Lecture notes for Ch. 10.1 - 10.5

Derivation of loss using "exact local method"

Assumptions: 1. Steady-state cost $J(u,d)$,
2. Quadratic cost
3. Linear model

1. Steady-state cost $J(u,d)$

$$\left.\begin{array}{l}\min\limits_{u,x} J(u,x,d)\\ f(x,u,d)=0\end{array}\right\} \quad \text{Eliminate } x: \quad \min_u J(u,d)$$

2. Quadratic cost (around u^*, d^*)

$$J(u,d) = J^* + \left(J_u^* \; J_d^*\right)^T \begin{pmatrix}\Delta u\\ \Delta d\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\Delta u\\ \Delta d\end{pmatrix}^T H^* \begin{pmatrix}\Delta u\\ \Delta d\end{pmatrix}$$

Nominal optimum, $J_u^* = 0$

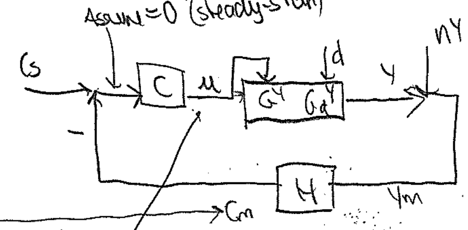
$$H^* = \begin{pmatrix} J_{uu}^* & J_{ud}^* \\ J_{ud}^{*T} & J_{dd}^* \end{pmatrix}$$

3. Linear measurement model

$$y = G^y u + G_d^y d \quad \text{(in deviation variables: } \Delta y, \Delta u, \Delta d)$$

Question: What is loss if we control $c = Hy$ ($\Delta c = H\Delta y$) at constant value, when there are disturbances?

Assume $=0$ (steady-state)



c is called z in book

Note: u varies to keep $c = c_s$ = constant

Need to compare with optimal case

Optimal input: Keep $J_u = 0$ (gradient) always

$$\underset{0}{J_u} = \underset{0}{J_u^*} + J_{uu}^* \Delta u_{opt}(d) + J_{ud}\Delta d$$

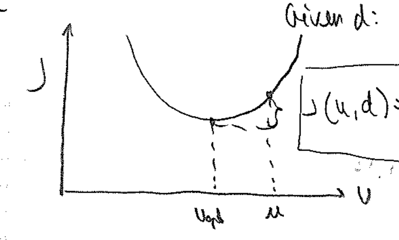
$$\Rightarrow \boxed{\Delta u_{opt}(d) = -J_{uu}^{-1} J_{ud}\, \Delta d}$$

$$\Delta y_{opt} = G^y \Delta u_{opt} + G_d^y \Delta d = \underbrace{\left(-G^y J_{uu}^{-1} J_{ud} + G_d^y\right)}_{F} \Delta d$$

Comment: In practise it is easier to find F by reoptimizing for each d:
$F = \Delta y_{opt}/\Delta d$

Evaluation of loss

Given d:



$$J(u,d) = J(u_{opt},d) + \underset{\substack{0\\ \text{(around } u_{opt})}}{J_u^T}(u-u_{opt}) + \frac{1}{2}(u-u_{opt})^T J_{uu}(u-u_{opt})$$

$$\boxed{\text{Loss} = J(u,d) - J(u_{opt},d) = \frac{1}{2}\tilde{z}^T\tilde{z} = \frac{1}{2}\|\tilde{z}\|_2^2 \quad \text{where } \tilde{z} = J_{uu}^{1/2}(u-u_{opt})}$$

Want to express $\tilde{z}$ as a function of d and n^y.

We have:

$$c = Gu + G_d d$$
$$c_{opt} = G u_{opt} + G_d d$$
$$\overline{c - c_{opt} = G(u - u_{opt})}$$

Thus: $\boxed{z = J_{uu}^{1/2} G^{-1}(c - c_{opt})}$ Note $G = \frac{\Delta c}{\Delta u} = \frac{\Delta c}{\Delta y}\frac{\Delta y}{\Delta u} = HG^y$

1. Here c is controlled at setpoint. Then
$c_m = c_s$ (assume perfect control at steady-state)
Also: $c_m = Hy_m = H(y + n^y) = \underset{c}{\underbrace{Hy}} + Hn^y$

$$\left.\right\} c = \underset{\substack{0\\ \text{(constant setpoint)}}}{c_s} - Hn^y$$

2. $c_{opt} = H y_{opt} = HF\Delta d$ (d)

So: $z = J_{uu}^{1/2}(HG^y)^{-1}(-Hn^y - HFd) = \overbrace{J_{uu}^{1/2}(HG^y)^{-1} H \left[FW_d \;\; W_{n^y}\right]}^{M(H)} \begin{bmatrix} d' \\ n^{y'} \end{bmatrix}$

Normalized disturbance and noise.

$$\left\| \begin{matrix} d' \\ n^{y} \end{matrix} \right\|_2 \leq 1$$

Both $\pm$ allowed so sign does not matter!!

Worst-case Z (worst-case loss)

$$\underbrace{\boxed{\max_{\left\| \begin{matrix} d' \\ n^{y} \end{matrix} \right\|_2 \leq 1} L = \frac{1}{2} \bar{\sigma}(M)^2}}_{\text{Want to select } H \text{ such that } \bar{\sigma}(M) \text{ is minimized.}}$$

$M = J_{uu}^{1/2} (HG^{y})^{-1} HY$

$Y = [F W_d \;\; W_{n^y}]$

noise magnitude

disturbance magnitude

SLIDES

Nullspace method

Special case: No noise ($n^y = 0$) and sufficient no. of measurements

Can find H such that $HF = 0$ (zero loss)

1. Convex formulation

$$\min \|HY\|_F$$
$$\text{s.t. } HG^{y} = J_{uu}^{1/2}$$

2. Analytical formula (provided Y full rank)

$$H^{T} = (Y Y^{T})^{-1} G^{y}$$