3.21 The moment generating function would be defined by $\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{tx}}{1+x^2}dx$. On $(0,\infty)$, $e^{tx} > x$, hence

$$\int_0^{\infty}\frac{e^{tx}}{1+x^2}dx > \int_0^{\infty}\frac{x}{1+x^2}dx = \infty,$$

thus the moment generating function does not exist.

3.23 a.

$$\int_\alpha^\infty x^{-\beta-1}dx = \frac{-1}{\beta}x^{-\beta}\bigg|_\alpha^\infty = \frac{1}{\beta\alpha^\beta},$$

thus $f(x)$ integrates to 1 .

b. $\mathrm{E}X^n = \frac{\beta\alpha^n}{(n-\beta)}$, therefore

$$\begin{aligned}
\mathrm{E}X &= \frac{\alpha\beta}{(1-\beta)} \\
\mathrm{E}X^2 &= \frac{\alpha\beta^2}{(2-\beta)} \\
\mathrm{Var}X &= \frac{\alpha\beta^2}{2-\beta} - \frac{(\alpha\beta)^2}{(1-\beta)^2}
\end{aligned}$$

c. If $\beta < 2$ the integral of the second moment is infinite.

3.25 Note that if T is continuous then,

$$
\begin{aligned}
P(t \leq T \leq t+\delta | t \leq T) &= \frac{P(t \leq T \leq t+\delta, t \leq T)}{P(t \leq T)} \\
&= \frac{P(t \leq T \leq t+\delta)}{P(t \leq T)} \\
&= \frac{F_T(t+\delta) - F_T(t)}{1-F_T(t)}.
\end{aligned}
$$

Therefore from the definition of derivative,

$$
h_T(t) = \frac{1}{1 - F_T(t)} = \lim_{\delta \to 0} \frac{F_T(t+\delta) - F_T(t)}{\delta} = \frac{F_T'(t)}{1 - F_T(t)} = \frac{f_T(t)}{1-F_T(t)}.
$$

Also,

$$
-\frac{d}{dt}(\log[1 - F_T(t)]) = -\frac{1}{1-F_T(t)}(-f_T(t)) = h_T(t).
$$