Nonlinear Filtering with eXogenous Kalman Filter and Double Kalman Filter

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Abstract— **We propose two variants of the Linearized Kalman Filter (LKF). The model linearization is made about an auxiliary state estimate that can be seen as an exogenous input to the LFK, and the resulting two-stage estimator is called an exogenous Kalman filter (XKF). Since the linearized model of the LKF does not depend on the estimate from the LKF, it follows from stability theory of cascades that the stability properties of the XKF are inherent from the auxiliary state estimator, since the nominal LKF is globally exponentially stable. In some cases, the use of nonlinear transforms and immersions can be used to render the nonlinear dynamics into a globally valid linear time-varying (LTV) form that can be used for design of the auxiliary state estimator. Even though the noise and disturbances may be influenced in a non-favorable way by such transforms, they allow the auxiliary state estimator to be designed using a time-varying Kalman filter, leading to a two-stage estimator called the Double Kalman Filter (DKF). The DKF is inherently globally exponentially stable for nonlinear systems that can be transformed into uniformly observable LTV systems. The XKF and DKF overcome the potential instability of the extended Kalman Filter (EKF) and similar algorithms that can result from inaccurate initialization, by applying a feed-forward/cascade structure instead of the feedback inherent in the EKF linearization. The method is illustrated using examples.**

I. INTRODUCTION

The Kalman Filter (KF) is an optimal state estimation algorithm that is widely used in many applications. It is a robust algorithm that is known to be globally exponentially stable when applied to uniformly observable linear time-varying (LTV) systems, and the process and measurement noise processes are white. However, in many applications there are nonlinearities that call for some extension or modifications to the KF, which has lead to a number of widely used approximate KF algorithms. The Extended KF (EKF), Unscented KF, Ensemble KF and particle filter are among the most well known. Global stability can no longer be guaranteed, in general, and existing stability analysis only gives implicit conditions that cannot be verified a-priori as they depend on initial errors and system trajectories, e.g. [1], [2]. Additionally, the errors and correlations due to linearization implies sub-optimal estimation accuracy, although in some applications the degree of local sub-optimality is small, [3].

The lack of global stability of the EKF (and other nonlinear KF approximations) is essentially caused by a feedback loop that is introduced when computing locally linear model approximations (linearizations), cf. Figure 1a. The key point is that the linearization is made about the current state estimate, and with a poor initialization of the state estimate the linearization will be poor, and the KF update may not be able to reduce the estimation error, and thus prevent convergence of the error.

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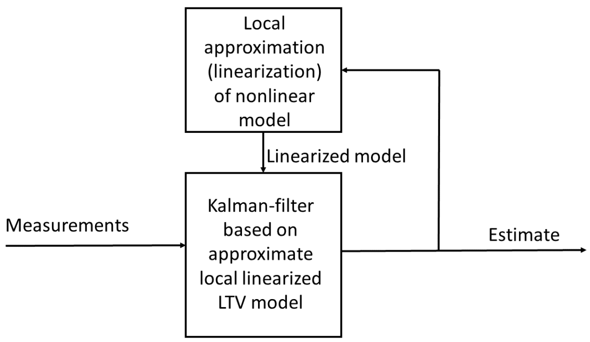
In this paper we study a recently proposed idea to overcome this problem, which is to linearize about another state estimate that is generated by an *auxiliary state estimator*, [4]. The auxiliary state estimator is an *exogenous input* to the LKF such that it does not depend of the LKF's own state estimate. For this reason, we call this two-stage estimation strategy an *eXogenous Kalman Filter* (XKF), cf. Figure 1b. The requirement to the auxiliary state estimator is essentially that it is globally convergent, and we make no specific requirements on the optimality or sensitivity with respect to noise. Certainly, the design of such an auxiliary state estimator is not trivial, and for many systems it may be impossible. Still, as we will show later in this article, there is a lot of freedom for the design of the auxiliary state estimator, and there are numerous approaches and applications of the XKF.

An important special case is when the nonlinear model can be transformed (via algebraic transforms, feedback, immersions or other mathematical tools) into a LTV model for the purpose of auxiliary state estimator design. In this case, another KF can be used as the auxiliary state estimator, and we therefore call this approach the *Double Kalman Filter* (DKF), cf. Figure 1c. The rationale for this approach is that the first-stage auxiliary KF based on the globally transformed LTV model will typically not achieve the same accuracy as the second-stage LKF based on the locally linearized model when considering the effect of measurement noise and process noise. In simple terms, the auxiliary first stage ensures the estimator's global convergence without aiming for optimal accuracy, while the second stage LKF is used to recover high estimation accuracy by considering the measurement noise and process noise.

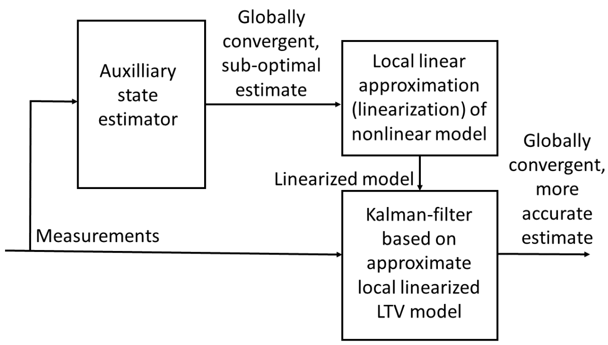
The XKF has been recently proposed [4]. The main novelty of the present paper is a discussion on the role of the auxiliary state observer. The introduction of the DKF as a new concept is the main contribution, generalizing ideas that were derived in the context of navigation application studies [5], [6].

II. DESIGN AND STABILITY OF XKF

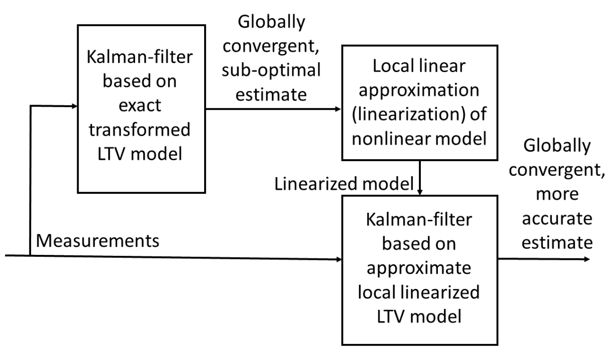
This section shows that the XKF – being the cascade of an auxiliary state estimator and a LKF – inherits the stability properties of the auxiliary state estimator. The analysis is developed in continuous time, based on [4], although we



(a) Extended Kalman Filter (EKF)



(b) eXogenous Kalman-filter (XKF)



(c) Double Kalman Filter (DKF)

Fig. 1. The figure illustrates the feedback structure of the EKF in comparison with the feed-forward/cascade structure of the XKF and DFK.

stress that the implementation should use standard discrete-time KF formulas.

Consider the nonlinear system

$$\dot{x}(t) = f(x(t),t) + G(t)w(t) \tag{1}$$

$$y(t) = h(x(t),t) + e(t) \tag{2}$$

where f, G, h are smooth vector-or matrix-valued functions, x is a bounded state vector, t is time, w is a vector of process disturbances, and e is a vector of measurement errors.

Let $\bar{x}$ be an estimate of x, given by an arbitrary auxiliary state estimator with bounded and convergent error $\breve{x}(t) = x(t) - \bar{x}(t)$. We will come back to the design of the auxiliary state estimation in Section III, and first we consider the design of a 2nd stage LKF, and note that $\bar{x}$ is an exogenous signal to this filter and can be used for linearization of the model in the LKF. A Taylor series expansion of (1) about the trajectory $\bar{x}(t)$ gives

$$\begin{aligned} \dot{x}(t) &= f(\bar{x}(t),t) + F(\bar{x}(t),t)\breve{x}(t) \\ &\quad + G(t)w(t) + q(x(t),\bar{x}(t),t) \end{aligned} \tag{3}$$

$$\begin{aligned} y(t) &= h(\bar{x}(t),t) + H(\bar{x}(t),t)\breve{x}(t) \\ &\quad + r(x(t),\bar{x}(t),t) + e(t) \end{aligned} \tag{4}$$

where $q(\cdot)$ and $r(\cdot)$ are higher-order terms, and

$$F(\bar{x},t) := \frac{\partial f}{\partial x}(\bar{x},t), \qquad H(\bar{x},t) := \frac{\partial h}{\partial x}(\bar{x},t)$$

Since $\bar{x}(t)$ is bounded, there exist constants $k_q, k_r > 0$ such that the higher order terms are bounded by

$$||q(t)|| \leq k_q ||\breve{x}(t)||^2, \qquad ||r(t)|| \leq k_r ||\breve{x}(t)||^2 \tag{5}$$

Whenever the error of the auxiliary state estimator converges, these terms vanishes asymptotically, so in the 2nd-stage LKF we neglect the higher order terms in (3)-(4) and design a LKF based on the truncated LTV model:

$$\begin{aligned} \dot{\hat{x}}(t) &= f(\bar{x}(t),t) + F(\bar{x},t)(\hat{x}(t) - \bar{x}(t)) \\ &\quad + K(t)\left(y(t) - h(\bar{x}(t),t) - H(\bar{x}(t),t)(\hat{x} - \bar{x}(t))\right) \end{aligned} \tag{6}$$

Disregarding for the moment the linearization errors q and r, and the fact that $\bar{x}(t)$ depends on the measurements, we recall that the KF is optimal under the assumption that w is white noise with covariance matrix Q, e is white noise with covariance matrix R, and w and e are uncorrelated. Then the time-varying gain satisfies $K(t) = P(t)H^T(\bar{x}(t),t)R^{-1}$ where P is the time-varying symmetric positive definite solution to the Riccati equation

$$\begin{aligned} \dot{P}(t) &= F(\bar{x}(t),t)P(t) + P(t)F^T(\bar{x}(t),t) \\ &\quad + G(t)QG^T(t) - K(t)RK^T(t) \end{aligned} \tag{7}$$

with $P(0)$ symmetric and positive definite, [7].

The estimation error of the 2nd stage LKF is $\tilde{x} := x - \hat{x} = \breve{x} + \bar{x} - \hat{x}$. From (3)-(4) and (6) it follows that the error dynamics is LTV with a perturbation:

$$\Sigma_1 : \qquad \dot{\tilde{x}}(t) = A(\bar{x}(t),t)\tilde{x}(t) + d(t) \tag{8}$$

with $A(\bar{x}(t),t) = F(\bar{x}(t),t) - K(t)H(\bar{x}(t),t)$ and $d(t) = q(x(t),\bar{x}(t),t) + K(t)r(x(t),\bar{x}(t),t) + K(t)e(t) + G(t)w(t)$. For the nominal case we have the following result that shows that the XKF inherits the stability properties of the auxiliary state estimator:

Theorem 1: Suppose there are no noises, i.e. $w = 0$ and $e = 0$, and assume

A1. The LTV system $(F(\bar{x}(t),t), G(t), H(\bar{x}(t),t))$ is uniformly completely observable and controllable.

A2. The nominal error dynamics Σ_2 of the auxiliary state estimator is Uniformly Globally Asymptotically Stable (UGAS), Semi-Globally Exponentially Stable (SGES), or Globally Exponentially Stable (GES).

A3. The LKF tuning parameters $P(0), Q, R$ are symmetric and positive definite.

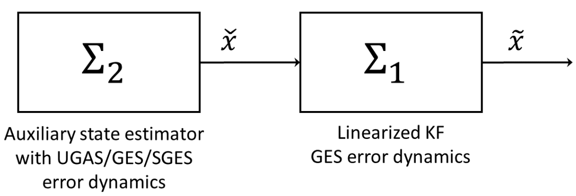


Fig. 2. Cascaded nominal error dynamics (no noise).

Then the origin $\check{x} = \tilde{x} = 0$ of the nominal error dynamics cascade $\Sigma_2 - \Sigma_1$ (see Figure 2) inherits the stability properties of Σ_2.

Proof: [4]: A2 implies boundedness of $F(\bar{x}(t),t)$, $G(t)$, and $H(\bar{x}(t),t)$. We can employ standard results on the KF [7], [8] in order to show global exponential stability (GES) of the origin of the nominal error dynamics (8) with $d = 0$. Note that $||d(t)|| \leq k_d ||\check{x}(t)||^2$ for some $k_d > 0$ due to (5). Since k_d is bounded and does not depend on $\tilde{x}$, the result follows from Theorem 2.1 and Proposition 2.3 in [9] since all their conditions are satisfied. ■

For the general case, when $w \neq 0$ and $e \neq 0$, one can show that bounded w and e give bounded estimation errors. It should be emphasized that even with a random Gaussian white noise assumption, the estimates of the state vector and covariance matrix may be biased and in general sub-optimal, like the EKF. The reason for this is the effect of the linearization error that is a random variable that may be non-white (due to errors resulting from the global nonlinear observer) and correlated with the measurements. Like with nonlinear filtering in general, best practice may be to investigate the errors using simulation.

III. THE AUXILIARY STATE ESTIMATOR

The main requirement to the auxiliary state estimator is that it should have strong (nominal) stability properties in the absence of noise, preferably global exponential stability. Its response to noise is considered less important in the context of XKF.

A. Nonlinear observers

One approach for the design of an auxiliary state estimator is the use of nonlinear observers. While the design of nonlinear observers with strong/global stability properties may not be trivial, or even possible, for every observable nonlinear system, there are still many specific systems and classes of systems for which nonlinear observer design is indeed possible, see e.g. [10], [11], [12].

B. The Double Kalman Filter

Another approach can be taken when the nonlinear dynamics can be transformed into a globally valid LTV system structure. In order to illustrate the main idea, consider one interesting special case, i.e. mechanical systems with position output measurement, where pseudo-measurements of velocity and other states may be generated by numerical differentiation and filtering of the position measurement. Clearly, this is not an optimal approach with respect to noise, but this is of less concern in this context as discussed above. For example, consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\alpha_1 x_2 - \alpha_2 x_2^2 - \alpha_3 x_2^3\\ &\quad +\beta_1 x_1 + \beta_2 x_1^2 + \beta_3 x_1^3 + u + w\\ y &= x_1 + e\end{aligned}$$

where x_1 is position, x_2 is velocity, u is a known input force, w is unknown process noise, y is position measurement with noise e. Using $x_1 = y - e$ and $x_2 = \dot{x}_1 = \dot{y} - \dot{e}$, this model can be exactly reformulated as

$$\begin{aligned}\dot{x}_2 &= -\alpha_1 x_2 - \alpha_2(\dot{y}-\dot{e})x_2 - \alpha_3(\dot{y}-\dot{e})^2 x_2\\ &\quad +\beta_1 x_1 + \beta_2(y-e)x_1 + \beta_3(y-e)^2 x_1 + u + w\end{aligned}$$

which can be written as

$$\begin{aligned}\dot{x}_2 &= -a_1(t)x_2 - a_2(t)x_2 - a_3(t)x_2\\ &\quad +b_1(t)x_1 + b_2(t)x_1 + b_3(t)x_1 + u + w + \varepsilon\end{aligned}$$

where the known time-varying coefficients are defined as

$$\begin{aligned}a_1(t) = \alpha_1, \quad a_2(t) = \alpha_2\dot{y}(t), \quad a_3(t) = \alpha_3\dot{y}(t)^2\\ b_1(t) = \beta_1, \quad b_2(t) = \beta_2 y(t), \quad b_3(t) = \beta_3 y(t)^2\end{aligned}$$

and all the influences of the noise e and its derivative $\dot{e}$ is lumped into the variable ε. When y and $\dot{y}$ are known time-varying signals, the resulting model is LTV and a globally exponentially stable auxiliary state estimator can be designed using the standard KF. We note that in the resulting DKF, the auxiliary first-stage KF tuning should explicitly account for the noises due to both e and $w + \varepsilon$. When ε depends on e and $\dot{e}$ as well as other signals in a non-trivial way, the optimal tuning of this auxiliary first-stage KF is not trivial.

Without going into further details we note that the above example can be generalized in a systematic way to larger classes of systems that could be expressed in the form

$$\dot{x} = A(z(t),t)x + G(z(t),t)w \tag{9}$$
$$y = C(z(t))x \tag{10}$$

where $z(t)$ are signals that are known at time t, typically inputs, measured outputs and their time-derivatives. One can also use rather general tools such as transforms into observer canonical forms using differential geometry [13], [14], or the use of differential algebra [15], and immersion [16].

The properties of the DKF can be summarized as follows:

Corollary 1: Assume an LTV model (9)-(10) is globally equivalent to the nonlinear system (1)-(2). Moreover, assume (9)–(10) is uniformly observable, and let an auxiliary state estimator be defined by a KF based on (9)-(10) and positive definite symmetric covariance matrices. If assumptions A1 and A3 of Theorem 1 holds, then the origin of the error dynamics of the DFK is GES.

Proof: Follows directly from Theorem 1 since A2 is satisfies by the origin of the auxiliary KF's error dynamics being GES. ■

The more extensive example presented next illustrates another design approach, where the physical and mathematical structure of a particular nonlinear system is exploited.

IV. DKF CASE STUDY: POSITIONING WITH PSEUDO-RANGE MEASUREMENTS

This example is based on [5]. Consider the problem of positioning during autonomous landing of a small fixed-wing Unmanned Aerial Vehicle (UAV). As an alternative to the use of differential/carrier-phase GNSS (e.g. [17]) it is of interest to investigate the use of low-cost radio navigation systems enabled by wireless network technology, [18]. Such radio systems typically provide measurement range up to 1000 m with range errors of about 10-50 cm, which may be sufficient for this purpose.

Let $p \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ denote the UAV's position and velocity, respectively, in the East-North-Up (ENU) coordinate frame. Assuming direct line-of-sight (LOS) path between the radio beacons and receiver, the range measurement model is

$$y_i = \rho_i + \beta + e_i, \quad \rho_i = ||p - p_i||_2 \tag{11}$$

for $i = 1,2,....,m$ where y_i is a pseudo-range measurement, p_i is the known position of the i-th beacon, m is the number of beacons, ρ_i is the geometric range along the LOS, and a common error (bias) $\beta := c\Delta_c$ where Δ_c is the clock synchronization error where c is the speed of light. The measurement model (11) is nonlinear due to the Euclidean norm $||\cdot||_2$. Using the method in e.g. [19], and neglecting for the moment the random noise e_i, one may transform the nonlinear algebraic equation (11) into a linear time-varying algebraic equation as follows:

$$(y_i - \beta)^2 = (p - p_i)^T (p - p_i) \tag{12}$$

Expanding and rearranging terms yields

$$y_i^2 - ||p_i||_2^2 = -2p_i^T p + 2y_i\beta + \left(||p||_2^2 - \beta^2\right) \tag{13}$$

Forming a difference, we can define a new measurement

$$z_i \quad := \quad y_i^2 - ||p_i||_2^2 - y_m^2 + ||p_m||_2^2 \tag{14}$$

for $i = 1,2,...,m-1$. Note that stacking instances of (14) into matrices and vectors and using (13) gives

$$z \quad = \quad 2Cx \tag{15}$$

where $x = [p;\beta]$ contains the unknown states, and the time-varying matrix $C \in \mathbb{R}^{(m-1)\times 4}$ is given by

$$C \quad := \quad \begin{bmatrix} -(p_1 - p_m)^T & y_1 - y_m \\ \vdots & \\ -(p_{m-1} - p_m)^T & y_{m-1} - y_m \end{bmatrix}$$

Assuming $m \geq 5$ and all transponders and the vehicle are not placed in the same plane, it can be established that C has full rank (equal to 4), which is sufficient for uniform observability also when this measurement equation is combined with a vehicle model such as a stochastic Markov model.

We note that in the difference formed in (14), the last (nonlinear) terms $(||p||_2^2 - \beta^2)$ of (13) are canceled. This is instrumental for transforming the model from nonlinear to a globally valid LTV model, but at the same time it represents a loss of information about the nonlinear relationship in the model. The transformation into an LTV model is enabled by the analytic redundancy provided by having at least five measurements, since there are only four unknowns to be estimated. However, the negative consequence of this is well known, e.g. [19], [20], [21], [22]: The estimation based on the quasi-LTV model (15) must be expected to be less accurate than by linearizing the original measurement equation (11) when considering the effect of noise. Nevertheless, as the example will show, the DKF does not suffer much from this since (15) is only used in the auxiliary state estimator while in the presence of noise the final stage LKF recovers performance close to the optimal by linearizing (11) about the auxiliary state estimate.

Four different estimators are simulated and compared:

- Algebraic estimator (no filtering) based on the globally valid quasi-linear algebraic model (15) using weighted least squares (WLS).
- The auxilliary state estimator (AKF) implemented as a KF with measurement model (15) using a stochastic Markov model of the UAV velocity.
- DKF combining the first stage AKF with an second stage LKF using the AKF estimates for linearization of (11), and the same Markov model of the UAV velocity as above.
- The Extended Kalman-filter (EKF) based on a linearization of (11) about its own estimate, and the same Markov model of the UAV velocity as above.

The parameters of the Markov model are tuned empirically based on the typical motions of the UAV, and the parameters of the measurement noise covariance matrices account for correlations between y_i and y_m according to (14), cf. [5].

Since the radio beacons need to be deployed at locations near the ground, the geography of typical UAV landing sites implies small vertical separation of the beacons even when placing them on masts, leading to a poorly conditioned position estimation problem. Moreover, since the UAV will land at ground level, its altitude will be similar to the vertical position of several of the beacons near the end of the final approach, i.e. just before touching the ground. This means that baselines will be crossed, and resolving ambiguity in the position estimate becomes non-trivial. The simulation example presented below as been deliberately chosen as a challenging (yet realistic) scenario in order to clearly illustrate the potential benefits of the proposed approach. In fact, the simulations show that the global convergence property of the proposed method is essential for safe navigation without any further sensors in the selected scenario, as well as similar scenarios that are tested but not reported.

We assume 6 radio beacons are distributed within an area with up to 800 m horizontal separation, and with up to 10 m vertical separation between the beacons. The beacons provide signals that are used for pseudo-range measurements with standard deviation $\sigma_y = 0.15$ m and sampling interval of 0.2 s, corresponding to a realistic hardware setup.

TABLE I

MONTE-CARLO SIMULATIONS, AVERAGED OVER 96 CASES WHERE ALL ESTIMATES CONVERGES.

Estimator	Horizontal st.dev	Vertical st.dev
Algebraic WLS	1.4812	8.5375
AKF	0.52605	2.5753
DKF	0.30178	0.90865
EKF	0.30484	0.87357

We simulate a small fixed-wing UAV trajectory in a final approach towards landing. Just before touching the ground, it is decided to abort the landing, so the vehicle climbs out and loiters at a holding position south of the landing target before going around to get in position for another approach. In a Monte-Carlo simulation with 100 scenarios, we simulate slight perturbations of the trajectory, due to wind and other random factors, as well as different measurement noise realizations.

During these simulations the EKF lost track of the correct global solution in 4 out of the 100 simulations. A representative example where this happens is shown in Figures 3 - 4. The horizontal position is accurately tracked by all estimators, and their differences in horizontal estimation accuracy can hardly been seen on this scale of presentation. However, the differences in performance are clearly apparent in the altitude estimate in Figure 3b, which shows that the AKF provides less accurate estimates than the DKF, and that the EKF diverges after $t = 42$ s.

The reason for the failure of the EKF is the existence of multiple local minimums in the nonlinear least-squares criterion that the EKF attempts to minimize. When the UAV's altitude is close to the height of the beacons, at $h = 5$ m and $h = 15$ m, it crosses their baselines. There, the minimums are so close to each other that the EKF is not able to distinguish between them at some point near $t = 42$ s, and chooses somewhat arbitrarily (i.e. dependent on the measurement noise) which minimum to track since there is conflicting information provided by the predictions of the dynamic model and the measurements. It can be seen in Figure 5 that loss of track is, in this case, not linked to any worse geometric conditioning of the problem than at other times. It is remarked that the problem might have been reduced or avoided by additional hardware, such as having beacons at higher altitude, or using additional sensors such as inertial sensors in order to improve the predictive capability of the model (see [6]), or altimeter or GNSS to explicitly resolve the ambiguity to ensure the integrity of the system. It is noted that the auxiliary AKF never loses track of the global solution, and consequently, the DKF never loses track of the global solution either.

Average estimation errors for the 96 cases where the EKF converges are summarized in Table I. It can be seen that the DKF typically performs just as well as the EKF despite sub-optimal accuracy in the auxiliary first-stage estimation. Hence, when the EKF does not experience divergence problems, the performances of the EKF and the DKF are close.

V. CONCLUSIONS

We study the eXogenous Kalman Filter (XKF) which is the cascade of an auxiliary state estimator with a LKF, where output of the auxiliary state estimator is an exogenous input only used for linearization in the LKF. It is shown that the XKF inherits the stability properties of the auxiliary state estimator due to their feed-forward/cascade structure. It is illustrated that for a relatively large class of nonlinear systems, the auxiliary state estimator can be a KF, leading to a two-stage estimator structure that we call a double KF (DKF). The first stage ensures GES, while the second stage contributes to improve estimation accuracy in the presence of measurement noise and process noise.

Examples illustrate that the estimation error of the multi-stage filter can be close to the perfectly linearized KF, and for a wide range of applications can the auxiliary state estimator be designed based on a globally valid LTV model achieved by from transformation of the nonlinear model without taking into account the influences of measurement noise and disturbances. This variant of the XKF is called the DKF - the Double Kalman Filter - since KF based on different models are used in the two KF stages.

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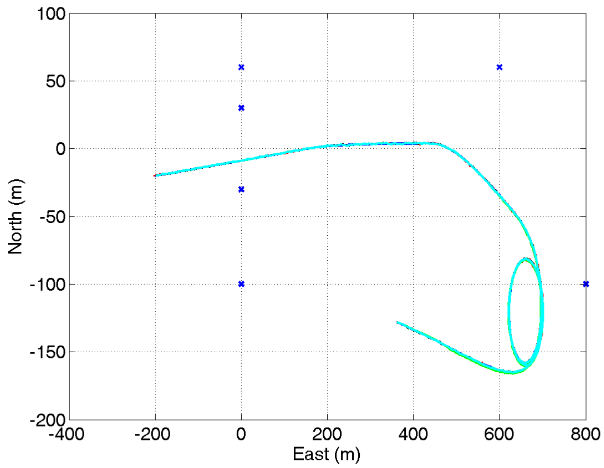
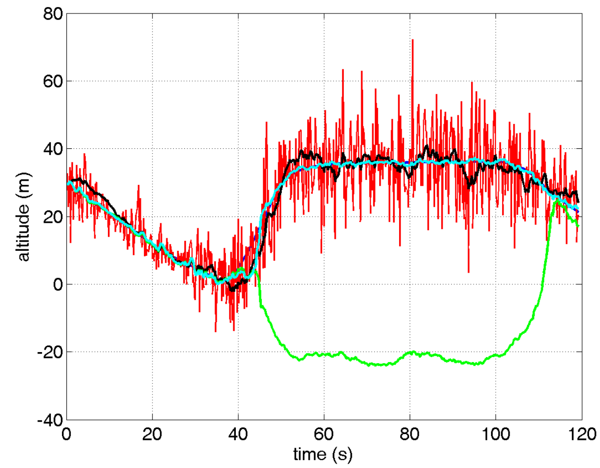


(a) Horizontal position (North-East)



(b) Vertical position (Altitude)

Fig. 3. True and estimated positions. The initial position is at -200 m East and -20 m North. The horizontal position of the six beacons are marked by blue points. Blue: Exact. Red: Algebraic WLS. Black: AKF. Cyan: DKF. Green: EKF. Note that the horizontal position estimates are very similar, and their differences cannot be clearly seen on the scale of this figure. The vertical position estimates are, however, clearly different.

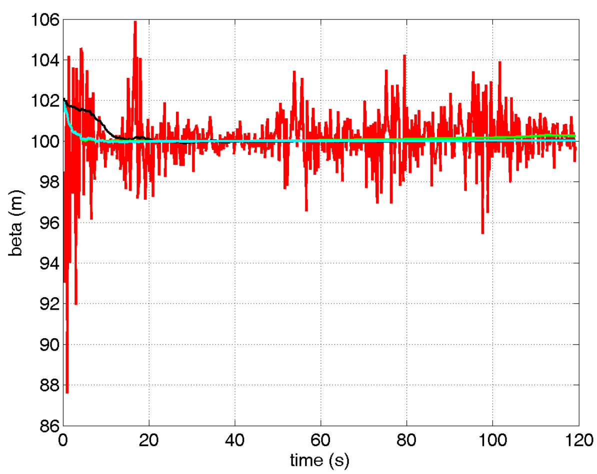


Fig. 4. Range bias β estimated with different observers. Blue: Exact ($\beta = 100$). Red: Algebraic WLS. Black: AKF. Cyan: DKF. Green: EKF.

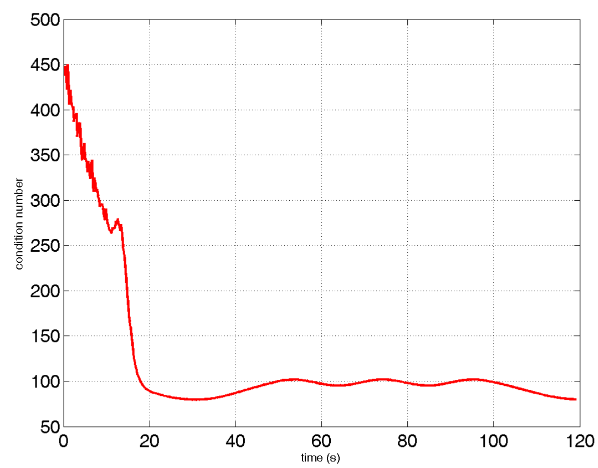


Fig. 5. Condition number of matrix C, i.e. ratio between largest and smallest singular value.

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