Norges teknisk–
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MIDTERM EXAM IN ST2201 MATHEMATICAL STATISTICS

Tuesday 13 March 2007
Time: 14:15–16:00

Tillatte hjelpemidler:
Alle trykte og håndskrevne hjelpemidler tillatt.
Alle kalkulatorer tillatt.

Oppgave 1

Let $X_1, ..., X_n$ be a sample from a distribution with pdf

$$f(x|\theta) = \begin{cases} C(\theta)e^{-x^2/2} & \text{for } x \geq \theta, \\ 0 & \text{otherwise}, \quad -\infty < \theta < \infty, \end{cases}$$

where

$$C(\theta) = \left(\int_\theta^\infty e^{-x^2/2} dx \right)^{-1}.$$

a) Find a one-dimensional sufficient statistic.

Solution. The likelihood function is

$$f(x_1, ..., x_n|\theta) = [C(\theta)]^n \exp\left[-\frac{1}{2} \sum_{i=1}^n x_i^2 \right] \prod_{i=1}^n I_{[\theta,\infty)}(x_i) =$$

$$= [C(\theta)]^n I_{[\theta,\infty)}(\min_i x_i) \exp\left[-\frac{1}{2} \sum_{i=1}^n x_i^2 \right].$$

Put

$$g(\min_i x_i, \theta) = [C(\theta)]^n I_{[\theta,\infty)}(\min_i x_i)$$

and

$$h(x) = \exp\left[-\frac{1}{2} \sum_{i=1}^n x_i^2 \right].$$

Due to the factorization theorem,

$$T(X) = \min_i X_i$$

is a (one-dimensional) sufficient statistic.

b) Find MLE of θ.

Solution. Rewrite the likelihood function as follows

$$f(X|\theta) = [C(\theta)]^n I_{(-\infty, \min_i X_i]}(\theta) \exp\left[-\frac{1}{2}\sum_{i=1}^{n} X_i^2\right].$$

MLE maximizes

$$[C(\theta)]^n I_{(-\infty, \min_i X_i]}(\theta).$$

But this function (as a function of θ) equals 0 for $\theta > \min_i X_i$ and is positive and increases for $\theta \leq \min_i X_i$ (because $C(\theta)$ increases in θ). Hence

$$\hat{\theta}_{MLE} = \min_i X_i.$$

Oppgave 2

Let $X_1, ..., X_n$ be a sample from a distribution with pdf

$$f(x|\theta) = \frac{1}{2}\theta^3 x^2 e^{-\theta x}, \quad x > 0,\ \theta > 0.$$

The prior for θ is a gamma distribution with parameters α and 1.

a) Find α if the Bayes estimator of θ has form

$$\hat{\theta}_B = \frac{3n+1}{1+\sum_{i=1}^n X_i}.$$

Solution. It is easy to see that the posterior is a gamma distribution with parameters

$$\alpha_1 = 3n + \alpha, \quad \beta_1 = \frac{1}{1+\sum_{i=1}^n X_i}.$$

The Bayes estimator is

$$\hat{\theta}_B = \alpha_1\beta_1 = \frac{3n+\alpha}{1+\sum_{i=1}^n X_i}.$$

Thus $\alpha = 1$.

b) Find $\hat{\theta}_{GMLE}$.

Solution.

$$\hat{\theta}_{GMLE} = (\alpha_1 - 1)\beta_1 = \frac{3n}{1 + \sum_{i=1}^{n} X_i}.$$