbb{R}$ such that $\chi\equiv 1$ on $(-\infty,1/2)$, $\chi\equiv 0$ on $(1,\infty)$ and $|\chi'|\leq 3$ on $\mathbb{R}$. Then

$$\chi_t:=\chi(t\log\frac{1}{\delta-\psi})\in C_0^\infty(\Omega,\mathbb{R}),\ \forall\,t>0. \tag{9.50}$$

Moreover,

$$|\overline{\partial}(\chi_t)|_{\hat{\omega}}\leq 3t, \tag{9.51}$$

on Ω. Since

$$(\chi_t^2\overline{\partial}\overline{\partial}^*\overline{\partial}\mathbf{a}, \overline{\partial}\mathbf{a}) = ||\chi_t\overline{\partial}^*\overline{\partial}\mathbf{a}||^2 - (2\chi_t\overline{\partial}\chi_t \wedge \overline{\partial}^*\overline{\partial}\mathbf{a}, \overline{\partial}\mathbf{a}), \tag{9.52}$$

by (9.49) and (9.51), we have

$$||\chi_t\overline{\partial}^*\overline{\partial}\mathbf{a}||^2 \leq 6t||\chi_t\overline{\partial}^*\overline{\partial}\mathbf{a}||. \tag{9.53}$$

Let t goes to zero, we know that

$$\overline{\partial}^*\overline{\partial}\mathbf{a} \equiv 0, \tag{9.54}$$

on Ω. Thus we have

$$\overline{\partial}\overline{\partial}^*\mathbf{a} = v, \tag{9.55}$$

on Ω. Put

$$u_\varepsilon = \overline{\partial}^*\mathbf{a}.$$

We know that $\overline{\partial}u_\varepsilon = v$ and

$$\int_\Omega |u_\varepsilon|^2_{\hat{\omega}} e^{-\psi_\varepsilon}\hat{\omega}_n \leq Q(\mathbf{a}, \mathbf{a}) \leq I(v). \tag{9.56}$$

Let ε go to zero, by taking the weak limit, we get an $(n, 0)$-current u such that $\overline{\partial}u = v$ in the sense of current and

$$\int_\Omega |u|^2_{\hat{\omega}} e^{-\psi_0}\hat{\omega}_n \leq I(v). \tag{9.57}$$

Again since $\overline{\partial}$ is elliptic on $(n, 0)$-forms, we know that the current u has a smooth representative $\mathbf{u}$ such that $\overline{\partial}\mathbf{u} = v$ and

$$\int_\Omega |\mathbf{u}|^2_{\hat{\omega}} e^{-\psi_0}\hat{\omega}_n \leq I(v). \tag{9.58}$$

Since $e^{-\psi_0}$ is not integrable near x, we know that

$$\mathbf{u}(x) = 0. \tag{9.59}$$

Put

$$\sigma = \partial f_1 \wedge \cdots \wedge \partial f_n - \mathbf{u}. \tag{9.60}$$

we know that

$$\overline{\partial}\sigma = 0, \ \ \sigma(x) \neq 0. \tag{9.61}$$

Put

$$\Omega_1 = \{\psi < \delta_1\}. \tag{9.62}$$

Choose $0 < \delta_1 < \delta$ small enough, one may assume that σ has no zero point in the closure of Ω_1. This time, put

$$v_j = \overline{\partial}(f_j\sigma), \ \ 1 \leq j \leq n, \tag{9.63}$$

and

$$\phi_\varepsilon = (n + 1)N\psi + (n + 1)\log(\psi + \varepsilon) - \log(\delta_1 - \psi). \tag{9.64}$$

Consider the complete J-Kähler form

$$\tilde{\omega} := i\partial\overline{\partial}\left(-\log(\delta_1 - \psi)\right). \tag{9.65}$$

Similar as before, for each j, we get a smooth $(n,0)$-form $\mathbf{u}_j$ on Ω_1 such that

$$\overline{\partial}\mathbf{u}_j = v_j, \;\; \mathbf{u}_j(x) = 0, \;\; d\mathbf{u}_j(x) = 0, \;\; 1 \leq j \leq n. \tag{9.66}$$

Put

$$\mathbf{u}_j = h_j \sigma, \;\; 1 \leq j \leq n. \tag{9.67}$$

Then

$$\overline{\partial}(f_j - h_j) = 0, \;\; h_j(x) = 0, \;\; dh_j(x) = 0, \;\; 1 \leq j \leq n. \tag{9.68}$$

We know that

$$z_j := f_j - h_j, \;\; 1 \leq j \leq n. \tag{9.69}$$

fit our needs. Thus the Newlander-Nirenberg theorem is proved.

9.2. **Vector bundle version of the Newlander-Nirenberg theorem.** In this section, we shall prove the following theorem:

Theorem 9.5. *Let E be a complex smooth vector bundle, say complex rank r, over a complex manifold X. Let D be a smooth connection on E. Denote by $D^{0,1}$ the $(0,1)$-part of D. If $(D^{0,1})^2 S \equiv 0$ on X for every smooth section S of E over X then for every $x \in X$ there exist r smooth sections, say $S_1, \cdots, S_r$, near x such that $D^{0,1}S_j = 0$ near x for each j and $\{S_j(x)\}_{1\leq j\leq r}$ defines a basis of E_x.*

Idea of the proof: Fix a Hermitian metric, say h, of E and define the "almost Chern connection", say D_h, on E with respect to $D^{0,1}$ and h, then use D_h to prove that the $\partial\overline{\partial}$-Bochner formula is true for smooth section of E. Finally, solve the $D^{0,1}$-equation with singular weight to get the sections that we need.

Step 1: Construct the almost Chern connection.

Let us fix a smooth Hermitian metric, say h, on E. They there is a natural sesquilinear product (see section 2), say $\{\cdot,\cdot\}$, on $\wedge^{\cdot,\cdot}T^*X \otimes E$ with respect to h. A connection D is said to be h-Hermitian if

$$d\{S,T\} = \{DS,T\} + (-1)^{\deg S}\{S,DT\}, \tag{9.70}$$

for every E-valued differential forms S,T. We shall prove that:

Lemma 9.6. *Let D be the connection in Theorem 9.5. Then there exists a unique h-Hermitian connection, say D_h, on E such that D_h is h-Hermitian and $D_h^{0,1} = D^{0,1}$. Moreover, $(D^{0,1})^2 = 0$ implies that $(D_h^{1,0})^2 = 0$.*

Proof. Let us define $D_h^{1,0}$ by requiring

$$\partial\{S,T\} = \{D_h^{1,0}S,T\} + (-1)^{\deg S}\{S, D^{0,1}T\}, \tag{9.71}$$

for every smooth E-valued differential forms S,T. Put

$$D_h = D^{0,1} + D_h^{1,0}.$$

Then one may verify that for every smooth differential form f on X, we have

$$D_h(f\wedge S) = df\wedge S + (-1)^{\deg f} f\wedge D_h S,$$

and

$$d\{S,T\} = \{D_hS,T\} + (-1)^{\deg S}\{S, D_hT\}. \tag{9.72}$$

Thus D_h fits our needs. Now it suffices to show that $(D_h^{1,0})^2 = 0$. Since $\partial^2 = 0$, by (9.71), if $(D^{0,1})^2 = 0$ then

$$0 = \partial\partial\{S,T\} = \{(D_h^{1,0})^2 S, T\}, \tag{9.73}$$

for every smooth E-valued differential forms S,T. Thus $(D_h^{1,0})^2 = 0$. □

Definition 9.4. *The curvature of the almost Chern connection D_h in the above lemma is defined to be $\Theta(E,h) := D_h^2$.*

Definition 9.5. *Let us write $D_h^{1,0} = \partial^E$ and $D^{0,1} = \overline{\partial}$.*

Step 2: $\partial\overline{\partial}$-Bochner formula for smooth sections of E.

Since a complex manifold is locallly Kähler. Let ω be a Kähler form on a pesudoconvex open neighborhood, say Ω, of x. Let γ be an arbitrary E-valued smooth $(n-q,0)$-form on Ω. Put

$$T = i^{(n-q)^2}\{\gamma,\gamma\}\wedge\omega_{q-1}, \;\; u = \gamma\wedge\omega_q. \tag{9.74}$$

By tha above lemma and *Step2* in the last section, still we have

$$i\partial\overline{\partial}T = \left(-2\mathrm{Re}\langle\overline{\partial}\overline{\partial}^* u, u\rangle + |\overline{\partial}^* u|^2 - |\overline{\partial}u|^2 + |(\partial^E)^* u|^2\right)\omega_n + i\Theta(E,h)\wedge T, \tag{9.75}$$

Step 3: Solve $D^{0,1}$-equation with "singular weight".

Let us choose a smooth basis, say $\{\sigma_1,\cdots,\sigma_r\}$ of E over a pesudoconvex open neighborhood, say Ω_1, of x such that

$$D_h\sigma_j(x) = 0, \; \forall\, 1\leq j\leq r. \tag{9.76}$$

Assume that Ω_1 is relatively compact in Ω and has global holomorphic coordinates, say $z_1,\cdots,z_n$, such that $z_j(x) = 0$ for each j. Put

$$\tau_j := D^{0,1}((dz_1\wedge\cdots\wedge dz_n)\otimes\sigma_j), \; \forall\, 1\leq j\leq r. \tag{9.77}$$

Similar as before, one may solve the $D^{0,1}$-equation with singular weight whose singular part is $n\log|z|^2$ and get r smooth E-valued $(n,0)$-form a_j, $1\leq j\leq r$, on Ω_1 such that

$$a_j(x) = 0, \;\; D^{0,1}a_j = \tau_j, \; \forall\, 1\leq j\leq r. \tag{9.78}$$

Put $a_j = (dz_1 \wedge \cdots \wedge dz_n) \otimes f_j$, then

$$S_j := \sigma_j - f_j,\ 1 \leq j \leq r. \tag{9.79}$$

are sections that we need.

Remark. *By Theorem 9.5, each $\{S_j\}_{1\leq j\leq r}$ gives a smooth local trivialization of E, and since each S_j lies in the kernel of $D^{0,1}$, we know that transition maps between these local trivializations are in fact holomorphic. Thus E has a holomorphic vector bundle structure.*

10. Berndtsson–Lempert approach to the Ohsawa–Takegoshi theorem

10.1. **L^2-extension of holomorphic sections from a smooth divisor.** Let $(L, e^{-\phi})$ be a holomorphic line bundle over an n-dimensional complex manifold X. Let $Y \subset X$ be a closed $(n-1)$-dimensional complex submanifold (i.e. a smooth divisor). Let u be a smooth section of $L + K_X$ over Y, we want to give a natural definition of the L^2-norm of u. The idea is to use the *polar function*.

Definition 10.1. *We call*

$$G := \log(|s|^2 e^{-\psi})$$

a Y-polar function, where s denotes the defining section of the line bundle $[Y]$, $e^{-\psi}$ is a smooth metric on $[Y]$. Sometimes we also write G as $G_{s,\psi}$. The associated G-norm of a smooth section u of $L + K_X$ over Y is defined as

$$||u||^2_G := \lim_{t\to-\infty} e^{-t}||\tilde{u}||^2_{G<t}, \quad ||\tilde{u}||^2_{G<t} := \int_{G<t} i^{n^2}\tilde{u}\wedge\overline{\tilde{u}}\, e^{-\phi},$$

where $\tilde{u}$ is any smooth L^2 section of $L + K_X$ which restricts to u on Y.

Lemma 10.1. *The G-norm does not depend on the choice of smooth extension $\tilde{u}$, moreover we have*

$$||u||^2_G = 2\pi \int_Y i^{(n-1)^2}\hat{u}\wedge\overline{\hat{u}}\, e^{-\phi+\psi},$$

if $G = G_{s,\psi}$ and $u = ds \wedge \hat{u}$ on Y, where $\hat{u}$ is a smooth section of $K_Y + (L - [Y])|_Y$.

Proof. Let U be a z-coordinate open subset of X with $s = z_n$ on U, then we have

$$U \cap \{G < t\} = \{z \in U : |z_n|^2 \leq e^{\psi(z)+t}\},$$

which gives

$$\lim_{t\to-\infty} e^{-t}\int_{G<t} i^{n^2} u\wedge\bar{u}\, e^{-\phi} = 2\pi\int_Y i^{(n-1)^2}\hat{u}\wedge\overline{\hat{u}}\, e^{-\phi+\psi},$$

since

$$\int_{|z_n|^2\leq e^{\psi(z',0)+t}} i\, dz_n\wedge d\bar{z}_n = 2\pi e^{\psi(z',0)+t},$$

where $(z') = (z_1, \cdots, z_{n-1})$. □

Remark: One way of looking at the above lemma is to use the following adjunction formula

$$(K_X+L)_Y \simeq K_Y + (L-[Y])|_Y.$$

Our starting point is the following direct consequence of Theorem 8.2.

Theorem 10.2. *Let $(L, e^{-\phi})$ be a holomorphic line bundle over an n-dimensional weakly pseudoconvex Kähler manifold (X,ω). Let $Y\subset X$ be a closed $(n-1)$-dimensional complex submanifold of X. Let $G=G_{s,\psi}$ be a Y-polar function. Fix $u\in H^0(Y,(K_X+L)|_Y)$ with $||u||_G<\infty$. If*

$$i\partial\overline{\partial}(\phi-\psi)>0$$

on X then for every relative compact open subset X_0 in X, u extends to $U\in H^0(X_0, K_X+L)$ such that

$$\int_{X_0} i^{n^2} U\wedge \bar{U}\, e^{-\phi}<\infty.$$

Proof. Since $||u||_G<\infty$, local holomorphic extension u_α and partition of unit λ_α together give a smooth extension

$$\tilde{u}:=\sum \lambda_\alpha u_\alpha$$

with

$$\int_{\rho<j} i^{n^2}\tilde{u}\wedge\overline{\tilde{u}}\, e^{-\phi}<\infty, \ \ \forall\, j\in\mathbb{N},$$

where ρ is a smooth psh exhaustion function of X. Then

$$\int_{\rho<j} |\overline{\partial}\tilde{u}|^2_\omega e^{-\phi-G}\omega_n<\infty, \ \ \forall\, j\in\mathbb{N}.$$

Solve $\overline{\partial}$-equation $\overline{\partial}a=\overline{\partial}\tilde{u}$ on $\{\rho<j\}$ with respect to the weight $\phi+G$, we get

$$\int_{\rho<j} i^{n^2} a\wedge\bar{a}\, e^{-\phi}\leq \int_{\rho<j} i^{n^2} a\wedge\bar{a}\, e^{-\phi-G}<\infty.$$

Since G is Y-polar, we have $a=0$ on Y. Fix j such that $X_0\subset\{\rho<j\}$, we know that

$$U:=\tilde{u}-a$$

fits out need. □

Remark: In case X is compact, the above theorem says that the L^2-extension problem is solvable for (L,Y) if $L-[Y]$ is ample. This fact can also be understood using the following canonical exact sequence

$$0\to K_X+L-[Y]\to K_X+L\to (K_X+L)|_Y\to 0,$$

which gives

$$H^0(X,K_X+L)\twoheadrightarrow H^0(Y,(K_X+L)|_Y)$$

since we have

$$H^1(X,K_X+L-[Y])=0$$

by the Kodaira vanishing theorem (see Theorem 4.6).

10.2. **Sharp Ohsawa–Takegoshi extension from a smooth divisor.** Using the complex Brunn–Minkowski theory [2], Berndtsson–Lempert [5] found a sharp Ohsawa–Takegoshi type effective version of Theorem 10.2. The following weakly pseudoconvex Kähler version of Theorem 3.8 in [5] is proved by Tai Terje Huu Nguyen in [16].

Theorem 10.3. *Let $(L, e^{-\phi})$ be a holomorphic line bundle over an n-dimensional weakly pseudoconvex Kähler manifold (X, ω). Let $Y \subset X$ be a closed $(n-1)$-dimensional complex submanifold of X. Let $G = G_{s,\psi}$ be a non-positive Y-polar function (see Definition 10.1). Assume that for some constant $0 < \varepsilon \leq 1$,*

$$i\partial\overline{\partial}(\phi - \psi) \geq \varepsilon \cdot i\partial\overline{\partial}\phi > 0 \tag{10.1}$$

on X. Then every $u \in H^0(Y, (K_X + L)|_Y)$ with $||u||_G < \infty$ extends to $U \in H^0(X, K_X + L)$ such that

$$\varepsilon \int_X i^{n^2} U \wedge \bar{U}\, e^{-\phi} \leq 2\pi \int_Y i^{(n-1)^2} \hat{u} \wedge \overline{\hat{u}}\, e^{-\phi+\psi},$$

where $\hat{u}$ is defined such that $ds \wedge \hat{u} = u$ on Y.

Proof. We shall only provide the main idea of the proof. The details can be found in [16]. Follow the proof of Theorem 3.8 in [5], one may consider the Hartogs domain

$$X_\phi := \{(z, v) \in L^* : |v|^2 e^\phi < 1\}$$

and use the following canonical isomorphism

$$K_{L^*} \simeq \pi^*(K_X + L),$$

where $\pi : L^* \to X$ denotes the canonical map from the dual bundle of L to the base manifold X. Via this isomorphism, one may identify sections of $K_X + L$ with sections of K_{L^*} whose coefficients depend only on X. Fix an arbitrary relatively compact weakly pseudoconvex open submanifold, say X_0, of X. By Theorem 10.2, u extends to a holomorphic section, say $\tilde{u}$, of $K_X + L$ over X_0 with $||\tilde{u}|| < \infty$. Let us identify $\tilde{u}$ as a section of K_{L^*}. Now it is enough to solve the problem on X_ϕ, which is weakly pseudoconvex in case $i\partial\overline{\partial}\phi \geq 0$. Put

$$Y_\phi := X_\phi \cap \pi^* Y.$$

Consider the following *quasi* Y_ϕ-polar function (we use "quasi" since G_ϕ also has singularity at $v = 0$, but with order $1 - \varepsilon < 1$)

$$G_\phi := (1 - \varepsilon) \log(|v|^2 e^\phi) + G.$$

Then the Berndtsson–Lempert method gives the following estimate for the minimal L^2 extension, say $\tilde{U}$, of $\tilde{u}|_{Y_\phi}$

$$\int_{X_\phi} i^{(n+1)^2} \tilde{U} \wedge \overline{\tilde{U}} \leq \lim_{t \to -\infty} e^{-t} \int_{\{G_\phi < t\} \cap X_\phi} i^{(n+1)^2} \tilde{u} \wedge \overline{\tilde{u}} := I.$$

Compute

$$\int_{X_\phi} i^{(n+1)^2} \tilde{U} \wedge \overline{\tilde{U}} = 2\pi \int_X i^{n^2} U \wedge \bar{U}\, e^{-\phi}.$$

Since $\varepsilon > 0$, the $(1-\varepsilon)\log|v|^2$ part of G_ϕ gives no contribution in the limit (try). Moreover, near a point where $v \neq 0$, $G_\phi < t$ is equivalent to that

$$\log|s|^2 \leq \psi - (1-\varepsilon)\log(|v|^2 e^\phi) + t,$$

which gives

$$I = 2\pi \int_{Y_\phi} i^{n^2} (dv \wedge \hat{u}) \wedge \overline{(dv \wedge \hat{u})}\, e^{\psi - (1-\varepsilon)\log(|v|^2 e^\phi)}.$$

Since for every fixed $z \in Y$,

$$\int_{|v|^2 e^{\phi(z)} < 1} (|v|^2 e^{\phi(z)})^{\varepsilon - 1}\, i dv \wedge d\bar{v} = \frac{2\pi}{\varepsilon} \cdot e^{-\phi(z)},$$

the theorem follows. □

Remark: In case X is Stein outside a divisor (e g X is projective), the following "singular" version of the above theorem also holds.

Theorem 10.4. *Let $(L, e^{-\phi})$ (ϕ can be non smooth now!) be a holomorphic line bundle over an n-dimensional complex manifold X. Assume that X is Stein or a closed complex submanifold of $\mathbb{P}^N$. Let $Y \subset X$ be a closed $(n-1)$-dimensional complex submanifold of X. Let $G = G_{s,\psi}$ be a non-positive Y-polar function (see Definition 10.1). Assume that for some constant $0 < \varepsilon \leq 1$,*

$$i\partial\bar{\partial}(\phi - \psi) \geq \varepsilon \cdot i\partial\bar{\partial}\phi \geq 0 \tag{10.2}$$

in the sense of current on X. Then every $u \in H^0(Y, (K_X + L)|_Y)$ with $||u||_G < \infty$ extends to $U \in H^0(X, K_X + L)$ such that

$$\varepsilon \int_X i^{n^2} U \wedge \bar{U}\, e^{-\phi} \leq 2\pi \int_Y i^{(n-1)^2} \hat{u} \wedge \bar{\hat{u}}\, e^{-\phi+\psi},$$

where $\hat{u}$ is defined such that $ds \wedge \hat{u} = u$ on Y.

Proof. If X is Stein then one may found a family of positively curved smooth metric ϕ_j on L whose decreasing limit is ϕ. Thus one may apply Theorem 10.3 to each ϕ_j and take the weak limit for $j \to \infty$. If X is a submanifold of $\mathbb{P}^N$, then one may choose $a \in \mathbb{C}^{N+1} \setminus \{0\}$ such that

$$Z_a \cap Y, \quad Z_a := \{[z_0 : \cdots : z_N] \in \mathbb{P}^N : a_0 z_0 + \cdots + a_N z_n = 0\}$$

defines a proper submanifold of Y. Since $\mathbb{P}^N \setminus Z_a$ is biholomorphic to $\mathbb{C}^N$, we know that $X \setminus Z_a$ is Stein and $Y \setminus Z_a$ is a closed Stein submanifold of $X \setminus Z_a$. Thus the projective case follows from the fact (try, start from the unit disc case!) that every L^2 holomorphic section on $X \setminus Z_a$ natural extends to an L^2 holomorphic section on X. □

10.3. **Berndtsson's subharmonicity theorem.** We shall follow Guan–Zhou's approach (see Guan–Zhou [11], see also [4] for the convex case) to show how to use Theorem 10.3 to prove a fundamental "subharmonicity" theorem in Berndtsson's complex Brunn–Minkowski theory.

Definition 10.2. *By a weakly pseudoconvex Kähler fibration we mean a holomorphic submersion*

$$p : \mathcal{X} \to \mathcal{B}$$

such that each point in the base manifold $\mathcal{B}$ has an open neighborhood whose preimage under p is weakly pseudoconvex Kähler.

Definition 10.3. *Let $p : \mathcal{X} \to \mathcal{B}$ be a weakly pseudoconvex Kähler fibration. We call*

$$f^t : C^\infty(X_t, K_{X_t} + L|_{X_t}) \to \mathbb{C}$$

a holomorphic family of currents with compact support if for every holomorphic section u of $K_{\mathcal{X}/\mathcal{B}} + \mathcal{L}$ over preimage of an arbitrary open set U in $\mathcal{B}$,

$$t \mapsto f^t(u^t), \quad u^t := u|_{X_t},$$

is holomorphic on U.

Theorem 10.5. *Let $p : \mathcal{X} \to \mathcal{B}$ be a weakly pseudoconvex Kähler fibration. Let $(\mathcal{L}, e^{-\phi})$ be a holomorphic line bundle over $\mathcal{X}$ with a smooth metric ϕ such that $i\partial\overline{\partial}\phi > 0$ on $\mathcal{X}$. Let f^t be a holomorphic family of currents with compact support. Put*

$$||f^t|| := \sup\left\{|f^t(u^t)| : u^t \in H^0(X_t, K_{X_t} + L|_{X_t}), \ \int_{X_t} i^{n^2} u^t \wedge \bar{u}^t \, e^{-\phi} = 1\right\}.$$

Assume that $||f^t||$ is upper semi-continuous. Then $\log ||f^t||$ is plurisubharmonic in t.

Proof. It suffices to show that $\log ||f^t||$ satisfies the sub-mean inequality on every (small) holomorphic disc in $\mathcal{B}$. By a complex affine transformation, it suffices to show that

$$\log ||f^0||^2 \leq \int_{\mathbb{D}_r} \log ||f^t||^2 \, \frac{idt \wedge \overline{dt}}{2\pi r^2}, \ \forall\, 0 < r < 1, \ \mathbb{D}_r := \{t \in \mathbb{C} : \ |t| < r\} \tag{10.3}$$

Let us take u^0 such that $||u^0|| = 1$ and $f^0(u^0) = ||f^0||_0$. By Theorem 10.3, we know that for each $r > 0$, u^0 extends to a holomorphic section, say u_r, on the preimage of $\mathbb{D}_r$ (notice that one may identify $K_{\mathcal{X}/\mathcal{B}} + \mathcal{L}$ with $K_{\mathcal{X}} + \mathcal{L}$ since now $K_{\mathcal{B}}$ is trivial) such that

$$\int_{|t|<r} ||u_r^t||^2 \, \frac{idt \wedge \overline{dt}}{2\pi r^2} \leq 1, \ u_r^t := u_r|_{X_t}. \tag{10.4}$$

Since $f^t(u_r^t)$ is holomorphic in t, we have

$$\log ||f^0||^2 = \log |f^0(u_r^0)|^2 \leq \int_{|t|<r} \log |f^t(u_r^t)|^2 \, \frac{idt \wedge \overline{dt}}{2\pi r^2}. \tag{10.5}$$

Hence $|f^t(u_r^t)| \leq ||f^t|| \cdot ||u_r^t||$ gives

$$\log ||f^0||^2 \leq \int_{|t|<r} \log ||f^t||^2 + \log ||u_r^t||^2 \, \frac{idt \wedge \overline{dt}}{2\pi r^2}. \tag{10.6}$$

Since $\log$ is concave, we know that

$$\int_{|t|<r} \log ||u_r^t||^2 \, \frac{idt \wedge \overline{dt}}{2\pi r^2} \leq \log\left(\int_{|t|<r} ||u_r^t||^2 \, \frac{idt \wedge \overline{dt}}{2\pi r^2}\right) \leq 0, \tag{10.7}$$

hence we have

$$\log ||f^0||^2 \leq \int_{|t|<r} \log ||f^t||^2 \, \frac{idt \wedge \overline{dt}}{2\pi r^2}.$$

from which (10.3) follows. The proof is complete. □

Remark 1: In order to verify that $||f^t||$ is upper semi-continuous, say at the origin of the unit disc, one may take u^{t_j} with $||u^{t_j}|| = 1$ such that $||f^{t_j}|| = |f^{t_j}(u^{t_j})|$ for every $t_j \to 0$. Taking a subsequence if necessary, one may assume that u^{t_j} convergence weakly to u^0 with $||u^0|| \leq 1$. If one could verify that

$$\lim f^{t_j}(u^{t_j}) = f^0(u^0),$$

they we know that

$$\limsup_{t \to 0} ||f^t|| \leq ||f^0||$$

and $||f^t||$ is upper semi-continuous at the origin.

Remark 2: In case $\mathcal{X}$ is Stein (or p is proper and the total space possesses a positive line bundle), the above theorem is also true for non-smooth ϕ with $i\partial\overline{\partial}\phi \geq 0$.

10.4. **Applications.** In this subsection, we shall discuss applications of Theorem 10.5. First, let us discuss possible generalizations of Theorem 10.3.

1. Let $(L, e^{-\phi})$ be a holomorphic line bundle over an n-dimensional weakly pseudoconvex Kähler manifold (X, ω). Assume that ϕ is smooth and $\omega_\phi := i\partial\overline{\partial}\phi > 0$.

2. We want to extend holomorphic sections of $K_X + L$ from a closed submanifold Y to X with good estimate. From Theorem 10.3, we need certain "polar function" G, which should at least satisfy the following conditions:

2a) $G \leq 0$ on X;

2b) for some $0 < \varepsilon \leq 1$, G is $(1-\varepsilon)\omega_\phi$-psh, i.e.

$$(1-\varepsilon) i\partial\overline{\partial}\phi + i\partial\overline{\partial}G \geq 0$$

in the sense of current on X;

2c) $Y \subset \{G = -\infty\}$ and G is smooth outside $\{G = -\infty\}$.

With the notation in the proof of Theorem 10.3, consider

$$G_\phi := (1-\varepsilon)\log(|v|^2 e^\phi) + G$$

on the Grauert tube X_ϕ in L^*. 2a) implies that G_ϕ is non-positive and by 2b) G_ϕ is psh on X_ϕ. For each $\tau \in \mathbb{C}$, let us define

$$X_\tau := \{x \in X_\phi : G_\phi(x) < t := \operatorname{Re}\tau\}$$

and

$$\mathcal{X} := \{(\tau, x) \in \mathbb{C} \times X_\phi : x \in X_\tau\}.$$

Lemma 10.6. *The projection mapping $p : \mathcal{X} \to \mathbb{C}$ defined by $p(\tau, x) = \tau$ gives a weakly pseudoconvex Kähler fibration.*

Proof. Notice that $\omega + i\partial\overline{\partial}|v|^2 e^{\phi}$ defines a Kähler form on L^*, hence $\mathcal{X}$ as a domain in $\mathbb{C} \times L^*$ is Kähler. By condition 2c), we know that

$$\tilde{G} := \mathrm{Max}\{-1, G_\phi(x) - \mathrm{Re}\,\tau\}$$

is smooth on $\mathcal{X}$, where Max is a regularized maximum function. For every open set U in $\mathbb{C}$, let $\psi(\tau)$ be a smooth subharmonic exhaustion function on U, we know that

$$-\log(1 - |v|^2 e^{\phi}) - \log -\tilde{G} + \psi(\tau)$$

defines a smooth psh exhaustion function on $p^{-1}(U)$. Hence $p^{-1}(U)$ is weakly pseudoconvex Kähler. The proof is complete. □

In order to apply Theorem 10.5, we need to construct a holomorphic family of currents with compact support

$$f^\tau : C^\infty(X_\tau, K_{X_\tau}) \to \mathbb{C}.$$

We shall use an approach due to Berndtsson–Lempert, for every smooth section g of $((K_{X_\phi})|_{Y_\phi})^*$ with compact support in Y_ϕ, we know that

$$f_g^\tau : F \mapsto \int_{Y_\phi} g(F|_{Y_\phi})\, dV, \quad \forall\, F \in C^\infty(X_\tau, K_{X_\tau}),$$

where dV is a fixed volume form on Y_ϕ, depends holomorphically on τ. By Remark 1 after the proof of Theorem 10.5, we know that $||f_g^\tau||$ is upper semi-continuous in τ and depends only on $t = \mathrm{Re}\,\tau$. Hence Theorem 10.5 implies that

Lemma 10.7. $\log ||f_g^t||$ *is a convex function of* $t \in \mathbb{R}$.

By the above lemma, if $\log ||f_g^t|| + t$ is bounded when $t \to -\infty$ then the above lemma implies that $\log ||f_g^t|| + t$, as a convex function, must be increasing, this is a key observation of Berndtsson–Lempert. In order to use this observation, notice that $f_g^t(F)$ depends only on $F|_{Y_\phi}$, in particular, for every $u \in H^0(Y_\phi, (K_{L^*})|_{Y_\phi})$, $f_g^t(u)$ is well defined and does not depend on t. Consider

$$||u||_{X_t} := \sup_{g \in C_0^\infty(Y_\phi, (K_{X_\phi}|_{Y_\phi})^*)} \frac{|f_g^t(u)|}{||f_g^t||},$$

the above observation implies that

$$e^{-t}||u||_{X_t}$$

is decreasing in t, we shall prove that:

Lemma 10.8. *Assume that* u *extends to an* L^2 *holomorphic section* U *of* K_{X_t} *over* X_t*, then* $||u||_{X_t}$ *is equal to the minimum* L^2*-norm* $||U||$ *of all possible* U*.*

Proof. Since $|f_g^t(u)| \leq ||f_g^t|| \cdot ||U||$ for all possible L^2-holomorphic extension U, we know that $||u||_{X_t} \leq \min ||U||$. To prove the identity, we shall use the Riesz representation theorem, which gives

$$\min ||U|| = \sup_{f \in H^*,\, f(H_0)=0} \frac{|f(U)|}{||f||},$$

where H denotes the space of L^2 holomorphic sections of K_{X_t} over X_t and H_0 is the space of forms in H that vanishes on Y_ϕ, H^* denotes the space of bounded $\mathbb{C}$-linear functionals on H. Notice that any such f can be approximated by f_g^t (otherwise there would exist an $F \in H \setminus H_0$ such that $f_g^t(F) = 0$ for all g, but this gives $F \in H_0$, a contradiction), hence we must have $||u||_{X_t} = \min ||U||$. . □

By the above lemma, if $\log ||f_g^t|| + t$ is bounded when $t \to -\infty$ and u extends to an L^2 holomorphic section U over X_ϕ, then we must have

$$\min ||U|| \leq \lim_{t\to-\infty} e^{-t} ||u||_{X_t}.$$

Let us consider a special case, when u comes from a section of $(K_X + L)|_Y$, then the coefficient of the minimal extension U_t on X_t of u would depend only on X, which gives

$$||u||^2_{X_t} = ||U_t||^2 \leq \int_{\{G_\phi < t\}\cap X_\phi} i^{(n+1)^2} U \wedge \bar{U},$$

notice that when $0 < \varepsilon < 1$, we have

$$\{G_\phi < t\} \cap X_\phi = \{(x, v) \in L^* : \log(|v|^2 e^\phi) < (1-\varepsilon)^{-1} \min\{0, t - G(x)\}\}$$

implies

$$\int_{\{G_\phi < t\}\cap X_\phi} i^{(n+1)^2} U \wedge \bar{U} = 2\pi \int_X i^{n^2} U \wedge \bar{U} e^{-\phi-(1-\varepsilon)^{-1}\max\{G-t,0\}}, \tag{10.8}$$

where U in the right hand side denotes the section of $K_X + L$ associated to U. The above formula suggests to prove Ohsawa–Takegoshi theorem directly using variation of weights instead of variation of domains! In fact the weight function

$$(1-\varepsilon)^{-1} \max\{G - t, 0\}$$

has already been used by Berndtsson–Lempert in [5]. But in [5], they finally use the limiting case ($\varepsilon \to 1$, thus variation of domains) to study the Ohsawa–Takegoshi extension theorem. Hence the main proof in [5] contains several technical reductions. We hope that our observation could simply their proof a little bit.

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