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LØSNINGSFORSLAG
EXAM IN ST2201 MATHEMATICAL STATISTICS
Wednsday 7 June 2006
Time: 09:00–14:00

Oppgave 1

Let $X_1, .., X_n$ be iid from pdf $f(x|\theta)$, and $\tau_1(\theta)$, $\tau_2(\theta)$ be two functions of θ. Suppose T_1 is the best unbiased estimator of $\tau_1(\theta)$, T_2 is the best unbiased estimator of $\tau_2(\theta)$, and that there exists a complete sufficient statistic. Prove that $T = T_1 + T_2$ is the best unbiased estimator of $\tau(\theta) = \tau_1(\theta) + \tau_2(\theta)$.

Solution. Let S be a complete sufficient statistic. Then T is a function of S (because, due to the Rao-Blackwell theorem, T_1 and T_2 are functions of S). T is an unbiased estimator of $\tau(\theta)$. But it follows from the Rao-Blackwell theorem and the uniqueness of the best unbiased estimator that for any function of the parameter there can be only one unbiased estimator which is a function of a complete sufficient statistic (If S is a complete sufficient statistic, and $T_1 = f_1(S), T_2 = f_2(S), ET_1 = \tau(\theta), ET_2 = \tau(\theta)$, then $0 = E(T_1 - T_2) = E(f_1(S) - f_2(S))$ and therefore $f_1(S) = f_2(S)$ a.s.), and it is the best unbiased.

Oppgave 2

Let $X_1, ..., X_n$ be iid from the distribution with pdf

$$f(x;\theta) = \theta(1+\theta)x(1-x)^{\theta-1},\ 0 < x < 1,\ 0 < \theta < 1.$$

a) Find a one-dimensional sufficient statistic.

Solution. $T(X) = \prod_{i=1}^{n}(1 - X_i)$ is a sufficient statistic (this directly follows from the factorization theorem).

b) Find MME and MLE of θ.

Solution.

$$\hat{\theta}_{MME} = \frac{2}{\bar{X}} - 2,$$

$$\hat{\theta}_{MLE} = \frac{-(S-2) + \sqrt{(S-2)^2 + 4S}}{2S}$$

where

$$S = \frac{1}{n}\sum_{i=1}^{n} \ln \frac{1}{1-X_i}.$$

c) Which of the following three functions of θ

$$\tau_1(\theta) = \frac{1}{\theta},\ \tau_2(\theta) = \frac{1}{\theta+1},\ \tau_3(\theta) = \frac{1}{\theta} + \frac{1}{\theta+1}$$

admits an efficient estimator (we call an unbiased estimator efficient if its variance coincides with the lower bound of the Cramer-Rao inequality)? Why? Find this estimator.

Solution. A function $\psi(\theta)$ admits an efficient estimator iff the score function $\partial \ln L/\partial\theta$ is represented in the form

$$\frac{\partial \ln L(\theta; X)}{\partial \theta} = c(\theta)(T(X) - \psi(\theta)).$$

Then $T(X)$ is the efficient estimator of $\psi(\theta)$. In our case the score function is

$$\frac{\partial \ln L(\theta; X)}{\partial \theta} = -n\left(-\frac{1}{n}\sum_{i=1}^{n}\ln(1-X_i) - \left(\frac{1}{\theta} + \frac{1}{\theta+1}\right)\right)$$

therefore $\tau_3(\theta) = \frac{1}{\theta} + \frac{1}{\theta+1}$ (and only this function of the three) admits an efficient estimator. The estimator is

$$T(X) = -\frac{1}{n}\sum_{i=1}^{n}\ln(1-X_i).$$

Oppgave 3

Let X_1, X_2 be iid $uniform(\theta, \theta+1)$, $\theta \geq 0$. For testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, consider two tests:

1) $\min\{X_1, X_2\} > a \Rightarrow H_1$,

2) $\max\{X_1, X_2\} > b \Rightarrow H_1$.

Let both tests have size α.

a) Find a and b.

Solution. $a = 1 - \sqrt{\alpha}$ (because $P(\min\{X_1, X_2\} > a) = P(X_1 > a)P(X_2 > a) = (1-a)^2$ under H_0), $b = \sqrt{1-\alpha}$ (because $P(\max\{X_1, X_2\} > b) = 1 - P(X_1 \le b)P(X_2 \le b) = 1 - b^2$ under H_0).

b) Find the power function $\pi_1(\theta)$ of the first test.

Solution. Let $\theta \ge 0$. Then $X_i - \theta \sim U[0,1]$ and therefore

$$\pi_1(\theta) = P_\theta(\min\{X_1, X_2\} > a) = P_\theta(X_1 - \theta > a - \theta)P_\theta(X_2 - \theta > a - \theta) =$$

$$= \begin{cases} (\theta + \sqrt{\alpha})^2 & \text{for } 0 \le \theta \le 1 - \sqrt{\alpha}, \\ 1 & \text{for } \theta > 1 - \sqrt{\alpha}. \end{cases}$$

c) Find the power function $\pi_2(\theta)$ of the second test.

Solution. Similarly (see part b)

$$\pi_2(\theta) = \begin{cases} 1 - (\sqrt{1-\alpha} - \theta)^2 & \text{for } 0 \le \theta \le \sqrt{1-\alpha}, \\ 1 & \text{for } \theta > \sqrt{1-\alpha}. \end{cases}$$

d) Compare the two tests. Is it true that the first test is uniformly more powerful than the second one?

Solution. $\pi_1(\theta) < \pi_2(\theta)$ for $0 < \theta < \sqrt{1-\alpha} - \sqrt{\alpha}$ and $\pi_1(\theta) > \pi_2(\theta)$ for $\sqrt{1-\alpha} - \sqrt{\alpha} < \theta < \sqrt{1-\alpha}$. Thus the two tests are not comparable.

Oppgave 4

Let $X_1, ..., X_n$ be iid from an exponential distribution $Exp(\theta)$ (pdf $\theta e^{-\theta x}$, $x > 0$, $\theta > 0$) Suppose that $\hat{\theta}_0$ is the Bayes estimator of θ based on this sample and on the improper natural non-informative prior, and $\hat{\theta}_\lambda$ is the Bayes estimator based on the proper exponential prior with density $\pi(\theta) = \lambda e^{-\lambda\theta}$, $\lambda > 0$. Prove that

$$\hat{\theta}_\lambda \to \hat{\theta}_0$$

in probability as $\lambda \to 0$ (n is fixed).

Solution.

$$\hat{\theta}_0 = \frac{n+1}{\sum_{i=1}^n X_i}, \quad \hat{\theta}_\lambda = \frac{n+1}{\lambda + \sum_{i=1}^n X_i}$$

that evidently implies that

$$\hat{\theta}_\lambda \to \hat{\theta}_0$$

in probability (and even a.s.) as $\lambda \to 0$.