

Control Allocation - A Survey

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Abstract

The control algorithm hierarchy of motion control for over-actuated mechanical systems with a redundant set of effectors and actuators commonly includes three levels. First, a high-level motion control algorithm commands a vector of virtual control efforts (i.e. forces and moments) in order to meet the overall motion control objectives. Second, a control allocation algorithm coordinates the different effectors such that they together produce the desired virtual control efforts, if possible. Third, low-level control algorithms may be used to control each individual effector via its actuators. Control allocation offers the advantage of a modular design where the high-level motion control algorithm can be designed without detailed knowledge about the effectors and actuators. Important issues such as input saturation and rate constraints, actuator and effector fault tolerance, and meeting secondary objectives such as power efficiency and tear-and-wear minimization are handled within the control allocation algorithm. The objective of the present paper is to survey control allocation algorithms, motivated by the rapidly growing range of applications that have expanded from the aerospace and maritime industries, where control allocation has its roots, to automotive, mechatronics, and other industries. The survey classifies the different algorithms according to two main classes based on the use of linear or nonlinear models, respectively. The presence of physical constraints (e.g input saturation and rate constraints), operational constraints and secondary objectives makes optimization-based design a powerful approach. The simplest formulations allow explicit solutions to be computed using numerical linear algebra in combination with some logic and engineering solutions, while the more challenging formulations with nonlinear models or complex constraints and objectives call for iterative numerical optimization procedures. Experiences using the different methods in aerospace, maritime, automotive and other application areas are discussed. The paper ends with some perspectives on new applications and theoretical challenges.

Key words: Over-actuated mechanical systems; Effectors; Actuators; Redundant inputs; Optimization; Generalized inverse; Saturation; Constraints; Fault-tolerant control; Re-configurable control.

1 Introduction

The objective of this survey is to give an overview of control allocation methods. It is not the objective of this paper to give a complete bibliography, but rather provide a subjective survey with an emphasis on recent developments within a common framework that is independent of the application domains where control allocation is conventionally used. The article is intended to encourage cross-disciplinary transfer of ideas and complement existing overview articles such as (Oppenheimer, Doman & Bolender 2010) that focus on aerospace applications of control allocation, and (Fossen & Johansen 2006) that focus on marine applications. In particular, there has recently been increasing interest in control allocation in the automotive and other industries where mechatronics prevail, which has led to increased research on nonlinear approaches to control allocation. Optimization-based allocation methods are emphasized since their computational complexity are already within the capabilities of today's off-the-shelf embedded computer technology, e.g. (Bodson 2002, Härkegård 2002, Johansen, Fossen & Berge 2004, Johansen, Fossen & Tøndel 2005, Petersen & Bodson 2006).

1.1 Over-actuated mechanical systems

Motion control systems are used to control the motion of mechanical systems such as vehicles and machines. *Effectors* are mechanical devices that can be used in order to generate time-varying mechanical forces and moments on the mechanical system, such as rudders, fins, propellers, jets, thrusters, and tires. *Actuators* are electromechanical devices that are used to control the magnitude and/or direction of forces generated by the individual effectors.

By mechanical design, there may be more effectors than strictly needed to meet the motion control objectives of a given application. Hence, in over-actuated mechanical systems, the controllability of the chosen states and outputs would also be achieved with less control inputs. An over-actuated mechanical design may be favorable due to several reasons:

- Need for effector redundancy in order to meet fault tolerance and control reconfiguration requirements.
- It may be desirable to choose a particular set of effectors rather than a smaller set of effectors for reasons such as

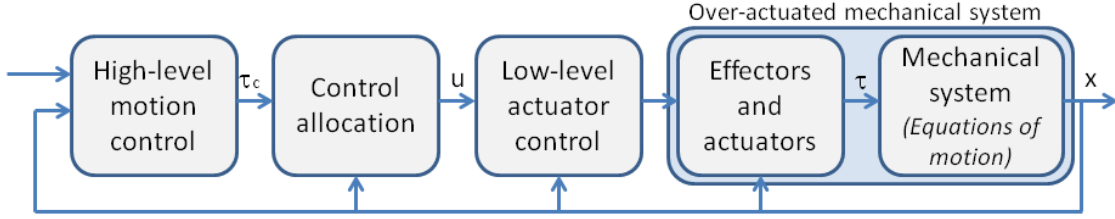


Fig. 1. Control system structure including control allocation. The vector τ_c denotes commanded virtual control effort (generalized forces), while τ are the actual allocated control effort.

cost, standardization, size, accuracy, dynamic response, flexibility, maintenance and mechanical design (see e.g. (Huston 2005)).

- Certain effectors can be shared among several control systems with different objectives, and therefore be redundant for the given motion control system. For example, a lateral stability control system for a car may use the four individual wheel brake actuators in order to set up a yaw moment while these actuators are primarily designed for the car's brake system to support the driver's control of longitudinal motion, see also (Valásek 2003).

The design of control algorithms for over-actuated mechanical systems is often divided into several levels. First, a high level motion control algorithm is designed to compute a vector of virtual inputs τ_c to the mechanical system. The virtual inputs are usually chosen as a number of forces and moments that equals the number of degrees of freedom that the motion control system wants to control, m , and such that the basic requirement of controllability is met. For a wide range of mechanical systems, this leads to a *dynamic model* that is linear in the virtual input

$$\dot{x} = f(x,t) + g(x,t)\tau \quad (1)$$

$$y = \ell(x,t) \quad (2)$$

where f, g, ℓ are functions, $x \in \mathbb{R}^n$ is the state vector, t is time, $y \in \mathbb{R}^m$ is the vector with the outputs that shall be controlled, and $\tau \in \mathbb{R}^m$ is the virtual input vector that should equal the output command τ_c of the high-level motion control algorithm, i.e. $\tau = \tau_c$. It is remarked that the model's linearity with respect to τ is not of importance for the control allocation design, but is convenient for the design of the high-level motion control algorithm design, although it still should consider constraints $\tau \in \mathbb{A}$, where \mathbb{A} is the attainable set of virtual controls.

Second, a control allocation algorithm is designed in order to map the vector of commanded virtual input forces and moments τ_c into individual effector forces or moments such that the total forces and moments generated by all effectors amounts to the commanded virtual input τ_c . This design is usually based on a *static effector model* in the form

$$\tau = h(u,x,t) \quad (3)$$

where h is a function, $u \in \mathbb{U} \subset \mathbb{R}^p$ is the control input, and

\mathbb{U} represents control constraints due to saturation and other physical constraints. Since the system is assumed to be over-actuated, we have $p > m$, such that the inverse problem of computing $u \in \mathbb{U}$ given $\tau = \tau_c$ is ill-posed because its solution is generally not unique. Commonly, effector models are linear in u such that

$$\tau = h(u,x,t) = B(x,t)u \quad (4)$$

Third, there may be a separate low-level controller for each effector that controls its actuators in order to achieve its desired force and moment.

The modular structure of the control algorithm is illustrated in the block diagram in Figure 1. This modularity allows the high-level motion control algorithm to be designed without detailed knowledge about the effector and actuator system. In addition to coordinating the effect of the different effectors in the system, issues such as effector/actuator fault tolerance, redundancy, and control constraints are typically handled within the control allocation module. Note that the effector model (3) is usually chosen to be static, so the low-level actuator control should handle the dynamic control of the actuators. Although the choice of a static effector model is common, it should be mentioned that it is also common that the control allocation algorithm is designed with actuator rate constraints in mind, and we will in later sections survey extensions where more sophisticated dynamic actuator models are integrated with the control allocation algorithm design.

It should also be mentioned that the design on the control allocation algorithm and the high-level motion control algorithm cannot always be independent. For example, it has been illustrated that the zero dynamics of the closed loop may depend on the control allocation design, such that a dynamic inversion type of control design approach may depend on the control allocation design to ensure stable zero dynamics (minimum phase response), see (Buffington & Enns 1996, Buffington, Enns & Teel 1998). On the other hand, it is also proven that in the framework of optimal control (LQ or nonlinear methods) the high-level motion control and control allocation can be separated (by choosing weight matrices appropriately) with no loss of control performance (Härkegård & Glad 2005). Moreover, as mentioned above and illustrated in (Page & Steinberg 1999), lack of feasibility

ity of the control allocation should be observed and handled by the high-level motion control algorithm in order to avoid unacceptable degradation of performance in such cases.

1.2 Control allocation introduction

The *primary objective* of a control allocation algorithm is to compute a control input $u \in \mathbb{U}$ that ensures that the commanded virtual control $\tau_c = h(u, x, t)$ is produced jointly by the effectors at all time t . This objective could fail to be met if the command τ_c require forces beyond the capabilities of the effectors due to saturation or other physical limitations. If a feasible $u \in \mathbb{U}$ cannot be found, the control allocation algorithm is usually required to degrade its performance objectives and search for a control input $u \in \mathbb{U}$ that minimizes the allocation error $\tau_c - \tau = \tau_c - h(u, x, t)$, in some sense. Usually, some kind of priority is involved such that the primary objective can be represented as an optimization problem

$$\min_{u \in \mathbb{R}^p, s \in \mathbb{R}^m} \|Qs\| \quad \text{subject to} \quad \tau_c - h(u, x, t) = s, \quad u \in \mathbb{U} \quad (5)$$

where s is some slack variable, $\|\cdot\|$ is some norm, and Q is some weight matrix that prioritizes the requirements that should be honored in case the commanded virtual input τ_c cannot be achieved. With the linear effector model (4) the problem can be written

$$\min_{u \in \mathbb{R}^p, s \in \mathbb{R}^m} \|Qs\| \quad \text{subject to} \quad \tau_c - B(x, t)u = s, \quad u \in \mathbb{U} \quad (6)$$

Actuator rate constraints can be included in the formulation by limiting the change in control Δu from the control u_ℓ from the last sampling instant to some set $\Delta u \in \mathbb{C}$. This leads to the formulation

$$\min_{u \in \mathbb{R}^p, s \in \mathbb{R}^m} \|Qs\| \quad \text{subject to} \\ \tau_c - h(u, x, t) = s, \quad u \in \mathbb{U}, \quad u = u_\ell + \Delta u, \quad \Delta u \in \mathbb{C} \quad (7)$$

In over-actuated mechanical systems, $\dim(u) > \dim(\tau)$, such that the solution to (5) or (7) is not unique. This is not a disadvantage, and offers the opportunity to introduce *secondary objectives*. Often, these are chosen from an operational perspective in order to minimize power or fuel consumption, minimization of actuator/effector tear and wear, or other criteria. In an optimization framework, this can be formulated by including a secondary cost function J into (5)

$$\min_{u \in \mathbb{R}^m, s \in \mathbb{R}^m} (\|Qs\| + J(x, u, t)) \quad \text{subject to} \\ \tau_c - h(u, x, t) = s, \quad u \in \mathbb{U}, \quad u = u_\ell + \Delta u, \quad \Delta u \in \mathbb{C} \quad (8)$$

Some generic cost functions that are commonly used are

$$J(x, u, t) = \frac{1}{2}(u - u_p)^T W(u - u_p) \quad (9)$$

$$J(x, u, t) = \|Wu\| \quad (10)$$

where $W \in \mathbb{R}^{p \times p}$ is a positive definite weighting matrix that prioritizes the different effectors, u_p is the preferred value of u (like zero deflection of control surfaces, trimmed flight conditions, last value $u_p = u_\ell$, or zero thrust), and $\|\cdot\|$ usually denotes 1-norm or ∞ -norm. Both W and u_p may be time-varying, and W should normally be chosen small compared to Q in order to reflect the fact that J represents an objective that is secondary to the primary objective of minimizing the slack variables weighted by Q . In later sections we will also comment upon additional and more application specific objectives and constraints, as well as formulations that accommodate alternative models with discrete variables or unknown parameters that need adaptation.

Although effector models must represent the actual physical forces and moments, they can be formulated and parameterized in many ways. Usually, one seeks effector models that are linear with respect to their inputs, leaving nonlinearities to be compensated for by nonlinear mappings, or in the low-level single effector / actuator control through inversion of monotone characteristics or linearizing feedback loops.

Optimization-based control allocation algorithms should be carefully chosen based on the properties of the effector model, objective function, constraints, and computational complexity. Explicit solutions can be found for linear effector models in combination with simple objective functions and constraints, while iterative numerical procedures are required for the more complicated problems. Particular attention will be needed for nonlinear and possibly non-convex optimization problems where issues such as numerical robustness, convergence to non-global minimums, computational complexity and reliability of the numerical software implementation should be addressed. These are key issues in the remaining sections of this survey.

We would like to emphasize that there exist several control allocation approaches that does not directly fit into the optimization formulation above. They will also be treated in this survey, and for consistency of the presentation they will still be presented in the context of optimization problems as far as possible.

1.3 Perspectives

This survey focuses on motion control for over-actuated mechanical systems, which is the conventional application area of control allocation. However, the principles of control allocation are general and not limited to motion control systems. Consequently, one does not need to restrict the virtual control input τ to be interpreted as generalized forces (forces and moments) and they may also represent quantities like energy and mass, for example. In particular, process plants are often characterized by excessive degrees of freedom for control. One example is allocation of gas lift rates in offshore oil production where the petroleum producing wells are coupled due to common pipelines and constraints on the available lift gas resources, (Camponogara & de Conto 2005, Bieker, Slupphaug & Johansen 2007).

In process control, any excessive degrees of freedom are commonly exploited via model predictive control (MPC) and real-time optimization, (Qin & Badgwell 2003, Garcia, Prett & Morari 1989), which are multi-variable optimization-based control strategies where the functionality of control allocation is inherently built into the optimal control formulation that is solved numerically online. Although control allocation usually has less ambitious objectives than MPC – recall the static effector model – we shall in later sections review how MPC can be used to solve the control allocation problem in motion control systems when actuator dynamics should be considered at the control allocation level. However, predictive control allocation has also been proposed in applications like engine management, (Vermillion, Sun & Butts 2011).

2 Control allocation for linear effector models

Most control allocation algorithms assume a linear effector model in the form

$$\tau = Bu \quad (11)$$

where $B \in \mathbb{R}^{m \times p}$ is a matrix, sometimes called the control effectiveness matrix, that describes the relationship between the control inputs u and the virtual control inputs τ (forces and moments) produced jointly by the effectors. For many applications, the matrix B will depend on the system state and time-varying parameters or inputs, cf. (4). For example, the aerodynamic lift and drag generated by an aircraft aileron, rudder, flap, spoiler, or other control surface, depends on vehicle velocity and angle of attack, in addition to nonlinearities due to the geometric shape of the control surface, e.g. (Da Ronch, Ghoreyshi, Vallepin, Badcock, Mengmeng, Ooppelstrup & Rizzi 2011). Fluid flow conditions of the effectors may depend on both the system state (e.g. velocity) and the effector state (e.g. slip angle or thrust vector direction), and lead to interactions between the effectors or between effectors and the body of the vehicle such as thruster-thruster interactions and thruster-hull interactions, (Fossen 2011), or change of flow pattern over a control surface (Oppenheimer et al. 2010). For example, deflection of an aerodynamic surface that is upstream of another aerodynamic surface may cause the forces and moments produced by the downstream effector to differ from those produced when the upstream effector is not deployed, (Oppenheimer & Doman 2007). Such issues will be further discussed in the Section 4 where application specific issues are surveyed. For now, we observe that since the control allocation problem is viewed as a static problem, the control allocation problem can utilize the time-frozen model (11) without any regard to how B will change with time, states and inputs. Consequently, the scheduled control effectiveness matrix B is updated at the next sampling instant and the control allocation problem is solved for the new scheduled matrix B . Hence, the formulation (11) does not exclude time-varying, parameter-varying, and linearized models. Robustness of the control allocation problem with respect to uncertainty in the scheduling of the B matrix as a function of some operating point variables is discussed in (Scottedward Hodel 2000).

The parameterization of the control input, i.e. the choice of elements of the u vectors in the model, is sometimes of great importance. In particular, it is usually desirable to choose the model such that it has the linear form (11) and thus avoiding more complex nonlinear control allocation methods. For vector thrust devices that can be commanded to produce thrust of varying amplitude and direction (in two or three degrees of freedom), the *extended thrust* formulation (Lindfors 1993, Sjørdalen 1997) leads to a linear model. Rather than parameterizing the thrust vector through its amplitude and some angle(s), its thrust components are decomposed on the body axes and treated as elements of the control input u . Depending on the actuator system, these components must usually be converted back to amplitude and angle(s) by suitable trigonometric transforms that describe the nonlinear relationships. Although this avoids a nonlinear model, the price to pay may be a more complicated formulation of amplitude and angular rate constraints in the control allocation formulation, e.g. (Johansen, Fuglseth, Tøndel & Fossen 2008).

2.1 Unconstrained linear control allocation

The main challenge of inverting the model (11) is that B is not a square matrix. Usually, for an over-actuated system, B will have full row rank (equal to $m < p$) and we will in general assume it has a non-trivial null space. This means there is an infinite number of vectors $u \in \mathbb{R}^p$ that satisfies (11) for any given $\tau \in \mathbb{R}^m$. The common way to deal with such extra freedom is to use generalized inverses (or pseudo-inverses), e.g. (Horn & Johnson 1985, Golub & van Loan 1983). Below, we present this approach in the context of minimizing a least-squares cost function.

Neglecting any saturation and rate constraints on the input u , and choosing for convenience a quadratic cost function that measures the cost of control action, leads to the control allocation cost function formulation

$$\min_{u \in \mathbb{R}^p} \frac{1}{2} (u - u_p)^T W (u - u_p) \quad \text{subject to } \tau_c = Bu \quad (12)$$

where $W \in \mathbb{R}^{p \times p}$ is a positive definite weighting matrix, and u_p is the preferred value of u . When B has full rank, this weighted least-squares problem has the following explicit solution

$$u = (I - CB)u_p + C\tau_c \quad (13)$$

where

$$C = W^{-1}B^T(BW^{-1}B^T)^{-1} \quad (14)$$

is a generalized inverse that can be derived from optimality conditions of (12) using Lagrange multipliers, see e.g. (Bordignon & Durham 1995, Virnig & Bodden 1994, Enns 1998, Snell, Enns & Garrard 1990, Durham 1993, Oppenheimer et al. 2010, Fossen & Sagatun 1991). For the special case $W = I$ and $u_p = 0$, the solution $u = B^+ \tau_c$ is defined by the Moore-Penrose pseudo-inverse, (Horn & Johnson 1985, Golub & van Loan 1983), given by $C = B^+ = B^T(BB^T)^{-1}$. Rank-deficiency of B means that no force or moment can be

generated in certain direction of the space \mathbb{R}^m where τ_c belongs. This means that all commands τ_c cannot be achieved, even without considering saturation. Although the mechanical design of the effectors and actuators will normally avoid a rank-deficient B -matrix, it might appear in special cases like singularities, effector or actuator failures, so the control allocation algorithm should be able to handle it in some applications. Several regularization methods could be applied, like a damped least-squares inverse

$$C_\varepsilon = W^{-1}B^T(BW^{-1}B^T + \varepsilon I)^{-1} \quad (15)$$

where $\varepsilon \geq 0$ is a small regularization parameter that must be strictly positive when B does not have full rank. Alternatively, a singular value decomposition (SVD) of the matrix $BW^{-1}B^T = U\Sigma V^T$ will characterize the directions where no generalized force can be produced, (Golub & van Loan 1983). The matrix $\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_p)$ contains the singular values. Inverting only the singular values that are non-zero (with some small margin $\delta > 0$), leads to the reduced rank approximation

$$\Sigma_\delta^{inv} = \text{diag}(\sigma_1^{-1}, \sigma_2^{-1}, \dots, \sigma_r^{-1}, 0, \dots, 0) \quad (16)$$

where r is the number of singular values larger than the regularization parameter δ , i.e. $\sigma_i \geq \delta$. This leads to the approximate inverse

$$C_\delta^{syd} = W^{-1}B^T V \Sigma_\delta^{inv} U^T \quad (17)$$

to be used instead of C in (13). The SVD can also be used when B has full rank, e.g. (Oppenheimer et al. 2010).

2.2 Constrained linear control allocation

The methods based on generalized inverses do not guarantee that constraints on the input $u \in \mathbb{U}$ are satisfied. A very simple solution to ensure satisfaction of saturation constraints (as well as rate constraints) is to saturate the u resulting from any of the unconstrained control allocation methods in Section 2.1. Obviously, this will normally mean that the allocated generalized force $\tau = B\text{Proj}_{\mathbb{U}}(u)$ may be different from the required/commanded force τ_c . Furthermore, the method does not guarantee that the allocated generalized force equals the required force whenever possible, or that the error between allocated and required generalized force is minimized in some sense whenever an exact allocation is not possible. In (Durham 1993) it is shown that no single generalized inverse (i.e. weight matrix W) can yield exact allocation whenever possible using simple saturation. Several constrained allocation methods are designed to give better solutions than simple saturation.

2.2.1 Redistributed pseudo-inverse and Daisy chaining

The first step of the redistributed pseudo-inverse method (see e.g. (Virnig & Bodden 1994, Shi, Zhang, Li & Liu 2010)) is

to solve the unconstrained control allocation problem, such as (12) (or a simpler version). If the solution satisfies the constraints, no further steps are needed. Otherwise, the unconstrained optimal vector u is projected onto the admissible set \mathbb{U} (i.e. saturated) to satisfy the constraints: $\bar{u} = \text{Proj}_{\mathbb{U}}(u)$. In order to reduce the gap between desired and allocated generalized forces, the unsaturated elements of the control vector \bar{u} are re-computed by solving a reduced problem using a reduced pseudo-inverse. More specifically, let $\bar{u} = (\bar{u}_C^T, \bar{u}_U^T)^T$ be decomposed into the saturated elements \bar{u}_C and unsaturated elements \bar{u}_U , and let $B = (B_C, B_U)$ be the associated decomposition of the B -matrix. Then $\bar{\tau}_C = B_C \bar{u}_C$ is the allocated generalized force due to the saturated controls, and the remaining controls u_U are redistributed by solving the redistribution equation

$$B_U u_U = \tau_c - \bar{\tau}_C \quad (18)$$

using the pseudo-inverse method. Then new elements of the sub-vector u_U may be saturated, and the redistribution procedure is repeated until either a feasible solution (that gives exact generalized force allocation) is found, or no further information improvement can be made. Although the method is simple, and often effective, it does neither guarantee that a feasible solution is found whenever possible, nor that the resulting control allocation minimizes the allocation error in some sense. There are examples (see e.g. (Bodson 2002)) that demonstrate that clearly sub-optimal control allocation can result.

The daisy chaining method (Adams, Buffington, Sparks & Banda 1994, Buffington & Enns 1996, Oppenheimer et al. 2010) offers a very simple alternative, but is often less effective than the above mentioned methods. This method groups the effectors into two or more groups that are ranked such that first the control allocation problem is solved for the highest prioritized group. If one or more effectors in that group saturates, the settings of the whole group is frozen. The gap between allocated and required generalized forces is then allocated by the second group. This is then repeated if a feasible solution is still not found, and there are more than two groups. Depending on the selected groups, this may lead to solutions where several effectors may not be fully utilized to minimize the allocation error, and can be sub-optimal compared to the redistributed pseudo-inverse.

2.2.2 Direct allocation

Some constrained control allocation methods are based on some scaling of the unconstrained optimal control allocation, such that the resulting control allocation is projected onto the boundary of the set of attainable generalized forces. In aerospace applications this is commonly referred to as the attainable moment set (AMS) since moments in 3-DOF are normally allocated. Here, the set of attainable generalized forces is denoted \mathbb{A} and is the set of vectors $\tau \in \mathbb{R}^m$ when the constrained optimization problem (e.g. (8)) has a feasible solution.

The direct allocation method (Durham 1993) starts with the unconstrained control allocation computed using some pseudo-inverse, e.g. $\tilde{u} = B^+ \tau_c$. If $\tilde{u} \in \mathbb{U}$ (i.e. satisfies the input constraints), no further steps are needed and we use $u = \tilde{u}$. Otherwise the method will search for another u that preserves the direction of τ_c but leads to an allocated generalized force Bu on the boundary of \mathbb{A} :

$$\max_{\alpha \leq 1} \alpha, \quad \text{subject to} \quad Bu = \alpha \tau_c, \quad \alpha \tau_c \in \mathbb{A} \quad (19)$$

where $\alpha \in [0, 1]$ is a scalar. Notice that when the set \mathbb{U} is polyhedral, then also \mathbb{A} is a polyhedral set. Solving the optimization problem (19) is not trivial for problems where the dimension of u is large, since there will be a significant amount of facets and vertices and it is not straightforward to identify which facet is intersected by the straight line from τ_c to the origin. Different numerical algorithms have been suggested, with different computational complexities. Improvements over the original algorithm (Durham 1993) are based on various data structures, enumerations and representations (Bordignon & Durham 1995, Durham 1994b, Durham 1994a, Durham 1999, Petersen & Bodson 2002) as well as linear programming (Bodson 2002, Oppenheimer et al. 2010).

2.2.3 Error minimization using linear programming

A powerful approach is to explicitly minimize the weighted error between the allocated virtual control input and the desired one. Extending the unconstrained optimization problem formulation (12) with input constraints leads to formulations such as (8). The constraint set \mathbb{U} is usually polyhedral, i.e. for some appropriate matrix A and vector b it can be represented as

$$\mathbb{U} = \{u \in \mathbb{R}^p \mid Au \leq b\} \quad (20)$$

Rate constraints \mathbb{C} can be formulated as a polyhedral set, too.

With the cost function defined using either 1-norm or ∞ -norm, this resulting problem is a linear program (LP) that can be solved using iterative numerical LP algorithms (e.g. (Paradiso 1991, Bodson 2002, Lindfors 1993, Bodson & Frost 2011)) by relatively straightforward reformulations into any of the standard LP forms via the introduction of additional variables. As an example, consider the 1-norm control allocation problem

$$\min_{u,s} \left(\sum_{i=1}^m q_i |s_i| + \sum_{j=1}^p w_j |u_j| \right) \quad (21)$$

subject to

$$Bu = \tau_c + s, \quad u_{min} \leq u \leq u_{max}, \quad \delta_{min} \leq u - u_\ell \leq \delta_{max} \quad (22)$$

With symmetric effectors and actuators, we have $u_{min} = -u_{max}$ and $\delta_{min} = -\delta_{max}$. Introducing auxiliary variables

$$s_i^+ = \begin{cases} s_i, & s_i \geq 0 \\ 0, & s_i \leq 0 \end{cases} \quad (23)$$

$$s_i^- = \begin{cases} -s_i, & s_i \leq 0 \\ 0, & s_i \geq 0 \end{cases} \quad (24)$$

$$u_i^+ = \begin{cases} u_i, & u_i \geq 0 \\ 0, & u_i \leq 0 \end{cases} \quad (25)$$

$$u_i^- = \begin{cases} -u_i, & u_i \leq 0 \\ 0, & u_i \geq 0 \end{cases} \quad (26)$$

we have $s_i = s_i^+ - s_i^-$, $|s_i| = s_i^+ + s_i^-$, $u_i = u_i^+ - u_i^-$, and $|u_i| = u_i^+ + u_i^-$. Stacking these variables into vectors s^+ , s^- , u^+ , u^- and defining $w = (w_1, \dots, w_p)^T$ and $q = (q_1, \dots, q_m)^T$ we get the following linear program

$$\min_{u^+, u^-, s^+, s^-} (w^T, w^T, q^T, q^T) \begin{pmatrix} u^+ \\ u^- \\ s^+ \\ s^- \end{pmatrix} \quad (27)$$

subject to

$$(B, -B, -I, I) \begin{pmatrix} u^+ \\ u^- \\ s^+ \\ s^- \end{pmatrix} = \tau_c \quad (28)$$

$$\begin{pmatrix} I & -I & 0 & 0 \\ -I & I & 0 & 0 \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \\ s^+ \\ s^- \end{pmatrix} \geq \begin{pmatrix} \max(u_{min}, \delta_{min} + u_\ell) \\ -\min(u_{max}, \delta_{max} + u_\ell) \end{pmatrix} \quad (29)$$

Other LP standard forms exists, and similar reformulations can be made, e.g. (Bodson 2002). The use of ∞ -norm will minimize the maximum effector use and therefore lead to a balanced use of effectors, (Frost, Bodson & Acosta 2009, Frost & Bodson 2010, Bodson & Frost 2011), and can also be reformulated into linear programs using similar augmentations with auxiliary variables.

The use of slack variables in the above formulations ensures that a feasible solution always exists. It should be mentioned that infeasibility handling can also be included via a two-level approach (e.g. (Oppenheimer et al. 2010, Bodson 2002)). While this might lead to reduced computational complexity on average, it may not contribute to reduced worst-case computational complexity that is usually the main concern in real-time implementations.

The most common numerical methods for linear programming are the simplex method, active set methods and interior point methods, (Nocedal & Wright 1999). The simplex method is studied for control allocation problems in (Bodson 2002), where the main conclusions are that the computational complexity is clearly within the capabilities of current embedded computer hardware technology. The simplex method iterates between vertices of the polyhedral set describing the set of feasible solutions, where at each iteration a system of linear equations corresponding to a basic solution is solved using numerical linear algebra. Since there is a finite number of basic solutions, the simplex algorithm is a combinatorial approach that finds the optimal solution in a finite number of iterations. The simplex algorithm usually beats the combinatorial complexity of the problem by trying to reduce the cost function at each iteration. Many numerically robust implementations of the simplex method exists, including portable C code, (Press, Flannery, Teukolsky & Vetterling 1992), which makes the approach fairly straightforward to apply in many embedded control platforms. However, there are some issues that may require special attention.

Although the simplex method tends to converge to the optimal solution within a number of iterations that is not bigger than the number of variables and constraints, (Press et al. 1992), it is hard to give a guaranteed limit on the number of iterations. Hence, the control allocation may have to accept some degree of sub-optimality since only a limited number of iterations may be allowed in a real-time implementation.

Degeneracies in the problem are characterized by constraints that are redundant, (Nocedal & Wright 1999). They may lead to non-uniqueness and singular linear algebraic inversion problems that in combination with numerical inaccuracies may require some additional considerations. Due to degeneracies, a change in the basic solution during one iteration may lead to a new basic solution where the cost remains the same. If particular care is not taken, a phenomena called cycling may arise, where the solver jumps back and forth between the same set of basic solutions forever without making any progress towards the optimum. As observed in (Bodson 2002), anti-cycling procedures are indeed needed since symmetric effectors may easily lead to degeneracies in the control allocation problem.

Fairly efficient general procedures are available to find a feasible initial point for starting the simplex method. In the control allocation problems above, the use of slack variables s makes it trivial to find a feasible initial point since there are no constraints on s , see also (Bodson 2002). Moreover, in control allocation problems, the solution from the previous sample is often a good initial guess for the current solution, since the problem parameters (including τ_c) often does not change significantly from one sample to the next. Still, one needs to have in mind that there may be exceptions due to failures or abrupt command changes. The number of iterations needed in the simplex algorithm may be reduced (at

least in average) by explicitly exploiting such information for warm start.

Optimal solutions of LPs are found at vertices of the feasible set. A consequence of this is that the LP-based method tends to favor the use of a smaller number of effectors, while methods based on a quadratic cost function and ∞ -norm tends to use all effectors, but to a smaller degree, (Bodson 2002). This seems to be the main reason why error minimization approaches are based on quadratic programming in most cases.

2.2.4 Error minimization using quadratic programming

With the common choice of 2-norm, the control allocation problems leads to a quadratic program (QP) that can be solved using numerical QP methods, e.g. (Härkegård 2002, Petersen & Bodson 2006, Petersen & Bodson 2005). As an example for such a formulation, consider the control allocation formulation

$$\min_{u,s} \left(\sum_{i=1}^m q_i s_i^2 + \sum_{j=1}^p w_j u_j^2 \right) \quad (30)$$

subject to

$$Bu = \tau_c + s, \quad u_{min} \leq u \leq u_{max}, \quad \delta_{min} \leq u - u_\ell \leq \delta_{max} \quad (31)$$

It can be transformed into a standard QP form without additional variables, where

$$\min_{u,s} \frac{1}{2} (u^T, s^T) H \begin{pmatrix} u \\ s \end{pmatrix} \quad (32)$$

subject to

$$(B, -I) \begin{pmatrix} u \\ s \end{pmatrix} = \tau_c \quad (33)$$

$$\begin{pmatrix} I, 0 \\ -I, 0 \end{pmatrix} \begin{pmatrix} u \\ s \end{pmatrix} \geq \begin{pmatrix} \max(u_{min}, u_\ell + \delta_{min}) \\ -\min(u_{max}, u_\ell + \delta_{max}) \end{pmatrix} \quad (34)$$

where $H = 2 \cdot \text{diag}(w_1, \dots, w_p, q_1, \dots, q_m)$. When all weights in the cost function are strictly positive (which they normally should be in a control allocation problem), the QP is strictly convex with H positive definite. With the use of slack variables, a feasible solution always exists and the problem always admits a unique optimal solution. It should be noted that several variations of the formulation can be made, including the use of a 1-norm for the slack variables which may have the advantage that with appropriate tuning the slack is zero whenever feasible, a property known as an exact penalty function, e.g. (Fletcher 1987).

QPs are usually solved using active set methods or interior-point methods, (Nocedal & Wright 1999), and both have been studied in the context of control allocation (Härkegård

2002, Petersen & Bodson 2005). An iterative fixed-point method has also been proposed and shown to be efficient, (Burken, Lu, Wu & Bahm 2001).

Active set methods are iterative methods, where at each iteration they improve their guess of the optimal active set, (Nocedal & Wright 1999). The optimal active set is the set of indices of the inequality constraints that are active (i.e. the inequality holds with equality) at the optimum. At each iteration, a search direction is computed based on the assumption of the working active set. The algorithm then searches for better solutions along this search direction, and either finds the optimum or detects how the working active set needs to be changed in order to make further progress towards the optimum. More specifically, consider the QP in standard form

$$\min_z \frac{1}{2} z^T H z, \text{ s.t. } Fz = \psi, Gz \leq \gamma \quad (35)$$

An active set algorithm for QP, as described in (Härkegård 2002), can be used to solve the control allocation problem at each sample.

Basic active set quadratic programming algorithm.

Initialization: Let z^0 be a feasible starting point for (35) (possibly based on the solution from the previous sample), and let the working active set \mathcal{W}^0 contain the indices of the active inequality constraints at z^0 . Set the iteration index $k = 0$.

Repeat:

- (1) Given z^k , find the optimal direction p^k by solving

$$\min_{p^k} \frac{1}{2} (z^k + p^k)^T H (z^k + p^k), \text{ s.t. } Fp^k = 0, G_{\mathcal{W}^k} p^k = 0$$

where $G_{\mathcal{W}^k}$ contains the rows of G indexed by the working active set \mathcal{W}^k .

- (2) If $z^k + p^k$ is feasible, then set $z^{k+1} = z^k + p^k$, and compute the vectors of Lagrange multipliers μ^k and λ^k associated with the equality and inequality constraints, respectively.
 - (a) If $\lambda^k \geq 0$, then z^{k+1} is the optimal solution. Terminate.
 - (b) Otherwise, remove the constraint associated with the most negative Lagrange multiplier in the vector λ^k from the working active set to define \mathcal{W}^{k+1} . Increment k and repeat.
- (3) Otherwise, apply a line search procedure in order to determine the maximum step $\alpha \geq 0$ such that $z^{k+1} = z^k + \alpha p^k$ is feasible. Add the primary bounding constraints to the working active set to define \mathcal{W}^{k+1} . Increment k and repeat.

□

Interior-point methods, on the other hand, replaces the inequality constraints with a barrier function that prevents the solution for going into the infeasible region, (Nocedal & Wright 1999). Newton's method is then applied to search towards the optimum of the unconstrained optimization problem resulting from this reformulation. For each iteration, the barrier function is reduced in order to allow the solver to approach the boundary of the feasible region in case the optimal solution is located there.

The active set methods tend to perform well in control allocation problems, (Härkegård 2002), while interior point methods have their advantage for larger-scale problems (Petersen & Bodson 2005). Active set methods have the advantage that their initialization can take advantage of the solution from the previous sample (known as warm start), which is often a good guess for the optimal solution at the current sample. This may reduce the number of iterations needed to find the optimal solution in many cases. Interior-point methods are generally initialized with points near the center of the feasible region and will always need a minimum number of iterations in order to converge due to the need to reduce the barrier function penalty in several steps. Warm start procedures are therefore difficult to implement for interior-point methods.

Like in LP, it is hard to give guarantees on the maximum number of iterations and computation time needed to find the optimal solution. Hence, some degree of sub-optimality may need to be accepted in order to respect limitations on computational resources in order to meet real-time computation constraints. Numerical challenges with degeneracies and cycling must be addressed also in QPs.

Several implementations of active set QP solvers were studied for control allocation problems in (Härkegård 2002), with fairly modest differences in computational complexity being observed. For non-real-time implementations, the Matlab toolbox QCAT (Härkegård 2004) is dedicated to QP-based control allocation and implements the methods of (Härkegård 2002, Petersen & Bodson 2005). For real-time implementation on embedded systems, there exists a few portable C code solvers such as the FORTRAN-to-C-converted active set solver QLD (Schittkowski 1986), the interior-point method automatic code generation tool CVXGEN (Mattingley & Boyd 2010), the active-set-like solver QPOASES (Potschka, Kirches, Bock & Schlöder 2010), and the conjugate gradient method, (Press et al. 1992). Recently, real-time certification and software for automated C-code generation of first-order (fast gradient) methods have become available, (Richter, Jones & Morari 2012).

2.2.5 *Explicit approaches to constrained error minimization*

Both linear and quadratic program formulations admits an explicit representation of the solution as a piecewise linear (PWL) solution function of τ_c . This PWL function is

determined using multi-parametric quadratic programming, (Johansen et al. 2005, Kvasnica, Grieder & Baotić 2004). Although the resulting PWL solution function may contain a large number of linear function pieces, each defined over a polyhedral region in \mathbb{A} , the online computations required to evaluate the PWL function are dramatically reduced by data structures such as binary search trees, (Johansen et al. 2005), and lattice representations, (Bayat, Johansen & Jalali 2012). Still, the online computer memory requirements may limit the applicability of this approach to problems where the B -matrix and constraint sets \mathbb{U} and \mathbb{C} are time invariant, and the requirements for fault tolerance and reconfigurability are simple, see e.g. (Spjøtvold & Johansen 2009). A decomposition approach that groups effectors into non-interacting groups can sometimes reduce the complexity through master- and slave-problems (Spjøtvold, Tøndel & Johansen 2006).

2.2.6 Dynamics and fault tolerance

An optimization-based control allocation method that is integrated with a parameter estimation scheme is described in (Casavola & Garone 2010). It leads to an adaptive solution that can accommodate an unknown time-varying B -matrix due to losses and faults. Control allocation is an effective approach to implement fault tolerant control. When effector or actuator faults are identified, they can be modeled as changes in the B -matrix of the constraints, or other parameters in the optimization problem. For example, an actuator that is locked in a faulty position could be systematically treated by setting the lower and upper constraint limits to the locked value. Alternatively, the preferred control vector could also be set to locked actuator values as proposed in (Schierman, Ward, Hull, Gandhi, Oppenheimer & Doman 2004). A systematic method that also changes the weights in the pseudo-inverse in order to ensure that faults are well distributed among the fault-free effectors without reconfiguring the high level controller is proposed in (Alwi & Edwards 2008).

A dynamic control allocation approach is presented in (Zaccarian 2009). It is designed to allocate the required control effort, while allocating the excessive degrees of freedom through a dynamic system that can be tuned for optimizing secondary objectives and constraints.

It is relatively straightforward to design a basic control allocation algorithm to comply with actuator rate constraints by incorporating this as a constraint or penalty on the change in control inputs from the previous sample to the current sample, see (7) and (Härkegård 2004). More sophisticated dynamic actuator models may be incorporated by using the MPC framework to solve the constrained control allocation problem (Luo, Serrani, Yurkovich, Doman & Oppenheimer 2004, Luo, Serrani, Yurkovich, Doman & Oppenheimer 2005, Vermillion, Sun & Butts 2007). MPC is an optimization-based control approach which can be used in control allocation, being able to handle actuator dynamics

as well as actuator saturation. MPC is a systematic design method that utilizes a model of the plant for predicting outputs and states. In control allocation this model describes the actuator dynamics. Using MPC the control allocation problem is solved on a future horizon, and the optimal solution is a future trajectory. Because of the predictive nature of the controller, the calculated control can pre-act to the actuator system dynamics to improve dynamic performance. On the negative side, MPC allocation requires significantly more computations than the static control allocation formulation since the number of optimization variables and constraints is a multiple of the prediction horizon, which may be a factor of 10-20 larger compared to the static problem. Still, it has been demonstrated in (Hanger, Johansen, Mykland & Skullestad 2011) that with efficient numerical QP solver software (Mattingley & Boyd 2010) that the real-time computations of MPC allocation with a linear dynamic actuator model can also be implemented with current off-the-shelf computer technology.

A similar strategy, that considers the current and past history of commanded virtual forces and moments, is given in (Venkataraman, Oppenheimer & Doman 2004). There, the control allocation problem is to solve for the control inputs so that some norm of the error between the achieved and desired moments is minimized. A computationally simpler strategy to include actuator constraints in the control allocation is via post-processing of the static control allocation, as proposed in (Oppenheimer & Doman 2004). The post-processor will over-drive the actuator in order to compensate for the dynamics of a first or second order linear actuator model.

3 Control allocation for nonlinear effector models

The general control allocation formulations (8) allows for a nonlinear effector model h , a non-quadratic cost function J , and constraint sets that are not polyhedral. Such nonlinear formulations may be necessary in order to achieve the desired performance, for example when time-varying linearization does not provide sufficiently accurate approximations, convex polyhedral sets do not accurately describe the effector limitations, or more complex cost functions are needed. Typical examples of nonlinearities are physically motivated cost functions, such as power consumption of a marine propeller that is a cubic function of its angular speed and a non-integer power function of its pitch angle, (Fossen 2011). Other examples are rudders-propeller pairs of ships, where the rudder is effective only when the propeller produces thrust forward, leading to non-convex constraint sets, (Johansen et al. 2008). Tires have highly nonlinear characteristics with saturations at high longitudinal and lateral slips, which may lead to nonlinear effector models, e.g. (Tjønnås & Johansen 2010). Nonlinearities are discussed further in Section 4 when describing application domains.

3.1 Nonlinear programming methods

The use of nonlinear programming for control allocation was proposed in (Poonamallee, Yurkovich, Serrani, Doman & Oppenheimer 2004, Johansen et al. 2004). In (Johansen et al. 2004) it was shown how formulations similar to (8) could be addressed by locally approximating the cost function by a quadratic cost function, and linearizing the constraints. This leads to a numerical method similar to sequential quadratic programming (SQP), except that the linear/quadratic approximation was made only once per sample. More specifically, consider the following nonlinear program resulting from a control allocation formulation

$$\min_{\zeta} J(\zeta), \text{ s.t. } F(\zeta) = 0, G(\zeta) \geq 0 \quad (36)$$

Algorithm: Sequential linearization and QP.

Initialize: Let ζ^0 be a feasible starting point for (36), possibly based on the solution from the previous sample.

Main computations:

- (1) Compute a 2nd order truncated Taylor expansion of J around ζ^0 using analytical or numerical differentiation.
- (2) Compute first order truncated Taylor expansions of F and G around ζ^0 using analytical or numerical differentiation.
- (3) Set $z = \zeta - \zeta^0$, and define the matrices of a QP approximation in the form (35) of the nonlinear program.
- (4) Solve the QP using an active set algorithm, and define an improved solution $\zeta^1 = \zeta^0 + z$.

□

In (Johansen et al. 2004), this strategy was studied in an example and shown to produce adequate results, with only minor increase in computational complexity (due to linearization and quadratic approximation) compared to a quadratic programming approach to linear control allocation. However, this conclusion can not be expected to generalize to arbitrary nonlinear control allocation problems. In particular, applications that require very large changes in allocated forces from one sample to the next may require several linear/quadratic approximations to be computed sequentially in order to achieve the necessary accuracy, see (Poonamallee et al. 2004). In a full SQP implementation, e.g. (Nocedal & Wright 1999), the linearization and QP steps in the above algorithm are repeated iteratively during one sampling instant until the optimality conditions are satisfied. With N iterations, the computation times would be roughly speaking N times the computation time of the sequential linearization and quadratic programming algorithm above.

Applications with strong nonlinearities may lead to non-convex cost or constraint functions such that the optimiza-

tion may get stuck in local minimums that may severely degenerate performance, or require additional computational mechanisms in order to find close to global optimal solutions. Unlike the linear control allocation case, there is little hope to find a general-purpose nonlinear programming algorithm and numerical software implementation for general nonlinear allocation problems.

While a general nonlinear optimization framework can accommodate any cost function, any model and any type of constraints, it is of great interest to study control allocation for specific classes of nonlinearities and constraints. By exploiting structural properties one may pursue the analysis of theoretical properties such as guaranteed convergence to optimal solutions without excessive amount of computations.

3.2 Mixed-integer programming methods

One particular model class leading to control allocation problem formulations that can be solved using mixed-integer linear programming (MILP) are piecewise linear functions (Bolender & Doman 2004a). Generally, a control allocation problem based on a piecewise linear effector model, piecewise linear cost function, and a constraint set that can be described as the non-convex union of polyhedral sets can be formulation as an MILP, see (Bemporad & Morari 1998, Heemels, Schutter & Bemporad 2001) for equivalence classes and how to formulate MILPs. While numerical MILP solvers are highly complex numerical software systems that may be difficult to verify and validate for use in a safety-critical real-time application, it can be noted that it has been demonstrated that simple enumeration methods in combination with numerical quadratic programming can be effective for solving practical non-convex control allocation problems where non-convex constraint sets are represented as the union of a small number of polyhedral sets, (Johansen et al. 2008, Ruth & Sørensen 2009).

3.3 Dynamic optimum-seeking methods

In (Johansen 2004) it was proposed to re-formulate the static nonlinear optimization formulations of control allocation, i.e. (5), (7), or (8), as a control Lyapunov-function and use constructive Lyapunov-design methods. In particular, it was assumed that the cost function $J'(x, u, t) = J(x, u, t) + \rho(u)$ was augmented with a barrier or penalty-function $\rho(\cdot)$ in order to enforce that input constraints $u \in \mathbb{U}$ are satisfied. Then the corresponding Lagrange function is formulated, with $\lambda \in \mathbb{R}^m$ being the vector of Lagrange multipliers

$$L(x, u, t, \lambda) = J'(x, u, t) + \lambda^T (\tau_c - H(u, x, t)) \quad (37)$$

Assuming that a Lyapunov function $V_0(x, t)$ for the high level motion control algorithm exists, the following control Lyapunov function is defined for the control allocation design

$$V(x, u, t, \lambda) = \sigma V_0(x, t) + \frac{1}{2} \left(\frac{\partial L^T}{\partial u} \frac{\partial L}{\partial u} + \frac{\partial L^T}{\partial \lambda} \frac{\partial L}{\partial \lambda} \right) \quad (38)$$

for some $\sigma > 0$. Requiring a negative time-derivative of V along trajectories of the system, one can derive a control allocation update algorithm on the form

$$\dot{u} = -\Gamma\alpha + \xi, \quad \dot{\lambda} = -K\beta + \phi \quad (39)$$

where Γ and K are symmetric positive definite gain matrices, and α, ξ, β and ϕ are signals defined in (Johansen 2004). The control allocation update law (39) will asymptotically track the optimal control allocation (assuming feasibility), while guaranteeing not to destabilize the closed-loop system. Notice that the latter is not an obvious feature due to the fact that this dynamic control allocation is only asymptotically optimal, and may deviate at every time-instant from the instantaneous optimal allocation of the corresponding static control allocation problem. This is leading to some loss of performance as shown in a case study in (Tavasoli & Naraghi 2011). The main advantage of the method is that no direct numerical optimization is needed (optimality tracking is built into the dynamic update law (39)) leading to modest computational complexity. Disadvantages of the method include possible convergence problems in case of non-convex cost function and constraints, similar to the nonlinear programming approach.

Actuator rate constraints can in some case be enforced and implemented by choosing the gain Γ sufficiently small, although there is no guarantee that they can be met if ξ is not small when the high level motion control algorithm requires fast changes in the virtual control that cannot be implemented with the given actuator system. An extension that leads to convergence to optimal control allocation in finite-time was proposed in (Liao, Lum, Wang & Benosman 2007), and the effects of internal dynamics and minimum phase properties when using dynamic inversion high level motion controllers were studied in (Liao et al. 2007, Benosman, Liao, Lum & Wang 2009).

The concept relies on a control Lyapunov-function, which allows for certain extensions to be made within the same framework. An adaptive approach where uncertain parameters θ in the effector model $h(u, x, t, \theta)$ are stably adapted using an adaptation law that is designed by augmenting the control Lyapunov-function in a standard way was proposed in (Tjønnås & Johansen 2005, Tjønnås & Johansen 2008). This framework was further extended to dynamically account for actuator dynamics within the control allocation in (Tjønnås & Johansen 2007b, Tjønnås & Johansen 2007a), and internal dynamics in the context of model reference adaptive control (Liao, Lum, Wang & Benosman 2009b, Liao, Lum, Wang & Benosman 2009a).

3.4 Direct nonlinear allocation

An extension of the method of attainable moment set computations and direct allocation for nonlinear effector models can be found in (Bolender & Doman 2004b). The methods rely of nonlinear programming and the ideas in this paper

can be traced back to (Doman & Sparks 2002, Venkataraman & Doman 2001).

4 Applications

4.1 Aerospace

In this section we consider aircraft and spacecraft control allocation problems separately.

4.1.1 Aircraft

In flight control applications the virtual control input τ is usually the moments about the roll, pitch and yaw axes. Conventional fixed-wing aircraft design and flight control systems are based on a relatively small number of control surfaces (effectors) that are dedicated to control each axis of the aircraft, like

- ailerons for roll control,
- an elevator for pitch control, and
- a rudder for yaw control.

The grouping of two or more control surfaces into a single effector by constraining them to move together is common in flight control, and is often called ganging, e.g. (Oppenheimer et al. 2010). Typically, left and right ailerons are constrained to deflect differentially, while right and left elevators are constrained to deflect equally. Assuming the actuators and effectors are fault-free and the above ganging scheme is used, even with five control surfaces there are only three effective effectors available to control the three axes and control allocation is not needed. However, many current aircraft designs have a larger number of control surfaces that can be used during normal or special conditions, such as vertical take-off-and-landing, or after failure of an actuator or effector. Depending on the type of aircraft, one may have many more effectors including

- V-tails that give coupled lateral and longitudinal forces,
- control surfaces like flaps, spoilers, and slats,
- tiltable propellers, and thrust vector jets

Control allocation is widely used with such designs in order to ensure optimal use of the effectors, including fault tolerant and robust control over a wide flight envelope, (Burken et al. 2001, Davidson, Lallman & Bundick 2001, Bolling & Durham 1997, Durham 1993, Huang, Liu & Zhu 2009). It is concluded in (Burken et al. 2001) that although simulations demonstrate success of the conventional flight control approach in many cases, the control allocation approach appears to provide uniformly better performance in all cases.

Effector models for aerospace applications are usually assumed in the linear form (11). As discussed in Section 2, the control effectiveness matrix B may depend on slowly varying variables such as altitude and velocity, and is therefore

scheduled as a function of these variables. It is also worthwhile to remark that nonlinear effector models tend to be better approximated using an affine model $\tau = Bu + b_0$ instead of the linear model (11), see (Doman & Oppenheimer 2002). The extensions are straightforward, so most control allocation designs proceed without loss of generality with the model (11).

All the (constrained) linear control allocation methods described in Section 2 are commonly found in the flight control literature, e.g. (Oppenheimer et al. 2010). The models and constraints are generally given by the physical characteristics of the effectors and actuators, while the choice of u_p and weighting matrices may reflect different objectives such as

- minimum wing loading,
- minimum control surface deflection,
- minimum radar signature,
- minimum drag,
- maximum lift, and
- rapid reconfigurability for fault tolerance,

and others, e.g. (Yang, Zhong & Shen 2009). Using the pseudo-inverse solution as a preference vector u_p allows one to analytically represent the control allocator in a robustness analysis of the system that is valid as long as no single axis is saturated and the commanded accelerations are feasible, (Schierman et al. 2004). This facilitates the verification and validation process that must be completed prior to flight testing when using optimization based control allocation methods.

A comprehensive comparison of performance of several state-of-the-art linear control allocation methods are provided by (Bodson 2002). One main conclusion is that the optimization-based methods tend to outperform the alternative methods proposed in the literature both in terms of avoiding unnecessary infeasibility and minimizing the use of control effort. The 2-norm (quadratic) formulation seems to be favorable over the 1-norm (linear) formulation since the solution tends to combine the use of all control surfaces (rather than just a few), (Petersen & Bodson 2006). The use of ∞ -norm will minimize the maximum effector use and therefore lead to a balanced use of effectors, which also has advantages for robustness to failure and nonlinearities, (Frost et al. 2009, Frost & Bodson 2010, Bodson & Frost 2011).

The underlying motivation for the direct control allocation method, (Durham 1993), is that in many applications (in particular aircraft) it is considered important to keep the direction of the allocated forces and moments equal to the command, in order to get graceful degradation of performance and handling qualities. Hence, it is practically motivated by a different objective than merely minimizing the error in allocated generalized forces, which may be important in some flight control applications.

Although the control allocation problem is in most cases decoupled from the high level motion control design, there

may be cases when the interactions between control and control allocation should be studied or accounted for. This includes cases when actuator dynamics is significant, e.g. (Oppenheimer et al. 2010), or the control allocation influences the zero-dynamics due to inversion-based control, e.g. (Buffington & Enns 1996, Buffington et al. 1998). An integrated approach to flight control and control allocation design is investigated in (Jung, Lowenberg & Jones 2006), while (Bosković & Mehra 2002) describes two control allocation methods that adapt the weight matrices of a pseudo-inverse like control allocation law in order to avoid saturation and rate constraints. Control allocation methods that explicitly take into account linear actuator dynamics have been proposed, using linear constrained MPC, (Luo et al. 2004, Luo et al. 2005, Luo, Serrani, Yurkovich, Oppenheimer & Doman 2007), or optimization in the framework of linear matrix inequalities (LMIs), (Kishore, Sen, Ray & Ghoshal 2008). In order to reduce effective time delay due to actuator rate limitations in case of quickly changing commanded moments, and thereby reduce the risk of pilot induced oscillations, (Yildiz & Kolmanovsky 2011) proposed and studied a control allocation method that also penalizes the difference between the time-derivatives of the commanded and allocated moments.

A linear programming approach to control allocation that accounts for interaction between control effectors due to aerodynamic couplings are studied in (Oppenheimer & Doman 2007). Constrained control allocation using nonlinear effector models have been studied using numerical nonlinear programming methods, (Ma, Li, Zheng & Hu 2008, Davidson et al. 2001).

Reconfiguration of the control allocation, via weight modifications (Zhou, Zhang, Rabbath & Theilliol 2010) or adaptation of effector model (Zhang, Suresh, Jiang & Theilliol 2007), was considered in in order to manage faults. Methods for stable adaptation of parameter uncertainty in the effector models due to failures, and associated control allocation strategies that will group effectors into a smaller set equivalent effectors, similar to ganging or daisy chaining, is studied in (Liu & Crespo 2010). Model predictive control is shown to be a powerful tool to model failures and a suitable basis for fault tolerant control allocation in (Joosten, van den Boom & Lombaerts 2008).

4.1.2 Spacecraft

Spacecraft have other actuators and effectors, either instead of, or in addition to control surfaces. These include reaction control jets, and reaction wheels. In addition, the energy consumption for control is generally a high priority objective of the control system design of spacecraft and will often need to be strongly considered in the control allocation strategy as well.

Flight tests with the Boeing X-40A reusable launch vehicle using reconfigurable control allocation based linear programming is reported in (Schierman et al. 2004). In case of

faulty locked actuators, it is in (Schierman et al. 2004) proposed to set the associated elements of the preferred control vector u_p to the locked actuator positions.

Fault tolerant control allocation for a planetary entry vehicle was investigated in (Marwaha & Valasek 2008), using mixed-integer linear programming of handle the quantized/discrete nature of pulsed reaction control jets. (Doman, Gamble & Ngo 2009) propose a control allocation approach to optimally combine the use of (discrete) pulsed reaction control jets with (continuous) control surfaces in spacecraft transitions from exo-atmospheric to endo-atmospheric flight, using mixed-integer linear programming.

Satellite systems often have redundant thrusters, where it is desirable to minimize energy consumption during a maneuver or attitude control. A linear programming control allocation approach is investigated in (Jin, Park, Park & Tahk 2006), where it is shown that it can reduce energy consumption compared to a simpler grouping strategy. A multi-saturation based model for a highly non-linear allocation function of micro-thrusters in a satellite is presented in (Boada, Prieur, Tarbouriech, Pittet & Charbonnel 2010). Quadratic programming is used for constrained thrust allocation of redundant satellites in (Fu, Cheng, Jiang & Yang 2011), with particular emphasis on fault tolerant reconfigurable control.

4.2 Ships, underwater vehicles and offshore vessels

In this section we first consider surface vessels in low-speed regimes where control surfaces are not effective unless actively excited by some generated flow. Next, we consider surface vessel maneuvering at high speed where control surface lift becomes significant, before we survey control allocation in multi vessel operations and underwater vehicles.

4.2.1 Station-keeping and low-speed maneuvering

Several types of ships and specialized vessels, such as semi-submersible platforms used in the petroleum industry, depend on thrust allocation control systems during certain modes of operation. This is in particular the case during dynamic positioning operations that include station keeping and low-speed maneuvering using joystick control or automatic tracking functionality, (Fossen 2011, Sørensen 2011). Such control systems often control the vessel in three degrees of freedom (surge, sway and yaw) and command the required surge and sway forces as well as yaw moment to the thrust allocation system. Dynamic positioning operations include drilling, offloading of cargo or petroleum at oilfield installations, pipe-laying, cable laying, seismic data acquisition, dredging, fire fighting and rescue, construction, diving support, and others. Thrust allocation for low-speed maneuvering is used for vessels ranging from cruise vessels, ferries, and tankers to smaller yachts, research and fishing vessels.

The thruster system can be implemented using various thrust producing devices that are effective in the low-speed regime:

- Main propellers gives positive or negative force in the longitudinal direction only, and possibly a small yaw moment if mounted off the longitudinal axis. The propeller thrust is usually controlled through its angular speed, a variable pitch angle, or both.
- Main propellers with rudders gives positive or negative force in the longitudinal direction. In addition, the rudder angle can be controlled to produce lateral forces and yaw moment when the propeller thrusts forward since the propeller slipstream is directed to flow at high speed past the rudder surface and can therefore produce a significant lateral force. When the propeller thrusts backwards, the rudder is not effective.
- Tunnel thrusters are propellers mounted in the lateral direction in tunnels through the ship hull. They produce lateral forces and yaw moments.
- Azimuth thrusters are propellers that can be turned to produce thrust in any direction in the horizontal plane. The propeller thrust is usually controlled through its angular speed, a variable pitch angle, or both. Since they are vector thrust devices, an azimuth thruster has two-degrees-of-freedom for the control allocation.
- Water jets and other propulsion devices and control surfaces are less commonly used.

Thrusters are commonly powered by electricity distributed from a power plant that may comprise one or more diesel-engine or gas turbine electric generators. Main propellers are sometimes directly driven by the engine. Safety and operational requirements require a high degree of redundancy to achieve the necessary fault tolerance. Typical requirements is that operations can continue uninterrupted for some time to allow them to be aborted safely after major failures such as loss of a single thruster, single generator set, a single electric switchboard, or a single engine room due to fire or flooding in a single compartment. Often, the worst case single point failure is loss of half of thrust capacity due to a switchboard short circuit failure, or fire or flooding in a machine room. Advanced vessel design with high redundancy tend to have four to eight thrust producing devices, where some are azimuth thrusters with two independent degrees-of-freedom for control. The thrust allocation algorithm therefore has many degrees of freedom in order to be capable to handle critical failures. The thruster system capacity is usually designed based on vessel capability requirements to withstand environmental forces such as wind, waves and currents, (Ruth & Sørensen 2009). Usually, wind loads are dominating.

Thrust allocation objectives and constraints that are commonly accounted for, (Fossen 2011, Johansen et al. 2008, Lindfors 1993, Webster & Sousa 1999, Sjørdalen 1997, Sørensen & Ådnanes 1997, Berge & Fossen 1997, Ruth, Sørensen & Perez 2007), include the following:

- Surge, sway and yaw control, usually with a priority on the yaw axis since loss of heading will usually imply loss of position under heavy wind conditions since ships are designed for minimum wind loads when heading up

against the wind.

- Thrusters have individual capacity constraints due to their power rating, but may also have coupled constraints if limited by the electric power available on a shared power bus.
- Rate constraints are generally important for the turning of azimuth thrusters and rudders' steering machine.
- Minimization of fuel consumption.
- Minimization of tear-and-wear on thrusters and generator sets due to time-varying control commands that must respond to the motion of the vessel caused by wind and waves.
- Avoiding too high variations in electric power consumption that may cause blackout due to over- or under-frequency protection of the weak electric power grid on an isolated ship or vessel.
- Sector constraints are sometimes imposed on azimuth thrusters in order to protect equipment (like subsea equipment lowered through a moon pool, or hydro-acoustic transceivers used for positioning), divers in the water, or to avoid thrust losses in nearby thrusters due to interactions caused when directing the slipstream of one thruster into the propeller disc of another thruster.
- Thrusters may be disabled and enabled dynamically in order to guarantee fault tolerance and operational flexibility.

Industrial solutions are described in (Jenssen 1980, Sjørdalen 1997, Sørensen & Ådnanes 1997, Jenssen & Realfsen 2006). A static QP-based strategy is described in (Jenssen 1980), while the method in (Sjørdalen 1997) utilizes pseudo-inverses, in combination with the extended thrust concept (Lindfors 1993). The control vector u consists of the horizontal plane thrust vector decomposed in the vessel xy -axes (horizontal plane) in order to allow linear models also with azimuth thrusters. In (Sjørdalen 1997), constraints are handled by saturation strategies in combination with filtering of azimuth angle commands that also serves the secondary objective of reducing thruster tear and wear, and the singular value decomposition is used to handle cases when temporary controllability is weak due to all thruster being aligned and can produce thrust all in the same direction. The interactions between the thrust allocation and low level thruster control strategies are studied in (Sørensen & Ådnanes 1997), which is particularly important in extreme seas where thrust losses can be large when the propeller ventilates and in-and-out-of-water effects may lead to propeller spin if properly addressed, (Smogeli & Sørensen 2009, Ruth, Smogeli, Perez & Sørensen 2009). Even at fairly low speeds, tunnel thrusters will significantly reduce their effect, which should be accounted for in scheduling of the control effectiveness matrix B , (Godhavn, Fossen & Berge 1998).

A practical strategy that explicitly optimize the thrust allocation in order to account for power generation constraints, variations in loads, and operational desires such as balancing the load on different electric bus segments and switchboards is described in (Jenssen & Realfsen 2006). An integrated approach to dynamic control of power plant as part of the thrust allocation strategy is studied in (Veksler, Johansen

& Skjetne 2012a, Veksler, Johansen & Skjetne 2012b), inspired by (Radan, Sørensen, Ådnanes & Johansen 2008) who studied the stabilizing effect on the electrical power plant. An similar industrial thrust allocation approach implementation with dynamic load control and prediction is described in (Mathiesen, Realfsen & Breivik 2012).

The allocation of control to rudders is particularly challenging due to their highly asymmetric characteristic (no effect when the propellers thrust backwards). Optimization-based approaches that consider the finite (usually small) number of combinations of propeller thrust directions have been proposed and successfully tested, (Lindegaard & Fossen 2003, Johansen et al. 2008, Johansen, Fuglseth, Tøndel & Fossen 2003). Similar strategies can be used for general non-convex thruster constraints, (Ruth & Sørensen 2009), e.g. due to forbidden sectors being less than 180 degrees.

In special situations, e.g. when there are primarily azimuth thrusters in use, an additional objective of thruster configuration singularity avoidance might be useful in order to avoid temporary loss of controllability when all thrusters point in more or less the same direction (Sjørdalen 1997, Johansen et al. 2004, Scibilia & Skjetne 2012).

Station keeping of ships and semi-submersible platform for long periods of time are sometimes implemented using mooring lines with thruster assisted position and heading control. This is commonly used for drilling units and floating production, storage and offloading units (FPSOs) operating in water depths of less than 500 meters. The thrust allocation must take into account the mooring line forces and provide assistance when needed to make corrections, for example in strong winds or after a mooring line break, e.g. (Nguyen & Sørensen 2009).

Control allocation for small-waterplane marine constructions such as semi-submersibles can obtain additional roll and pitch damping using a conventional thruster system. This is possible for constructions with large draft and beam relative the length since controllability depends on moment arms in roll and pitch (Sørensen & Strand 2000). In this case the thrust allocation scheme should not only allocate forces and moment in the horizontal plane (surge, sway and yaw), but also allocate moments in roll and pitch.

4.2.2 High-speed maneuvering and ship autopilots

Ship autopilots conventionally use rudders to meet heading control objectives, while they may also use additional control surfaces such as fins and azimuth propellers (azipods) which calls for control allocation solutions. It is also possible to use rudders for roll damping alone or in combinations with controllable fins (see (Perez 2005) and references therein). The penalties for the use of rudders and fins must be included in the control objective together with penalties and criteria for accurate steering and roll damping. This is an over-actuated control allocation problem except for ships

equipped with one single rudder for simultaneously heading and roll damping, that is a under-actuated rudder-roll damping system that depend on frequency separation of these functions with rudder-roll damping at high frequencies (see (Fossen 2011) and references therein).

Severe instances of parametric rolling of ships can be avoided by specifying the control objectives of the speed and heading autopilots such that the frequency of excitation is changed via the Doppler-shift of the encounter frequency (Holden, Breu & Fossen 2012). The optimal frequency is found by using MPC or extremum seeking control, and nonlinear control allocation is used to compute the desired speed and heading angle based on a penalty function designed such that the encounter frequency never is equal to two times the natural frequency in roll, (Breu, Feng & Fossen 2012). This is the condition for parametric resonance.

4.2.3 Multi vessel operations

Control allocation strategies have also been proposed for multi-vessel operations, where several tug-boats cooperatively generate forces and moments in order to tow a floating structure. This is formulated in a straightforward manner in the control allocation framework by incorporating the constraints on the tug-boats capacity and direction, (Esposito, Feemster & Smith 2008, Feemster & Esposito 2011). Its implementation requires a supervisory strategy that coordinates the tub-boats that operate as effectors/actuators in this framework.

4.2.4 Maneuvering of underwater vehicles

Highly maneuverable underwater vehicles, either ROVs (Remotely Operated Vehicles) or AUVs (Autonomous Underwater Vehicles), are often controlled using compact electrically driven thrusters and fins. The thrust allocation problem is similar to a dynamically positioned surface vessel, including cases when also vertical forces are controlled using thrusters in addition to buoyancy control. Commonly used methods include pseudo-inverses, redistributed pseudo-inverses or simple optimization formulations, (Fossen & Sagatun 1991, Indiveri & Parlangeli 2006). Aspects of fault-tolerant control by saturation mechanisms and appropriate weighting of the pseudo-inverse is studied in (Sarkar, Podder & Antonelli 2002).

4.3 Automotive and ground vehicles

This section starts with an overview of yaw stability control allocation, before considering electric vehicles, rollover prevention systems, and mobile robots.

4.3.1 Yaw stability control

Active safety systems like the electronic stability control (ESC) are now common in production cars, and shown to

have tremendous life-saving effect when skids may occur due to evasive maneuvers, slippery surface, or too high speed in curves, e.g. (Lie, Tingvall, Krafft & Kullgren 2005, Ferguson 2007). The ESC detects deviation between the actual lateral motion of the vehicle and the drivers intention, usually by comparing the vehicle's lateral acceleration and yaw rate with information computed from the steering wheel angle commanded by the driver. In case of a significant difference, the ESC will automatically take action to counteract skidding by actuating a yaw moment to correct the skidding motion of the case, (van Zanten 2000, Mokhiamar & Abe 2002). In most vehicles, the four brakes are actuated independently in order to set up such a moment, possibly in combination with engine torque reduction. Increasing the longitudinal wheel slip by setting up a longitudinal braking force will effectively reduce the lateral friction forces, and both these phenomena contribute to generate a change in the yaw moment.

This leads to a control allocation problem, where the constrained forces and moments generated by the four brakes must be coordinated to generate the desired yaw moment while at the same time minimize other forces generated by the tires in order to avoid unintended side effects or discomforting the driver. For small control actions, the main challenge is that brake forces are uni-directional. However, for large control actions, the problem is much more challenging since the tire nonlinearities due to the saturating characteristics of tire friction forces must be taken into account. This saturation level depends strongly on the ground surface and tires, both being uncertain time-varying properties to the control system. Moreover, anti-lock braking systems (ABS) may be activated and act as a limitation to the achievable performance, since the ABS may limit the longitudinal slip in order to maintain high lateral friction, which may be a conflicting objective in some cases when lateral stability is lost. Furthermore, the load distribution on the tires may be far from even due to large accelerations, and this must also be considered when allocating forces to each wheel. In order to maximize the region of stability, these nonlinear effects are important to consider.

Some electronic stability control systems also use active steering to manipulate the yaw moment, where an electric motor on the steering column may add actuation in addition to the driver's command, (Ackermann & Bunte 1996, Ackermann 1996). There has also been proposed systems that have additional redundancy by combining active steering and active braking, (Yu & Moskwa 1994, Guvenc, Acarman & Guvenc 2003, Wang & Longoria 2006), which also achieves additional control authority and an opportunity to enhance the region of stability.

Due to the strong nonlinearities and dynamic constraints, the use of nonlinear constrained control allocation techniques will generally be desired for lateral vehicle control. However, in order to avoid the online computational burden of nonlinear programming, several simplified approaches have been proposed. The effector mapping from longitudi-

nal tire slips and slip angles are linearized in (Wang, Solis & Longoria 2007) and an accelerated fixed-point iteration algorithm is studied as a computationally efficient alternative to quadratic programming. (Plumlee, Bevely & Hodel 2004). A commonly used control allocation objective is to minimize friction forces, for example the adhesion potential characterized using friction ellipse models for each individual tyre, (Knobel, Pruckner & Bünte 2006). Using linearization of the model $\tau = G\varphi(u, x, \theta)$, where θ are time-varying parameters, the constrained least-squares problem of allocation error minimization can be solved using numerical online quadratic programming (Andreasson & Bünte 2006, Tagesson, Sundström, Laine & Dela 2009). The effect of vehicle handling performance of weighting matrix coefficients on the control allocation performance is studied in (Mokhiamar & Abe 2004, Mokhiamar & Abe 2006), when using a control allocation method that assumes unconstrained optimization. Nonlinearities and uncertainty is with these approaches handled within low-level actuator/effector controllers that can also provide time-varying constraint limits (such as estimated maximum tire/road friction coefficient) to the control allocation.

A nonlinear programming approach to nonlinear constrained control allocation for yaw stabilization is taken in (Tøndel & Johansen 2005), where computational efficiency is achieved through an approximative multi-parametric nonlinear programming algorithm that pre-computes a piecewise linear function that can be evaluated online using binary search tree data structures in order to approximate the optimal solution. The nonlinear optimizing control allocation method of (Hattori, Koibuchi & Yokoyama 2002, Ono, Hattori, Muragishi & Koibuchi 2006) minimizes work load of each tire, assuming all wheels can be actuated independently with respect to steering and brake/traction. Theoretical convexity properties of the optimization problem are studied.

Fault tolerant control with respect to brake failures is studied in (Hac, Doman & Oppenheimer 2006), where the main objective of the linear programming based control algorithm during the failure mode is to redistribute the control tasks to the functioning actuators, so that the vehicle performance remains as close as possible to the desired performance in spite of a failure.

The dynamic adaptive nonlinear control allocation method is studied for yaw stabilization in (Tjønnås & Johansen 2010) (see section 3.3), where a combination of braking and front-wheel steering is used for actuation. Estimation of maximum tire-road friction coefficient is an integral part of the adaptive control allocation strategy. In (Tavasoli & Naraghi 2011), the performance of the method is further compared to a (static) nonlinear programming approach. A simpler gradient-based dynamic control allocation approach is shown to be effective in (Gerard & Verhaegen 2009). It should also be mentioned that model predictive control designs, incorporating dynamics vehicles models and actuators models, are effective methods for solving the combined motion control and control allocation problem for vehicle dynamics control, e.g.

(Borrelli, Falcone, Keviczky, Asgari & Hrovat 2005), although the computational complexity is much more of a challenge than with a control allocation design.

4.3.2 *Electrical propulsion*

Electrically powered ground vehicles may have in-wheel electric motors that combine drive and regenerative brake functions, possibly in combination with friction brakes, and the trend is towards highly over-actuated vehicles, (Valásek 2003), with extended stability, (Jonasson, Andreasson, Solyom, Jacobson & Trigell 2011). Control allocation can be used to coordinate the electric motors in individual combinations of drive/brake mode while optimizing energy efficiency. The method proposed in (Fredriksson, Andreasson & Laine 2004, Chen & Wang 2011) utilizes linear/quadratic approximations to formulate an approximate control allocation problem that can be solved numerically online with computational efficiency.

4.3.3 *Rollover prevention*

Although yaw-stabilizing ESC does not explicitly consider the risk of roll-over, it is widely acknowledged that an ESC function will reduce the risk of rollover since it tends to reduce lateral accelerations that is a main cause of rollover accidents. Further enhancements can be achieved if rollover prevention functionality can be considered as an integral part of the vehicle dynamics control system. Using brake and steering actuators, the control allocation approach is extended by incorporating roll moment allocation together with yaw moment allocation in the control allocation, (Lu, Messih, Salib & Harmison 2007, Gerdes & Carlson 2003). Using polyhedral approximation of the friction ellipsoids, a quadratic programming approach is taken to allocation error minimization in (Schofield, Hägglund & Rantzer 2006). In (Alberding, Tjønnås & Johansen 2009), the dynamic nonlinear control allocation approach of (Tjønnås & Johansen 2010) is extended to include rollover prevention objectives.

Active suspension actuators have also been proposed for the control of ground vehicle yaw dynamics (Wang, Crolla, Wilson & Xu 2005), although they are not currently common in production vehicles.

4.3.4 *Mobile robots*

Traction control for mobile robots that operate off-road is considered in (Waldron & Hubert 1999, Waldron & Abdallah 2007). Based on the geometry of the problem, several simple (unconstrained) closed-form computationally efficient control allocation strategies for wheeled and legged mobile robots with active suspension are derived and compared in (Waldron & Abdallah 2007), while pseudo-inverse type control allocation strategies are studied in (Waldron & Hubert 1999, An & Kwon 2010). A linear programming solution to control allocation for wheeled mobile robots is

presented in (Feng, Xu, Li & Sun 2010). Nonlinearities, uncertainty and additional complexity in these approaches is to a large extent handled in the low-level controllers at each actuator/effector.

4.4 Other application areas

Legged walking robots require coordination of the dynamic or periodic motion of each leg. The force distributing control allocation algorithm should take into account energy-efficiency and contact friction between the leg and ground which is non-zero only for a fraction of a cycle, e.g. (Klein & Chung 1987, Sreenivasan, Waldron & Mukherjee 1996). A nonlinear programming approach, simplified by a pseudo-inverse calculations of initial solution guess, is presented in (Jung & Baek 2000). The control effectiveness matrix B is time-varying due to the cyclic contact patten of the walk.

Control allocation has been used in the development flapping wing micro air vehicle control methods. In (Oppenheimer, Doman & Sighthorsson 2011) they describe a method to control 5 degrees of freedom using two physical actuators that drive flapping wings. The six variables parameterize the periodic motion of two independently flapping wings that in turn control five degrees of freedom.

Multi-agent swarms (like formations of mobile robots) are considered in a fairly general context in (Pedrami, Wijendra, Baxter & Gordon 2009). Control allocation strategies based on pseudo-inverses and nonlinear programming are investigated.

The control allocation problem when using a large-scale distributed array of air-jet actuators is studied in (Jackson, Fromherz, Biegelsen, Reich & Goldberg 2001, Fromherz & Jackson 2003). In order to achieve computational efficiency to allow real-time implementation at high update frequencies in case of thousands of independent actuators, they empirically compare optimal solutions with approximate solution that depend on hierarchical decomposition into actuator groups. A similar approach was taken in (Singla & Junkins 2007), where a hierarchical decomposition and re-parameterization using basis-function leads to computational complexity reduction of the control allocation computations. These methods also allows parallelization so the algorithm can be distributed on multiple processors.

Over-actuated mechanical design are increasing in popularity in automotive, aerospace and maritime industries, and not only humanoid walking robots, but emerging concepts like robotic snakes (e.g. (Liljebäck, Stavadahl, Pettersen & Gravadahl 2011)), and robotic fish (e.g. (Liang, Wang & Wen 2011)) with highly redundant and over-actuated bio-inspired locomotion mechanisms will for sure benefit from further research on control allocation.

5 Conclusions

Control allocation is fairly well understood and established for linear models. Numerical optimization seems to offer many advantages, while their main challenge is the numerical implementation, in particular computational complexity, verification and validation.

For nonlinear models, the numerical optimization approach is also highly promising, and less alternatives exist. It is, however, much more challenging than with linear models due to the possibility of local minimums in addition to the even stronger challenges of computational complexity, numerical sensitivity, verification and validation. Alternative formulations, including asymptotically optimal methods based on Lyapunov-design have been proposed, and the area of nonlinear control allocation is currently an active area of research.

The main driving force of nonlinear control allocation research seems to be applications in the automotive and aerospace industries, and other emerging areas where mechatronics is used.

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