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Order Reduction and Output Feedback Stabilization of an Unstable CFD Model

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Abstract

This paper extends earlier results on (optimal) control of nominally stable models in computational fluid dynamics by means of reduced-order models on state-space form. We consider stabilization of a computational fluid dynamics model of an unstable *system. A stabilizing controller is found based on optimal control design for the reduced-order model and then applied to the full model, where it is shown through simulations to stabilize the system.*

1. Introduction

Over the last decade, control of fluid flow has gained an increasing amount of interest from the control community. Control-system designers have turned their attention to both internal flow (flow inside bounded regions, such as within engines or turbomachinery) and external flow (flow around vehicles or bodies). A review of advances made in the intersection between control theory and fluid mechanics can be found in [8].

Distributed systems such as flow dynamics are modeled mathematically by a set of partial differential equations (PDEs). In order to simulate such systems, the PDEs are usually discretized using a set of tools known as computational fluid dynamics (CFD) ([9], [2], [17]). Although acceptable for simulation purposes, the resulting discretized models are often of such high order that it may be infeasible to use them in controller design. For instance, numerical simulation of the full three-dimensional Navier-Stokes equations is still too costly for the purpose of optimization and control of unsteady flows [16].

There is, however, a potential for using model-order reduction techniques to reduce the complexity of the models. Model reduction is a powerful tool, and a comprehensive treatment of approximation of large-scale dynamical systems can be found in the recent book [3]. Although many methods have been successfully applied in control system design, methods such as Hankel model-order reduction and balanced truncation are too computationally demanding for systems of such high order as those encountered in many CFD applications.

Proper orthogonal decomposition (POD) has emerged as a popular alternative that is still applicable even for very high-order systems, using the method of snapshots that will be further presented in section 2. This approach has been considered for active control purposes by numerous authors, among them [12, 5, 16, 7, 6, 1].

One open issue in model-order reduction methodology is the generation of reduced-order models that are suitable for (optimal) control [19]. In [5] and [4] it was shown that the reduced-order models of a CFD-based model can have a state-space structure, which is convenient for controller design. This was also exploited in [10] for optimal control of a 2D plate. While the CFD models in these cases were nominally stable, in this paper we extend the analysis to *unstable* models. This contribution demonstrates the possibility to design stabilizing controllers for a class of systems that would otherwise be very computationally demanding or maybe even infeasible.

The paper is organized as follows. In section 2 the proper orthogonal decomposition is presented. The full CFD model of the unstable system is presented in section 3.1, and the reduced-order model is derived in section 3.2. A controller is designed in section 4 and validated in section 5, and concluding remarks can be found in section 6.

Throughout this paper, we denote by $\rho(A)$ the spectral radius of a matrix A, that is, the largest modulus of an eigenvalue.

2. POD Model-Order Reduction

First introduced independently by Karhunen [11] and Loève [13], POD is sometimes called the

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Karhunen-Loève expansion. When first applied in the context of fluid mechanics in [14], it was used to study turbulent flows. Applicable even for complex non-linear problems, POD makes it possible to keep the essential dynamics of the original model in order to perform controller design. Given the dynamical system to be approximated

$$\dot{x}(t) = f(x(t), u(t)) \quad (1a)$$
$$y(t) = g(x(t), u(t)), \quad (1b)$$

where $x \in \mathbb{R}^n$ denote the system states, $u \in \mathbb{R}^m$ are the inputs to the system and $y \in \mathbb{R}^p$ are the system outputs. It is assumed that the state $x(t)$ can be approximated as a linear combination of r basis vectors

$$x \approx \Phi\hat{x}, \quad (2)$$

where $\hat{x} \in \mathbb{R}^r$ is the reduced state and $\Phi \in \mathbb{R}^{n\times r}$ is a projection matrix containing as columns the r basis vectors $\phi_1, \phi_2, \ldots, \phi_r$. Substituting (2) into (1) and requiring the resulting residual to be orthogonal to the space spanned by Φ gives the reduced-order model

$$\dot{\hat{x}} = \Phi^T f(\Phi\hat{x}(t), u(t)) \quad (3a)$$
$$\hat{y}(t) = g(\Phi\hat{x}(t), u(t)), \quad (3b)$$

where $\hat{x} = \Phi^T x \in \mathbb{R}^r$ is the reduced state and $\hat{y} \in \mathbb{R}^p$ is the output of the reduced model. Several model reduction algorithms use the general projection framework just described; however, they differ in the way the projection matrix Φ is computed. The POD procedure proceeds as follows. Collect a finite number of samples $x(t_i)$ of (1a) for $t = t_1, \ldots t_M$ in a matrix of snapshots

$$\mathscr{X} = \left[x^1, x^2, \ldots, x^M\right] = \left[x(t_1), x(t_2), \ldots, x(t_M)\right], \quad (4)$$

where the columns $\left\{\mathscr{X}_{\cdot,j}\right\}_{j=1}^M$ can be thought of as the spatial coordinate vectors of the system at time step t_j. The rows $\{\mathscr{X}_{i,\cdot}\}_{i=1}^n$ describe the time trajectories of the system evaluated at different locations in the spatial domain [12]. The snapshots may be taken from physical experiments or from computer (CFD) simulations.

The reduced-order model will capture only the dynamics present in this data, and so the choice of snapshot data is critical. Suitable inputs should therefore be used to excite the system, so that the desired characteristics are present in the data.

For a given number of basis vectors r, the POD basis is found by minimizing the error E between the original snapshots and their representation in the reduced space, defined by

$$E = \sum_{i=1}^{M} \left[x(t_i) - \tilde{x}(t_i)\right]^T \left[x(t_i) - \tilde{x}(t_i)\right], \quad (5)$$

where $\tilde{x}(t_i) = \Phi\Phi^T x(t_i)$. The minimizer Φ is then found as the set of left singular vectors of the snapshot matrix $\mathscr{X}$, which is conveniently computed using the singular value decomposition of $\mathscr{X}$,

$$\mathscr{X} = \Phi\Sigma\Psi^T, \quad (6)$$

where the columns of $\Phi = [\phi_1, \ldots, \phi_n]$ form the optimal orthogonal basis for the space spanned by $\mathscr{X}$, Φ and Ψ are unitary matrices (i.e. $\Phi^{-1} = \Phi^T, \Psi^{-1} = \Psi^T$) and Σ is a diagonal matrix with the singular values σ_i of $\mathscr{X}$ on the diagonal. The r most significant basis functions are associated with the r largest singular values σ_i, $i = 1, \ldots r$, of $\mathscr{X}$. If the singular values σ_i fall of rapidly in magnitude, a reduced-order model may be constructed that captures the most salient characteristics of the snapshot data. A rigorous treatment of the connection between SVD and POD can be found in [18].

3. Case Study: Heated Plate

3.1. CFD Model

To demonstrate how an unstable system can be stabilized using POD and feedback control, we study heat conduction in a plate. The plate is $1\,\text{m} \times 1\,\text{m}$, defining the two-dimensional computational domain $\Omega = [0,1] \times [0,1]$ depicted in figure 1. The plate is insulated along the boundaries, apart from the center of each boundary, where four flux actuators are located. This defines Neumann boundary conditions on all boundaries. The temperature $T(t,x,y)$ of the plate is governed by the unsteady linear two-dimensional heat equation

$$\rho c_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + k\frac{\partial^2 T}{\partial y^2} + S, \quad (7)$$

where ρ and c_p are the density and specific heat capacity of the plate, respectively, and k is the thermal conductivity, that is assumed to be uniform over the computational domain and independent of temperature. Note that x now and in the following denotes a spatial coordinate and no longer the state variable. The source term $S \triangleq S_c + S_T$ is a term containing heat sinks and sources. In the present problem, convective heat transfer to the surroundings gives rise to a sink term

$$S_c = hA(T - T_\infty) \;\; [\text{W}], \quad (8)$$

where h is the convective heat transfer coefficient, A is the heat transfer area of the surface and T_∞ is the ambient temperature. Due to electric current, the plate is subject to an internal temperature-dependent heat source

$$S_T = k_1 T \;\; \left[\text{W/m}^3\right],$$

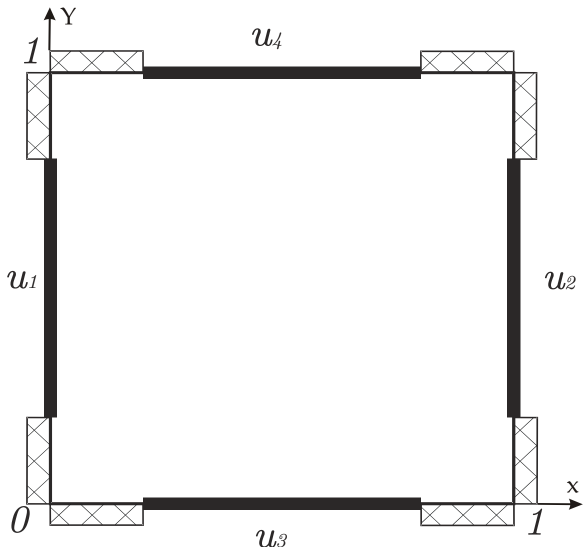

Figure 1. Sketch of plate with actuators on boundaries (bold lines).

where $k_1 > 0$, at all points except from the boundary. Intuitively, this positive feedback from the temperature to the source may lead to a physically unstable system if the convective heat loss to the surroundings is not large enough. An increase in temperature will then lead to a stronger source, which again increases the temperature, and so on.

Discretizing the governing equation by the finite volume method, (7) is integrated over each control volume CV and over the time interval from t to $t+\Delta t$, to obtain [17]

$$\int_{CV}\left(\int_t^{t+\Delta t}\rho c_p\frac{\partial T}{\partial t}dt\right)dV = \int_t^{t+\Delta t}\int_{CV}\left(k\frac{\partial^2 T}{\partial x^2}\right)dV dt + \int_t^{t+\Delta t}\int_{CV}\left(k\frac{\partial^2 T}{\partial y^2}\right)dV dt + \int_t^{t+\Delta t}\int_{CV} S\, dV dt,$$

where the order of integration has been changed for the first term. Using the numerically unconditionally stable backward Euler (fully implicit) temporal discretization and n grid points over the spatial domain Ω, the system (7) can be written as a system of n equations on the form

$$a_P T_P = a_W T_W + a_E T_E + a_S T_S + a_N T_N + a_P^0 T_P^0 + S_u, \quad (9)$$

where the a's are coefficients and T_P is the temperature at the grid point (point P) under consideration at time step $k+1$. S_u and S_P arise from discretizing the source

ρ	c_p	k	h	T_∞	k_1
1000	1000	1000	100	293	1000

Table 1. Numerical values of parameters.

term S as

$$\Delta V \cdot S = S_u + S_P T_P, \quad (10)$$

where S_P is included in a_P. Using the convenient compass notation, T_W, T_E, T_S and T_N are the temperatures at the west, east, south and north adjacent grid points, respectively, at time step $k+1$. T_P^0 is the temperature at grid point P at time step k. Collecting the temperature at all grid points in a row vector $T(k) \in \mathbb{R}^n$ leads to a discrete linear system of the form

$$\begin{aligned} ET(k+1) &= \bar{A}T(k) + \bar{B}u(k) + \bar{V}, \\ y(k) &= \bar{C}T(k), \end{aligned} \quad (11)$$

where $E \in \mathbb{R}^{n\times n}$ is a penta-diagonal matrix containing the coefficients a_p, a_W, a_E, a_S and a_N and $\bar{A} \in \mathbb{R}^{n\times n}$ is a diagonal matrix with a_P^0 on the main diagonal. $\bar{B} \in \mathbb{R}^{n\times m}$ contains the contributions from the inputs, while the constant source terms give rise to a constant term $\bar{V} \in \mathbb{R}^n$.

To validate that the plate model is unstable, we rearrange system (11) into regular state-space form by inverting E and multiplying throughout (11) to obtain

$$\begin{aligned} T(k+1) &= AT(k) + Bu(k) + V, \\ y(k) &= CT(k), \end{aligned} \quad (12)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $V \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $C \in \mathbb{R}^{p\times n}$. The matrix E is indeed invertible in this case, since it is strictly diagonally dominant, and consequently non-singular. This is not necessarily possible in general but it is often the case in simple 1D or 2D problems. It allows us to establish that the system matrix A, given the numerical parameter values in table 1, has an unstable eigenvalue just outside the unit circle,

$$\rho(A) > 1. \quad (13)$$

When the system matrices are of very high order, inverting E, determining eigenvalues of the A and designing a stabilizing controller are computationally demanding tasks. This motivates the search for a reduced-order model.

3.2. Reduced-Order Model

The PDE (7) is discretized using 50 grid points in both the x- and y-direction. This gives in total 2500 states in the full model (12). To construct a model of reduced order, (12) is simulated for $M = 600$ time steps, thus forming the matrix of snapshots $\mathscr{X}$. During this simulation the inputs are varied randomly taking moderate step changes over a suitable range to excite as much of the system dynamics as possible. SVD of the snapshot matrix is performed, and the singular values are considered in order to form the POD basis Φ_r, as depicted in figure 2. As can be seen from the figure the

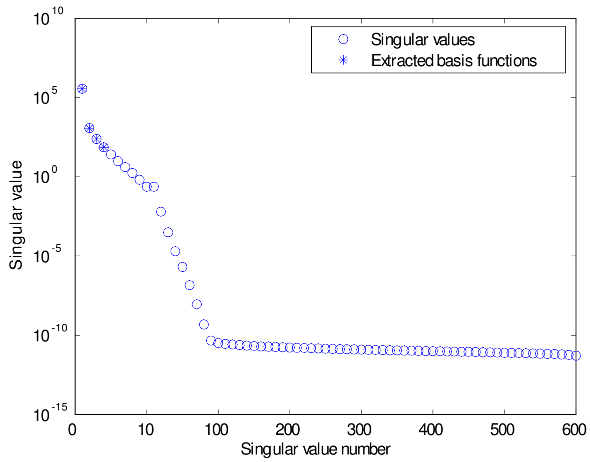

Figure 2. Singular values of the snapshot matrix. The $*$'s indicate singular values corresponding to the extracted basis functions. Note that the ordinate axis is logarithmic.

singular values fall off quite rapidly, and many of the singular values are close to zero, indicating that the basis functions corresponding to those singular values can be omitted without loss of information. There is no systematic approach to establish how many basis functions that should be included in Φ_r. The heuristic criterion

$$P = \frac{\sum_{i=1}^{r} \sigma_i^2}{\sum_{i=1}^{M} \sigma_i^2}, \tag{14}$$

gives an indication on how much of the energy that is conserved in the reduced-order model. If $P \approx 1$ most of the energy is captured in the first r basis functions, indicating a fairly accurate reduced-order model [5]. If we choose $r = 4$ basis functions $P = 99.99\%$. Moreover, if the reduced-order model has four states the number of states in the reduced-order model is equal to the number of inputs. Consequently, the reduced-order model is fully actuated, which might be favorable when tracking a reference profile for the complete state. The reduced-order model is seen to be controllable and hence also stabilizable. It should be noted that this does not guarantee the existence of a controller designed for the reduced-order system that also stabilizes the full-order model.

The basis Φ_r defines the linear dimensionality-reducing mapping $H : \mathbb{R}^n \mapsto \mathbb{R}^r$, such that

$$\hat{T} = H(T) = \Phi_r^T T, \tag{15}$$

and the expansion $G : \mathbb{R}^r \mapsto \mathbb{R}^n$ such that

$$T = G\left(\hat{T}\right) = \Phi_r \hat{T}. \tag{16}$$

By requiring that the residual is orthogonal to the space spanned by the reduced basis and inserting $T = \Phi_r \hat{T}$ into (11) we get

$$\Phi_r^T E \Phi_r \hat{T}(k+1) = \Phi_r^T A \Phi_r \hat{T}(k) + \Phi_r^T B u(k) + \Phi_r^T V. \tag{17}$$

Defining $E_r \triangleq \Phi_r^T E \Phi_r$ allows us to write

$$\begin{aligned} \hat{T}(k+1) &= E_r^{-1}\Phi_r^T A \Phi_r \hat{T}(k) + E_r^{-1}\Phi_r^T B u(k) \\ &\quad + E_r^{-1}\Phi_r^T V, \end{aligned} \tag{18}$$

where E_r is invertible since E, Φ^T and Φ_r are all nonsingular. This yields the reduced-order model on discrete state-space form

$$\begin{aligned} \hat{T}(k+1) &= \hat{A}\hat{T}(k) + \hat{B}u(k) + \hat{V}, \\ \hat{y}(k) &= \hat{C}\hat{T}(k), \end{aligned} \tag{19}$$

where $\hat{T} \in \mathbb{R}^r$, $u \in \mathbb{R}^m$, $\hat{y} \in \mathbb{R}^p$, $\hat{A} = E_r^{-1}\Phi_r^T A \Phi_r \in \mathbb{R}^{r\times r}$, $\hat{B} = E_r^{-1}\Phi_r^T B \in \mathbb{R}^{r\times m}$, $\hat{V} = E_r^{-1}\Phi_r^T V \in \mathbb{R}^r$ and $\hat{C} \in \mathbb{R}^{p\times r}$. In this example, $r = m = 4$. To ensure tracking for the plate temperature, we set C to be the $n \times n$ identity matrix. Consequently, $\hat{C} \in \mathbb{R}^{n\times r}$.

The reduced-order model (19) is unstable since

$$\rho\left(\hat{A}\right) > 1. \tag{20}$$

Note that the general POD procedure does not automatically preserve stability properties during the reduction process. Nominally stable models may result in unstable reduced-order models, and vice verca. In order to be able to replace the analysis of the full model by analysis of the reduced-order model it is important that the stability properties are well reflected in the reduced-order model. This is the subject of on-going research. One criterion for preserving stability properties in POD is presented in [15]. The result is however not applicable to models of very high order.

The reduced-order state $\hat{T}(k)$ is estimated online through a linear observer of the form

$$\hat{T}(k+1) = \left(\hat{A} - L\hat{C}\right)\hat{T}(k) + \hat{B}u(k) + \hat{V} + Ly(k),$$

where $y(k)$ is the output from the high-order CFD model and L is chosen such that $\rho\left(\hat{A} - L\hat{C}\right) < 1$.

4. Controller Design

Feedback control is performed by use of heat flux actuators on parts of the boundary of the domain, shown as the bold lines in figure 1. The control objective is to reach a constant temperature reference T_d while at the same time rejecting disturbances. The reference temperature T_d is set to be a uniform temperature of $300\,^\circ$K.

Since the full model is too large for controller design the reduced-order model is analyzed instead. The reduced-order reference $\hat{T}_d$ is found as $\hat{T}_d = \Phi_r^T T_d$. Given the unstable reduced-order model (19), the control objective is to stabilize the system around the reference temperature. Defining the tracking error as

$$e(k) \triangleq \hat{T}_d - \hat{T}(k), \tag{21}$$

the control input is chosen as

$$u = Ke = K\left(\hat{T}_d - \hat{T}(k)\right), \tag{22}$$

where K is chosen such that $\rho\left(\hat{A} - \hat{B}K\right) < 1$. The controller gain K is taken to be the solution to the linear quadratic regulator problem with infinite horizon, i.e. the constant matrix K such that the state-feedback law

$$u(k) = -K\hat{T}(k) \tag{23}$$

minimizes the cost function

$$\mathscr{J}_{LQ} = \sum_{k=k_0}^{\infty} \left\{ \hat{T}^T(k) Q \hat{T}(k) + u(k)^T R u(k) \right\}, \tag{24}$$

subject to the state dynamics

$$\hat{T}(k+1) \quad = \quad \hat{A}\hat{T}(k) + \hat{B}u(k). \tag{25}$$

The matrices Q and R are weighting matrices of suitable dimensions.

Taking into consideration that the reduced-order model is merely an approximation, the controller should include integral action in order to minimize the steady-state tracking error. To do this in a straightforward way, we define the augmented state

$$\tilde{T}(k) \triangleq \begin{bmatrix} \hat{T}(k) \\ u(k-1) \end{bmatrix} \in \mathbb{R}^{r+m}, \tag{26}$$

giving an augmented state-space model

$$\begin{aligned} \tilde{T}(k+1) &= \tilde{A}\tilde{T}(k) + \tilde{B}\Delta u(k) + \tilde{V}, \\ \tilde{y}(k) &= \tilde{C}\tilde{T}(k), \end{aligned} \tag{27}$$

where

$$\begin{aligned} \tilde{A} &\triangleq \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}, & \tilde{C} &\triangleq \begin{bmatrix} C & 0 \end{bmatrix}, \\ \tilde{B} &\triangleq \begin{bmatrix} B \\ I \end{bmatrix}, & \tilde{V} &\triangleq \begin{bmatrix} V \\ 0 \end{bmatrix}, \end{aligned} \tag{28}$$

and $\Delta u(k) = u(k) - u(k-1)$. In this augmented state-space model, integral action is built-in, and the input increment $\Delta u(k)$ is found as

$$\Delta u(k) = K\left(\hat{T}_d - \hat{T}(k)\right),$$

where K is the feedback gain matrix found above.

5. Numerical Simulation

Initially, the plate temperature is at rest at, and equal to the ambient temperature at 293 K. At $t = 0$ the inner source is switched on. Without control the temperature of the plate is strictly increasing. The plate temperature is shown for four different time instants in figure 3. If the simulation is run for a longer period of time the temperature continues to increase, illustrating the instability of the system.

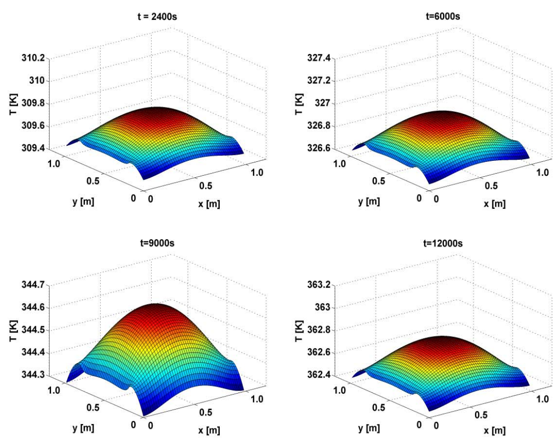

Figure 3. Plate temperature without control, shown for $t = 2400,\ 6000,\ 9000$ **and** $12000\,\text{s}$**.**

Now, the full CFD-model is simulated with the controller designed for the reduced-order model in section 4. The weighting matrices Q and R in (24) are set to $Q = diag\{50, 50, 50, 50\}$ and $R = diag\{0.0001, 0.0001, 0.0001, 0.0001\}$. The system is stabilized, and it is simulated until steady-state is reached, after approximately $t = 100$ minutes. The largest steady-state error is close to 3 K, as shown in figure 4.

It is seen that although the original CFD model is symmetric, the controller based on the reduced-order model does not manage to exploit this symmetry, since the symmetry is not preserved in the model-order reduction scheme.

The simulation of the full closed-loop CFD model illustrates that the plant has been stabilized.

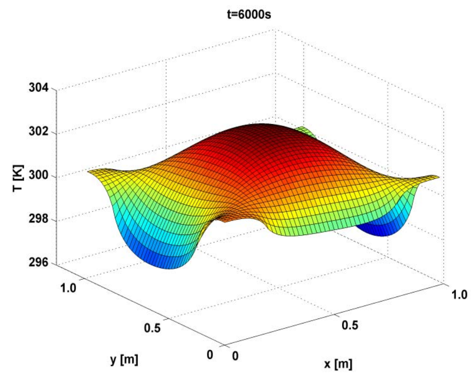

Figure 4. Steady state temperature, shown here for $t = 6000\,\mathrm{s}$.

6. Concluding Remarks

In this paper we have demonstrated, using a case study, that a CFD-model of an unstable system can be stabilized through model-order reduction and a controller designed for the reduced-order model. This makes it possible to design stabilizing controllers for systems that would otherwise be very computationally demanding.

Further and on-going work include stability analysis of reduced-order models of more general PDE-models, and development of model-based reduction methodology for control applications. The 2D plate model analyzed here is viewed as a stepping stone in this work.

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References

[1] K. Afanasiev and M. Hinze. Adaptive control of a wake flow using proper orthogonal decomposition. In M. Dekker, editor, *Lecture Notes in Pure and Applied Mathematics*, Shape Optimization and Optimal Design, 2001.

[2] J. D. Anderson. *Computational Fluid Dynamics*. McGraw-Hill International Editions, 1995.

[3] A. C. Antoulas. *Approximation of Large-Scale Dynamical Systems*. Advances in Design and Control, SIAM, Philadelphia, 2005.

[4] P. Astrid. *Reduction of process simulation models*. PhD thesis, Eindhoven University of Technology, 2004.

[5] P. Astrid, L. Huisman, S. Weiland, and A. C. P. M. Backx. Reduction and predictive control design for a computational fluid dynamics model. In *Proc. 41st IEEE Conf. on Decision and Control*, volume 3, pages 3378–3383, Las Vegas, NV, 2002.

[6] J. A. Atwell, J. T. Borggaard, and B. B. King. Reduced order controllers for burgers' equation with a nonlinear observer. *Applied Mathematics and Computational Science*, 11(6):1311–1330, 2001.

[7] P. Benner and J. Saak. A semi-discretized heat transfer model for optimal cooling of steel profiles. In P. Benner, V. Mehrmann, and D. Sorensen, editors, *Dimension Reduction of Large-Scale Systems*, volume 45 of *Lecture Notes in Computational Science and Engineering*, pages 353–356. Springer-Verlag, Berlin/Heidelberg, Germany, 2005.

[8] T. R. Bewley. Flow control: new challenges for a new Renaissance. *Progress in Aerospace Sciences*, 37(1):21–58, 2001.

[9] J. H. Ferziger and M. Peric. *Computational Methods for Fluid Dynamics*. Springer Verlag, 3rd edition, 2002.

[10] M. Hazenberg, P. Astrid, and S. Weiland. Low order modeling and optimal control design of a heated plate. In *Proc. European Contr. Conf.*, Cambridge, 2003.

[11] K. Karhunen. Zur spektral theorie stochasticher prozesse. *Ann Acad Sci Fennicae, Ser A1 Math Phys*, 34:1–7, 1946.

[12] K. Kunisch and S. Volkwein. Control of the Burgers Equation by a Reduced-Order Approach Using Proper Orthogonal Decomposition. *Journal of Optimization Theory and Applications*, 102(2):345–371, August 1999.

[13] M. Loeve. Functiona aleatoire de second ordre. *Revue Science*, 84:195–206, 1946.

[14] J. L. Lumley. The structure of inhomogeneous turbulent flow. In *Atmospheric Turbulence and Wave Propagation*, 1967.

[15] S. Prajna. POD Model Reduction with Stability Guarantee. In *Proc. 42nd IEEE Conf. on Decision and Control*, Hawaii, 2003.

[16] S. S. Ravindran. A reduced-order approach for optimal control of fluids using proper orthogonal decomposition. *International Journal for Numerical Methods in Fluids*, 34:425–448, 2000.

[17] H. K. Versteeg and W. Malalasekera. *An Introduction to Computational Fluid Dynamics*. Longman, 1995.

[18] S. Volkwein. Proper orthogonal decomposition and singular value decomposition. Technical Report 103, SFB-Preprint, 1999.

[19] K. E. Willcox, O. Ghattas, B. van Bloemen Waanders, and B. Bader. An Optimization Framework for Goal-Oriented, Model-Based Reduction of Large-Scale Systems. In *44th IEEE Conference on Decision and Control*, Seville, Spain, 2005.