

Derivation of loss using "exact local method"

1. Steady-state cost $J(u,d)$
2. Quadratic cost (around $u_d,d$)
3. Linear measurement model

**Assumptions:**
1. Steady-state cost $J(u,d)$
2. Quadratic cost
3. Linear model

1. Steady-state cost $J(u,d)$

$$\min_{u} J(u,x,a) \quad \text{subject to} \quad J(x,u,d) = 0$$

2. Quadratic cost (around $u_d,d$)

$$J(u,d) = J^* + (u^* - u_d)^T H_d (u_d - u)$$

**Notational optimum:**

$$H = H_d$$

3. Linear measurement model

$$y = G^T u + G_d d$$

**Question:** What is the loss if we control $C = H y$ (at steady-state) at constant value, when there are disturbances?

![Diagram of control system with annotations]

**Evaluation of loss**

**Optimal input:** Keep $u = 0$ always

$$J_u = J^* + J_{u,y} + J_{u,d} d$$

$$= 0$$

$$\Rightarrow J_{u,y} d = -J_{u,d}$$

$$J_{u,y} = G^T H_d + G_d d$$

**Comment:** In practise, it is easier to find $F$ by requiring that for each $d$

$$F = G_d d$$

**Loss:**

$$J(u,d) - J(u,y,d) = \frac{1}{2} E y^2 - \frac{1}{2} E H_d y^2$$

**Wanted:**

$$z = J_{u,y}$$

We have

$$C = G_u + G_d d$$

$$C_{u,y} = G_{yy} + G_{yd} d$$

$$C_{u,d} = G_{yd}$$

Thus:

$$z = \frac{1}{2} E y^2 (C_{u,d} - H_d)$$

Note $G = \frac{dC}{du}$: $\frac{dy}{dC}$ $\frac{dC}{dd} = H G_d$

1. Here, $C$ is controlled at steady state. Then

$$C_{u,y} = C_{u,d}$$

2. Also:

$$C_{u,y} = H_d y_d = H (y + r) - H y_d$$

So:

$$z = \frac{1}{2} E y^2 (H G_d - H y_d) = \frac{1}{2} E H (y + r)^2$$
Normalized disturbance and noise:
\[ \left\| \Delta \right\|_2 \leq \frac{\bar{\varepsilon}}{2} \]

Both \( \varepsilon \) and \( \Delta \) are allowed, so sign does not matter!!

Worst-case \( \varepsilon \) (worst-case loss)

\[
\max L = \frac{1}{2} \bar{\varepsilon} (M^2)
\]

\[
\left\| \Delta \right\|_2 \leq \frac{\bar{\varepsilon}}{2}
\]

\[ M = \frac{H}{\bar{\varepsilon}} H^T Y \]
\[ Y = [F_{\text{est.}}, \text{noise magnitude}] \]

\[ \text{desired magnitude} \]

\[ \text{estimated magnitude} \]

Need to select \( H \) such that \( \bar{\varepsilon}(M) \) is minimized.

Null space method:

Special case: No noise \((\varepsilon = 0)\) and sufficient no. of measurements

Can find \( H \) such that \( H^T Y = 0 \) (zero loss)

1. Convex formulation:

\[ \min \left\| H^T \right\|_F \]
\[ \text{s.t. } H^T Y = 0 \]

2. Analytical formula (provided \( Y \) full rank):

\[ H^T = (Y Y^T)^{-1} G \]

Null space method

Special case: No noise \((\varepsilon = 0)\) and sufficient no. of measurements