

Robust and Stochastic MPC: Are We Going In The Right Direction?

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Abstract: Motivated by requirements in the process industries, the largest user of model predictive control, we re-examine some features of recent research on this topic. We suggest some proposals are too complex and computationally demanding for application in this area and make some tentative proposals for research on robust and stochastic model predictive control to aid applicability

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1. INTRODUCTION

My purpose in this paper is not to present some new theory or procedure; rather my aim is to discuss some difficulties or obstacles that impede the successful application of Model Predictive Control. These difficulties are both theoretical and practical. Our subject now has an excellent foundation created by many researchers. This foundation is not threatened. However, in my opinion, some research does not address industrial needs sufficiently well and there are some topics for which more research is needed. Using a recent review (Mayne (2014)) and a recent paper on model predictive control in industry (Forbes et al. (2015)) some areas of current research that need further attention or redirection are described. Main attention is given to robust and stochastic model predictive control because these forms of control often require the on-line solution of complex optimal control problems. It is hoped that this paper will stimulate further research in these areas.

2. BACKGROUND

The system to be controlled is usually described by

$$x^+ = f(x, u) \tag{1}$$

if there is no disturbance or by

$$x^+ = f(x, u, w) \tag{2}$$

if a disturbance w is present. The state $x \in \mathbb{R}^n$, the control $u \in \mathbb{R}^r$ and the disturbance $w \in \mathbb{R}^p$. Model uncertainty is described in the usual way by

$$x^+ = f(x, u, w) \tag{3}$$

$$y = h(x) \tag{4}$$

$$w = \Delta(y(\cdot)) \tag{4}$$

Δ is a causal input-output operator representing the unmodelled dynamics with input $y(\cdot)$ and output w ; Δ does not necessarily have a finite-dimensional state representation.

The output $y \in \mathbb{R}^s$ and Δ is an operator representing the unmodelled dynamics that, at time t , maps the output sequence $\{\dots, y(-1), y(0), y(1), \dots, y(t)\}$ (over the interval

$(-\infty, t]$) into $w(t)$. The system is usually subject to some constraints, i.e. the control u is required to lie in a compact set $\mathbb{U} \subset \mathbb{R}^m$ and the state may be required to lie in a closed set $\mathbb{X} \subset \mathbb{R}^n$. The equilibrium (target) state-control pair (\bar{x}, \bar{u}) is required to be such that (\bar{x}, \bar{u}) lies in the interior of $\mathbb{X} \times \mathbb{U}$. In addition the state x is required to lie in the closed set $\mathbb{X} \subset \mathbb{R}^n$. In addition, the finite horizon optimal control problem $\mathbb{P}_N(x)$ solved on-line (N is the horizon) may require the terminal state to lie in the compact set $X_f \subset \mathbb{R}^n$; this is a constraint on the optimal control problem and is *not* a system constraint. In robust model predictive control it is assumed that the disturbance w takes values in the compact set $\mathbb{W} \subset \mathbb{R}^p$ that contains the origin in its interior. In stochastic model predictive control $\{w(t)\}$ is a random process, a sequence of independent, identically distributed random variables taking values in a not necessarily compact set $\mathbb{W} \subset \mathbb{R}^p$. In the stochastic case it is assumed that there is an underlying probability space with probability measure P .

The decision variable for the optimal control problem varies considerably. In conventional model predictive control in which the system is described by (1), the decision variable is the control sequence $\mathbf{u} = \{u(0), u(1), \dots, u(N-1)\} \in \mathbb{R}^{Nm}$; this is one of the big attractions of model predictive control since off-line determination of a control law $\kappa : \mathbb{R}^n \rightarrow \mathbb{R}^m$, a complex task, is replaced by on-line determination of a control sequence \mathbf{u} for each encountered value of the state x . The decision variable \mathbf{u} is also employed fairly often in the literature on robust on stochastic model predictive control. In order to overcome the disadvantages, discussed below, of using \mathbf{u} as a decision variable for robust or stochastic model predictive control, a feedback policy $\pi \triangleq \{\mu_0(\cdot), \mu_1(\cdot), \dots, \mu_{N-1}(\cdot)\}$, a sequence of measurable control laws, is also employed; for each i , $\mu_i : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Optimizing over arbitrary functions is obviously too complex so π is often parameterized by a vector $v = (v_0, v_1, \dots, v_{N-1})$ with $\mu_i(x) \triangleq \theta(x, v_i)$; e.g. $\theta(x, v_i) = \sum_{j \in J} v_i^j \phi_j(x)$ in which $\{\phi_j(\cdot) \mid j \in J\}$ is a set of pre-specified functions. When the system $f(\cdot)$ is

linear, a common choice is $\mu_i(x) = \theta(x, v_i) = v_i + Kx$, K chosen so that $A + BK$ is stable, a parameterization suggested by Rossiter *et al* Rossiter et al. (1998). The decision variable \mathbf{u} may be regarded as a degenerate policy in which $\mu_i(x) = \theta(x, v_i) = v_i = u_i$ for all i , all x . Let Π denote the class of policies π defined above, i.e. $\Pi \triangleq \{\pi = \{\mu_0(\cdot), \mu_1(\cdot), \dots, \mu_{N-1}(\cdot) \mid \mu_i(x) = \theta(x, v_i), i = 0, 1, \dots, N-1\}\}$. Optimizing with respect to $\pi \in \Pi$ is equivalent to optimizing with respect to the vector sequence $\mathbf{v} = \{v_0, v_1, \dots, v_{N-1}\}$.

2.1 Definition of cost function $V_N(x, \mathbf{u})$ or $V_N(x, \pi)$

We next define the cost function that is optimized to determine the current control. For nominal model predictive control, in which the system is assumed to satisfy (1), $x^{\mathbf{u}}(i; x)$ denotes the solution of (1) at time i given that the initial state is x at time 0 and the control is \mathbf{u} . For robust or stochastic model predictive control, in which the system is assumed to satisfy (2), $x^\pi(j; x, \mathbf{w})$ denotes the solution of

$$x(i+1) = f(x(i), \mu_i(x(i)), w(i)), i = 0, 1, \dots, N-1 \quad (5)$$

at time j given that the initial state is $x(0) = x$ and the control policy is $\pi = \{\mu_0(\cdot), \mu_1(\cdot), \dots, \mu_{N-1}(\cdot)\} \in \Pi$. The definition of cost depends on the type of model predictive control: conventional, robust or stochastic:

1. Conventional MPC:

$$V_N(x, \mathbf{u}) \triangleq \sum_{i=0}^{N-1} \ell(x^{\mathbf{u}}(i; x), u(i)) + V_f(x^{\mathbf{u}}(N; x)) \quad (6)$$

2a: Robust MPC - Nominal cost:

$$V_N(x, \pi) \triangleq \sum_{i=0}^{N-1} \ell(x^\pi(i; x, \mathbf{0}), \mu_i(x^\pi(i; x, \mathbf{0}))) + V_f(x^\pi(N; x, \mathbf{0})) \quad (7)$$

2b: Robust MPC - Worst case cost:

$$V_N(x, \pi) \triangleq \max_{\mathbf{w} \in \mathbb{W}^N} \sum_{i=0}^{N-1} \ell(x^\pi(i; x, \mathbf{w}), \mu_i(x^\pi(i; x, \mathbf{w}))) + V_f(x^\pi(N; x, \mathbf{w})) \quad (8)$$

in which $\mathbf{0} \triangleq \{0, 0, \dots, 0\}$ is a sequence of zero vectors.

3a. Stochastic MPC - Nominal cost:

$$V_N(x, \pi) \triangleq \sum_{i=0}^{N-1} \ell(x^\pi(i; x, \mathbf{0}), \mu_i(x^\pi(i; x, \mathbf{0}))) + V_f(x^\pi(N; x, \mathbf{0})) \quad (9)$$

3b. Stochastic MPC - Expected cost:

$$V_N(x, \pi) \triangleq E_{|x} \sum_{i=0}^{N-1} \ell(x^\pi(i; x, \mathbf{w}), \mu_i(x^\pi(i; x, \mathbf{w}))) + V_f(x^\pi(N; x, \mathbf{w})) \quad (10)$$

in which $E_{|x}(\cdot) \triangleq E(\cdot \mid x)$ and E is expectation under P , the probability measure of the underlying probability space.

2.2 Definition of constraint set $\mathcal{U}_N(x)$ or $\Pi_N(x)$

Constraints also depend on the type of model predictive control that is employed:

1. Nominal MPC: For each x , $\mathcal{U}_N(x)$ is the set permissible control sequences \mathbf{u} . Each $\mathbf{u} \in \mathcal{U}_N(x)$ satisfies:

$$u(i) \in \mathbb{U}, x^{\mathbf{u}}(i; x) \in \mathbb{X}, \forall i \in \mathbb{I}_{0:N-1}, \\ x^{\mathbf{u}}(N; x) \in X_f \quad (11)$$

It is assumed here and in the sequel that $X_f \subset \mathbb{X}$.

2. Robust MPC: For each x , $\Pi_N(x)$ is the set of permissible control policies. Each $\pi = \{\mu_0(\cdot), \mu_1(\cdot), \dots, \mu_N(\cdot)\} \in \Pi_N(x)$ satisfies:

$$\mu_i(x^\pi(i; x, \mathbf{w})) \in \mathbb{U}, x^\pi(i; x, \mathbf{w}) \in \mathbb{X}, \forall i \in \mathbb{I}_{0:N-1}, \\ x^\pi(N; x, \mathbf{w}) \in X_f, \forall \mathbf{w} \in \mathbb{W}^N \quad (12)$$

in which $\mathbb{I}_{a:b} \triangleq \{a, a+1, \dots, b-1, b\}$.

3. Stochastic MPC: Because the probability density of the disturbance w does not have finite support, it is impossible to satisfy the state and terminal constraints almost surely. To obtain a meaningful optimal control problem, it is necessary to ‘soften’ the state and terminal constraints. In contrast, for process control applications, the control constraint must *always* be satisfied, a requirement sometimes ignored in the literature. Two methods for ‘softening’ the constraint have been used in the literature. In the first (Primbs and Sung (2009)), ‘hard’ constraints of the form $x(w) \in \mathbb{X}$ for all $w \in \mathbb{W}$ are replaced by the average constraint $E(x(w)) \in \mathbb{X}$. In the second (Kouvaritakis et al. (2010); Prnadini et al. (2012)) the constraint $x(w) \in \mathbb{X}$ for all $w \in \mathbb{W}$ is replaced by $P(x(w) \in X) \geq 1 - \varepsilon$. Hence, the constraints employed in the optimal control problem solved on-line take the form

$$\mu_i(x^\pi(i; x, \mathbf{w})) \in \mathbb{U}, E_{|x}(x^\pi(i; x, \mathbf{w})) \in \mathbb{X} \quad \forall i \in \mathbb{I}_{0:N-1}, \\ E_{|x}(x^\pi(N; x, \mathbf{w})) \in X_f \quad \forall \mathbf{w} \in \mathbb{W}^N, \quad (13)$$

in which $E_{|x}(\cdot) \triangleq E(\cdot \mid x)$ when average constraints are employed, or

$$\mu_i(x^\pi(i; x, \mathbf{w})) \in \mathbb{U} \quad P_{|x}(x^\pi(i; x, \mathbf{w})) \in \mathbb{X} \geq 1 - \varepsilon \\ \forall i \in \mathbb{I}_{0:N-1}, P_{|x}(x^\pi(N; x, \mathbf{w})) \in X_f \quad \forall \mathbf{w} \in \mathbb{W}^N \quad (14)$$

in which $P_{|x}(\cdot) \triangleq P(\cdot \mid x)$ when probabilistic constraints are employed. Let $\Pi_N(x)$ denote the set of policies $\pi \in \Pi$ satisfying the appropriate constraints, average or probabilistic. The possibility of satisfying the *hard* control constraint, which is necessary in process control applications, is discussed below.

2.3 Definition of constraint set $\mathcal{U}_N(x)$ or $\Pi_N(x)$

Constraints also depend on the type of model predictive control that is employed:

1. Nominal MPC: For each x , $\mathcal{U}_N(x)$ is the set permissible control sequences \mathbf{u} . Each $\mathbf{u} \in \mathcal{U}_N(x)$ satisfies:

$$u(i) \in \mathbb{U}, x^{\mathbf{u}}(i; x) \in \mathbb{X}, \forall i \in \mathbb{I}_{0:N-1}, \\ \text{and } x^{\mathbf{u}}(N; x) \in X_f \quad (15)$$

It is assumed here and in the sequel that $X_f \subset \mathbb{X}$.

2. Robust MPC: For each x , $\Pi_N(x)$ is the set of permissible control policies. Each $\pi = \{\mu_0(\cdot), \mu_1(\cdot), \dots, \mu_N(\cdot)\} \in \Pi_N(x)$ satisfies:

$$\mu_i(x^\pi(i; x, \mathbf{w})) \in \mathbb{U}, x^\pi(i; x, \mathbf{w}) \in \mathbb{X}, \forall i \in \mathbb{I}_{0:N-1}, \\ x^\pi(N; x, \mathbf{w}) \in X_f, \forall \mathbf{w} \in \mathbb{W}^N \quad (16)$$

in which $\mathbb{I}_{a:b} \triangleq \{a, a+1, \dots, b-1, b\}$.

3. Stochastic MPC: Because the probability density of the disturbance w does not have finite support, it is impossible to satisfy the state and terminal constraints almost surely. To obtain a meaningful optimal control problem, it is necessary to ‘soften’ the state and terminal constraints. In contrast, for process control applications, the control constraint must *always* be satisfied, a requirement sometimes ignored in the literature. Two methods for ‘softening’ the constraint have been used in the literature. In the first (Primbs and Sung (2009)), ‘hard’ constraints of the form $x(w) \in \mathbb{X}$ for all $w \in \mathbb{W}$ are replaced by the average constraint $E(x(w)) \in \mathbb{X}$. In the second (Kouvaritakis et al. (2010); Prnadini et al. (2012)) the constraint $x(w) \in \mathbb{X}$ for all $w \in \mathbb{W}$ is replaced by $P(x(w) \in \mathbb{X}) \geq 1 - \varepsilon$. Hence, the constraints employed in the optimal control problem solved on-line take the form

$$\begin{aligned} \mu_i(x^\pi(i; x, \mathbf{w})) \in \mathbb{U}, E_{|x}(x^\pi(i; x, \mathbf{w})) \in \mathbb{X} \quad \forall i \in \mathbb{I}_{0:N-1}, \\ E_{|x}(x^\pi(N; x, \mathbf{w})) \in X_f \quad \forall \mathbf{w} \in \mathbb{W}^N, \end{aligned} \quad (17)$$

in which $E_{|x}(\cdot) \triangleq E(\cdot | x)$ when average constraints are employed, or

$$\begin{aligned} \mu_i(x^\pi(i; x, \mathbf{w})) \in \mathbb{U} \quad P_{|x}(x^\pi(i; x, \mathbf{w})) \in \mathbb{X} \geq 1 - \varepsilon \\ \forall i \in \mathbb{I}_{0:N-1}, P_{|x}(x^\pi(N; x, \mathbf{w})) \in X_f \quad \forall \mathbf{w} \in \mathbb{W}^N \end{aligned} \quad (18)$$

in which $P_{|x}(\cdot) \triangleq P(\cdot | x)$ when probabilistic constraints are employed. Let $\Pi_N(x)$ denote the set of policies $\pi \in \Pi$ satisfying the appropriate constraints, average or probabilistic. The possibility of satisfying the *hard* control constraint, which is necessary in process control applications, is discussed below.

2.4 Definition of nominal optimal control problem $P_N(x)$

For nominal model predictive control, the optimal control problem, $P_N(x)$, that is solved on-line is:

$$P_N(x) : \quad V_N^0(x) = \min_{\mathbf{u} \in \mathcal{U}_N(x)} V_N(x, \mathbf{u}) \quad (19)$$

with $V_N(\cdot)$ defined in (6) and $\mathcal{U}_N(x)$ defined in (11). The optimising control sequence is $\mathbf{u}^0(x) = \{u_0^0(x), u_1^0(x), \dots, u_{N-1}^0(x)\}$. The control applied to the system is the first element of this sequence, i.e. $u = \mu_N(x) \triangleq u_0^0(x)$.

For robust and stochastic model predictive control, the optimal control problem is

$$P_N(x) : \quad V_N^0(x) = \min_{\pi \in \Pi_N(x)} V_N(x, \pi) \quad (20)$$

with $V_N(\cdot)$ defined in (7) - (8) for robust model predictive control and in (9)-(10) for stochastic model, predictive control. The constraint set $\Pi_N(x)$ is defined in (12) for robust model predictive control and in (13) or (14) for stochastic model predictive control. The optimising control policy is $\pi^0(x)$. The control applied to the system is $\mu_0^0(x)$ where $\mu_0^0(x)$ is the first element of the optimising policy $\pi^0(x)$.

3. STABILISING CONDITIONS

3.1 Conventional Model Predictive Control

Because the optimal control problem $P_N(x)$ has finite horizon, the resultant receding horizon control law is not necessarily stabilising. Conditions that ensure stability

are well known and take two different forms: addition to the optimal control problem of a suitable terminal cost $V_f(\cdot)$ and terminal constraint $x(N) \in X_f$ ensures closed-loop stability as shown, for example, in (Mayne et al. (2000)); alternatively closed-loop stability may be achieved by a suitably large value of the horizon (Grüne and Pannek (2011)). One argument for using the latter is that a terminal constraint is not employed in most process control applications. However any hard state constraint, including a terminal constraint, is avoided in the process industry since it may lead, because of unmodelled factors, to infeasibility of $P_N(x)$ necessitating an extra recovery mechanism. It therefore makes no sense to omit the terminal constraint if other hard state constraints are included in the problem formulation. For each (x, j) , let X_j denote the feasible set for $P_j(x)$. If there are no hard constraints, $X_N = \mathbb{R}^n$ and recursive feasibility is ensured. Otherwise, if an appropriate terminal cost and constraint are employed, the equilibrium state is asymptotically stable with a region of attraction X_N . Also the feasible sets $\{X_j\}$ are forward nested, satisfying $X_f = X_0 \subset X_1 \subset \dots \subset X_{N-1} \subset X_N$, thereby ensuring recursive feasibility (in the absence of unmodelled factors). It is shown in (Mayne (2013)) that if recursive feasibility is required when hard state constraints are employed it is necessary that $P_N(x)$ has an explicit or implicit terminal constraint that is control invariant. Let $P_N^*(x)$ denote the optimal control problem when the terminal constraint is removed. It is shown in (Limon et al. (2006)) that there exist a sequence of sets $\{X_N^*, X_{N-1}^*, \dots, X_1^*, X_0^* = X_f\}$ that is forward nested and satisfies $X_i^* \subset X_i$ for all i . The terminal state of the solution to $P_i^*(x)$ lies in X_f for all $x \in X_N^*$ so that the equilibrium state is asymptotically stable with a region of attraction X_N^* . In this case we refer to $x(N) \in X_f$ as an implicit terminal constraint. If all state constraints are soft, $P_N^*(x)$ is feasible for all $x \in \mathbb{R}^n$.

3.2 Robust model predictive control

Similar considerations apply when there is a bounded additive disturbance provided that the terminal cost and terminal constraint are replaced by robust versions; the terminal cost is now required to be a *robust* local Lyapunov function and the terminal constraint set to be *robustly* control invariant (Mayne et al. (2000)), i.e. for each $x \in X_f$ there exists a $u \in \mathbb{U}$ such that $x^+ = f(x, u, w) \in X_f$ and $V_f(x^+) + \ell(x, u) \leq V_f(x)$ for all $w \in \mathbb{W}$. It is possible to determine a $V_f(\cdot)$ and X_f with these properties if the linearization of $f(\cdot)$ at the nominal ($w = 0$) equilibrium state is stabilizable, $f(\cdot)$ is sufficiently smooth, $\ell(\cdot)$ is positive definite and \mathbb{W} is sufficiently small.

3.3 Stochastic model predictive control

Because the probability density of the disturbance w does not necessarily have finite support, the problem of finding a terminal cost and constraint that ensure closed-loop stability (in the stochastic sense) has not yet been satisfactorily resolved although efforts in this direction have been made. It is clearly impossible to find a terminal cost $V_f(\cdot)$ and compact X_f satisfying the conditions for robust model predictive control given above. Primbs and Sung in (Primbs and Sung (2009)) consider control of a

stochastic linear system with multiplicative noise $C(x, u)w$ that tends to zero in magnitude as (x, u) tends to zero. As discussed above, hard constraints are softened so that the terminal constraint $x(N) \in X_f$, for example, is replaced by $E(x(N) | x(0) = x) \in X_f$. Primbs and Sung assume the existence of a linear terminal controller $\kappa_f(\cdot)$ such that the terminal cost $V_f(\cdot)$ is a stochastic Lyapunov function satisfying, for all $x \in \mathbb{R}^n$,

$$E(V_f(x^+) | x) + \ell(x, \kappa_f(x)) \leq V_f(x) \quad (21)$$

in which $x^+ = f(x, u, w) = Ax + Bu + C(x, u)w$ with $u = \kappa_f(x)$ and $V_f(\cdot)$ is quadratic and positive definite, a requirement similar to the control Lyapunov property employed in robust model predictive control. However, unlike in robust model predictive control, we cannot assume that $X_f \triangleq \{x | V_f(x) \leq \alpha\}$ for some $\alpha > 0$ is robustly control invariant. Instead, Primbs and Sung assume *inter alia* that, if the current state x and policy $\pi = \{\mu_0(\cdot), \mu_1(\cdot), \dots, \mu_{N-1}(\cdot)\}$ satisfy the terminal constraint $E(x^N(N; x) | x) \in X_f$, so does $(x, \tilde{\pi})$ where $\tilde{\pi} \triangleq \{\mu_0(\cdot), \mu_1(\cdot), \dots, \mu_{N-1}(\cdot), \kappa_f(\cdot)\}$ from which it follows that $V_{N+1}^0(x) \leq V_N^0(x)$, i.e., as in the robust case, the value function decreases monotonically with the horizon N . However, to establish recursive feasibility, terminal constraint difficulties require modification of the model predictive algorithm; the modified algorithm switches permanently to the optimal infinite horizon unconstrained policy once the state leaves a given sub-level set of $V_N^0(\cdot)$. Similar difficulties occur when probabilistic constraints are employed. More recent work therefore assumes that the random disturbance w is bounded (Kouvaritakis et al. (2010)) or that there are no state (or terminal) constraints (Chatterjee and Lygeros (2015)). The latter paper is interesting because it introduces advanced results on stochastic stability that should underlie all research on stability of stochastic model predictive control. Interestingly the optimal control problem considered in this paper has a terminal cost $V_f(\cdot)$ that bears some resemblance to that employed in robust model predictive control. The condition is the existence of a terminal control law $\kappa_f(\cdot)$, a positive constant b and a bounded terminal set X_f such that

$$\begin{aligned} \sup_{x \in X_f} E_x V_f(x^+) + \ell(x, \kappa_f(x)) &\leq V_f(x) + b \\ E_x V_f(x^+) + \ell(x, \kappa_f(x)) &\leq V_f(x) \quad \forall x \notin X_f \end{aligned}$$

in which $x^+ \triangleq f(x, \kappa_f(x), w)$. This condition is similar to a global Control Lyapunov Function for the set X_f ; the condition is strong if the open-loop system $x^+ = f(x, u, 0)$ is unstable.

One concludes that there is not yet a fully satisfactory treatment of stabilizing terminal conditions for stochastic model predictive control. Can satisfactory terminal conditions be obtained or not?.

4. THE ON-LINE OPTIMAL CONTROL PROBLEM $\mathbb{P}_N(X)$

4.1 Conventional model predictive control

In conventional model predictive control, the optimal control problem $\mathbb{P}_N(x)$ for linear systems with quadratic costs is a standard quadratic program if the terminal constraint

set is polytopic or omitted; if omitted an implicit terminal constraint (confining the initial state to belong to an appropriate set) or a sufficiently large horizon can ensure closed-loop stability. Polytopic terminal constraint sets, widely used in the literature, cannot be reliably computed for large problems. For many applications, especially in process control, soft state constraints are usually employed and the terminal constraint omitted. Large problems encountered in industry (state dimensions in the hundreds, thousands of constraints) can be handled. For nonlinear systems, a global solution to $\mathbb{P}_N(x)$ cannot be guaranteed necessitating the use of suboptimal model predictive control (Scokaert et al. (1999)).

4.2 Potential objections to robust and stochastic model predictive control

The explosive growth in the use of model predictive control is largely due to the process industries where it is now very widely used. Unlike the situation in the automotive industry where sophisticated design effort can be devoted to the development by experts of a controller that is widely used, the large number and diversity of processes require management of the plant and controller by operators. Maintenance of the model predictive controller is, therefore, the responsibility of non-experts who, therefore, should be able to understand what the controller is doing. It has been noted (Qin and Badgwell (2003)) that the increasing complexity of model predictive control is affecting serviceability and maintainability. Increased complexity requires process engineers to possess increased skills but there is a paucity of engineers with these skills. One way to help operators understand the controller is to provide predicted trajectories of key variables together with constraints that these trajectories should satisfy (Forbes et al. (2015)). Some of the proposals for robust and stochastic model predictive control will be examined in the light of these requirements.

4.3 Robust model predictive control

Complexity The complexity of the optimal control problem may be seen by inspection of (7) and (8), which define the cost function $V_N(\cdot)$, and (12), which defines the constraints. The decision variable has dimension Nq where q is the dimension of parameterization employed in each control law $\mu_i(\cdot)$; this can be as low as Nm , the dimension of the decision variable in conventional model predictive control if the system $f(\cdot)$ is linear and the common parameterization $u = Kx + v$ is employed. The number of control constraints is Nq_u if q_u is the number of constraints that define \mathbb{U} and if the \mathbf{u} is used as the decision variable; because of the disturbance \mathbf{w} , this number could be considerably higher if a policy π is employed as the decision variable. The number of state constraints in $Nq_x q_w$ if q_x is the number of constraints that define \mathbb{X} and q_w is the number of possible realizations of the disturbance sequence $\mathbf{w} \in \mathbb{W}^N$; q_w can be as low as NV where V is the number of vertices of the set \mathbb{W} if the system $f(\cdot)$ is linear but could otherwise be infinite. If min-max model predictive control is employed, i.e. if the cost is defined by (8) and if the optimal control problem is reformulated as $\min_{\pi} h$ subject to the *additional* constraint

$$\sum_{i=0}^{N-1} \ell(x^\pi(i; x, \mathbf{w}), \mu_i(x^\pi(i; x, \mathbf{w})) + V_f(x^\pi(N; x, \mathbf{w})) \leq h \quad \forall \mathbf{w} \in \mathbb{W}^N$$

The number of constraints in $\mathbb{P}_N(x)$ is therefore increased by q_e , the number of possible realizations of the disturbance sequence \mathbf{w} , which can be infinite if the system is nonlinear. If the optimization is over a feedback policy π , it is not obvious how the basis functions $\{\phi_j(\cdot)\}$ should be chosen. When the system is nonlinear, the resultant optimal control problem, in which a control law must be chosen for *each* $j \in \{1, 2, \dots, N-1\}$ seems to be more complex than the problem of choosing offline a *single* control law $\kappa(\cdot)$ for the nonlinear system, a problem that model predictive control was intended to avoid!

The optimal control problem is considerably more complex than that for conventional model predictive control even if the system $f(\cdot)$ is linear. If the system is nonlinear, the problem is too complex to be solved exactly and scenario-based optimization techniques (Bernardini and Bemporad (2009); Calafiore and Fagaino (2013)) may have to be employed to get an approximate solution.

Prediction accuracy A poor prediction accuracy has two consequences:

1: In the nonlinear case, the optimal control problem is made more difficult and possibly infeasible. One reason is that it is unlikely that a suitably accurate parameterization of π can be easily found; probably because of this, most papers propose optimization over control sequences \mathbf{u} rather than over control policies π . Because of this, the estimated spread of the predicted trajectories can easily become excessive even to the extent of making the optimal control problem $\mathbb{P}_N(x)$ infeasible. This is easily seen if $f(\cdot)$ is open-loop unstable and optimization is over control sequences. In such a case, the policy π that is determined (if possible) will *not* converge to the optimal policy determined by dynamic programming as the horizon N tends to infinity, unlike the case when conventional model predictive control is employed. A consequence is that the predicted trajectories can differ considerably from the actual trajectories, again unlike the case when conventional model predictive control is employed.

2: The inaccuracy of the predicted trajectories means that reliable estimates cannot be provided for the operators.

Unlike in conventional model predictive control, the accuracy of the estimated trajectories does not improve with horizon length N if the optimal control problem is solved by optimization over control sequences or parsimonious parameterizations of π .

4.4 Stochastic model predictive control

The cost function $V_N(x, \pi)$ is either expressed as a nominal cost (9), which is independent of the disturbance sequence \mathbf{w} , or as an expectation (10). As discussed above, constraints are defined as expectations (13) or probabilistically (14). When the system is linear, the decision variable is a control sequence, and the additive disturbance is Gaussian, the expectations can be computed. Otherwise, computationally expensive Monte Carlo simulations have to be performed. To avoid generalities, we examine two

proposals that exploit recent advances in stochastic optimization (Calafiore and Campi (2006); Campi and Garatti (2008)).

In (Calafiore and Fagaino (2013)), the problem of controlling the system $x^+ = A(\theta)x + B(\theta)u + C(\theta)w$ is considered, in which θ is a vector of uncertain parameters and \mathbf{w} is a sequence on independent, identically distributed random variables. The system is subject to the state constraint $x \in \mathbb{X}(\theta)$ and the control constraint $u \in \mathbb{U}(\theta)$. The assumptions, which are rather strong for stochastic, as compared with robust, model predictive control, are that $A(\theta), B(\theta)$ and $C(\theta)$ are bounded and that $w \in \mathbb{W}$ with \mathbb{W} a bounded set, that $(A(\theta), B(\theta))$ is a stabilizable pair for all θ . These assumptions are strong enough to ensure the existence of robust terminal constraint set X_f as defined towards the end of §3.2. The control parameterization $u = Kx + v$ is employed so that the system description becomes $x^+ = A_K(\theta)x + B(\theta)v + C(\theta)w$ with $A_K(\theta) \triangleq A(\theta) + B(\theta)K$. The problem is formulated as a min-max problem, minimizing the maximum over all realizations of (θ, \mathbf{w}) of a cost $J(x, \mathbf{v}, (\theta, \mathbf{w}))$ in which the stage cost is $\ell(x, v) = d(x, X_f) + |v|_{\mathbb{R}}^2$. Suitably transcribed, an approximate version of the problem is:

$$\min_s \{c's \mid h(x, s, (\theta, \mathbf{w})^i) \leq 0, \forall i \in \mathbb{I}_{1:M}\} \quad (22)$$

in which $\mathbb{I}_{1:M} \triangleq \{1, 2, \dots, M\}$, the decision variable is $s \triangleq (\mathbf{v}, z, q)$ with $z \in \mathbb{R}$ an upper bound to $J(x, \mathbf{v}, (\theta, \mathbf{w})^i)$ for all $i \in \mathbb{I}_{1:M}$, and $q \in \mathbb{R}$ is a slack variable for each of the constraints specifying $x \in \mathbb{X}$, $x \in X_f$, $v + Kx \in \mathbb{U}$ and $(\theta, \mathbf{w})^i$ is the i th realization of (θ, \mathbf{w}) ; the solution to the approximate problem minimizes the maximum of J over M realizations rather than over all realizations of (θ, \mathbf{w}) . the function $h(\cdot)$ is convex in (x, s) so the useful theory in (Calafiore (2010)) can be employed. Given any $\varepsilon \in (0, 1)$ (e.g. $\varepsilon = 10^{-9}$), an integer M can be computed such that the solution to the problem (18) satisfies all the constraints (slackened constraints in this case) with probability not less than $1 - \varepsilon$. the crucial fact is that N grows at most logarithmically with $1/\varepsilon$ so that almost certain (probability greater than $1 - \varepsilon$) satisfaction of constraints can be achieved with *relatively* modest values of M . With receding horizon control, i.e. with current control $u = v^0(x) + Kx$, the state is steered to X_f with probability not less than $1 - \varepsilon$ along a trajectory that satisfies the state and control constraints, again with probability not less than $1 - \varepsilon$. However even relatively modest values on M can make the use of this controller for typical process control problems problematic. A modest process control problem has state dimension $n = 150$, control dimension $m = 50$, $n_x \geq 50$ (soft) state constraints (constraints on outputs), m control constraints. Even for conventional model predictive control, a problem of this size necessitates choosing a horizon $N \leq 10$ for the associated deterministic control problem in order to contain the complexity of the optimal control problem. For the corresponding uncertain problem, the number of state constraints is roughly Mn_x where M is several thousand and $n_x \approx 50$, giving rise to approximately 100,000 constraints in the optimal control problem. A large process control problem could have $m = 283$ and $n > n_x = 603$ ((Qin and Badgwell, 2003, Table 6)). Also, slackened constraints cannot be employed for control constraints that are ‘hard’. To reduce the complexity of the

optimal control problem, a novel procedure is proposed in (Zhang et al. (2013)) where the controller policy parameterization $u(i) = v_i + M_i(w(0), w(1), \dots, w(i-1))'$ due to (Goulart et al. (2006)) is used. The advantage of this parameterization is that it converts the problem into determination of a feedforward rather than a feedback controller. The scenario approach (Campi and Garatti (2008)) is first employed to select, for each $i \in \mathbb{I}_{0:i-1}$, a parameterized convex set \mathcal{W}_i with the property that the probability that $\{w(0), w(1), \dots, w(i-1)\} \in \mathcal{W}_i$ is not less than $1 - \varepsilon_i$ is specified in (Campi and Garatti (2008)). In the second stage, robust model predictive control is employed to minimize, at each $i \in \mathbb{I}_{0:i-1}$, the cost for the associated optimal control problem subject to the constraint $\{w(0), w(1), \dots, w(i-1)\} \in \mathcal{W}_i$. The procedure requires much lower computation time (on the problem with state dimension 3 examined in the paper) than direct use of the controller parameterization policy of (Goulart et al. (2006)). Nevertheless the computation times and storage requirements do not appear to make the method suitable for process control problems of the complexity routinely encountered.

5. POSSIBLE ALTERNATIVE

5.1 Introduction

It seems therefore that, at present, model predictive control that requires optimization subject to satisfaction of constraints for all realizations, or a realistic sample, of disturbance trajectories is not practical at present for robust model predictive control of nonlinear systems or stochastic model predictive control of linear or nonlinear systems. This is a consequence of the complexity of the resultant optimal control problem. It is the purpose of this section to propose an alternative that has the advantage that the on-line optimal control problem is considerably simpler but that also has some disadvantages. The main objective of the proposed alternative is to replace searching over disturbance trajectories in the on-line optimal control problem by a suitable offline optimization problem that determines parameters which ensure the simpler on-line controller is satisfactory.

The underlying idea (Mayne et al. (2011)) is very simple. Suppose the system is described as usual by $x^+ = f(x, u, w)$ with $x \in \mathbb{X}$, $u \in \mathbb{U}$ and $w \in \mathbb{W}$. At the initial state x , a *nominal* optimal control problem $\mathbb{P}_N^{\text{nom}}(x)$ with tightened constraints on z and v (e.g. $z \in 0.5\mathbb{X}$, $v \in 0.5\mathbb{U}$) is solved to steer the initial state $x(0)$ of the *nominal* system $z^+ = f(z, v, 0)$ to the origin in N steps yielding the *central* control and state trajectories $\mathbf{v}^* \triangleq \{v^*(0), v^*(1), \dots, v^*(N)\}$ and $\mathbf{z}^* \triangleq \{z^*(0) = x(0), z^*(1), \dots, z^*(N) = 0\}$. The terminal equality constraint $z(N) = 0$ is chosen to simplify the choice of terminal conditions for the model predictive controller, considered next, that attempts to steer the state of the uncertain system $x^+ = f(x, u, w)$ back to the central trajectory.

At each subsequent ‘event’ (x, t) (state x , time t) the following *nominal* optimal control problem $\mathbb{P}_N^0(x, t)$ is solved:

$$V_N^0(x, t) = \min\{V_N(x, t, \mathbf{u}) \mid \mathbf{u} \in \mathbb{U}^N\} \quad (23)$$

with $V_N(\cdot)$ defined by

$$V_N(x, t, \mathbf{u}) \triangleq V_f(x(N)) + \sum_{i=0}^{N-1} \ell(x(i) - z^*(t+i), u(i) - v^*(t+i)) \quad (24)$$

Let $\mathbf{u}^0(x, t) \triangleq \{u^0(0; x, t), \dots, u^0(N-1; x, t)$ denote the optimizing control sequence $\mathbf{x}^0(x, t)$ the associated state sequence and let $\kappa_N(x, t) \triangleq u^0(0; x, t)$. It is easily seen that $V_N^0(x, t) = 0$ if $x = z^*(t)$, i.e. $V_N^0(x, t) = 0$ for all (x, t) lying on the central state trajectory and positive elsewhere. Note, there are no state or terminal constraints in the problem $\mathbb{P}_N^0(x, t)$. The lack of a terminal constraint may be overcome. A local Lyapunov function $V_f(\cdot)$ may be determined in the usual way, e.g. as in (Michalska and Mayne (1993)), together with an associated control invariant set $X_f \triangleq \{x \mid V_f(x) \leq \alpha\}$ for some $\alpha > 0$. It can be shown, following (Limon et al. (2006)) that, for all $c > 0$, all (x, t) in a compact neighbourhood \mathcal{N} of the tube $\mathcal{T} \triangleq \{(x, t) \mid V_N^0(x, t) \leq c\}$ there exists a $\beta > 0$ such that $x^0(N; x, t) \in X_f$. It then follows, under the usual assumptions including twice differentiability of $f(\cdot)$ and $\ell(\cdot)$, that, for all $c > 0$, there exist constants $c_2 > c > 1 > 0$, and $\gamma \in (0, 1)$ such that:

$$c_1|x - z^*(t)|^2 \leq V_N^0(x, t) \leq c_3|x - z^*(t)|^2 \quad (25)$$

$$V_N^0(x^+, t^+) \leq \gamma V_N^0(x, t) \quad (26)$$

for all $(x, t) \in \mathcal{N}$. It follows from (25) that any initial state x with $V_N^0(x, 0) \leq c$ of the *nominal* system $x^+ = f(x, \kappa_N(x, t), 0)$, $t^+ = t + 1$ will be steered towards the trajectory $\mathbf{z}^0 \triangleq \{z^*, 0, 0, \dots\}$ and, thus, to the origin exponentially fast. Also

$$z^*(i) = 0, v^*(i) = 0, \forall i \geq N \quad (27)$$

$$V_N^0(x, t) = V_N^*(x) \triangleq V_N^0(x, T) \quad \forall x, \forall t \geq T \quad (28)$$

Hence this strategy reverts to conventional model predictive control for $t \geq N$. The use of this control strategy for robust and stochastic model predictive control will be examined next.

5.2 Robust model predictive control

For this problem it is customary to assume that the disturbance w is bounded, i.e. $w \in \mathbb{W}$, a compact set constraining the origin in its interior. Using the same control law $\kappa_N(\cdot)$ defined above, the disturbance w obviously prevents convergence of the solution $x^{\kappa_N, \mathbb{W}}(\cdot; x)$ of $x^+ = f(x, \kappa_N(x, t), w)$, $t^+ = t + 1$, with state x at time 0, to \mathbf{z}^0 . However, because $V_N^0(x, t)$ increases as x diverges from $z^0(t)$, because \mathbb{W} is compact and because $x \mapsto V_N^0(x, t)$ is uniformly Lipschitz continuous (if $f(\cdot)$ and $\ell(\cdot)$ are), (25), which holds for the nominal system, is replaced by:

$$V_N^0(x^+, t^+) \leq \gamma V_N^0(x, t) + c_3|w| \quad (29)$$

for the uncertain system. Consequently there exists a tube $\mathcal{T} \triangleq \{(x, t) \mid V_N^0(x, t) \leq c\}$, for some $c > 0$, such that any initial event (x, t) in a compact neighbourhood of \mathcal{T} is steered by the controller $\kappa_N(\cdot)$ to \mathcal{T} and, once in \mathcal{T} , the event (x, t) remains there.

When the system $f(\cdot)$ is linear, the cross section of the tube at time t , i.e. the set $S_c(t) \triangleq \{x - z^0(t) \mid (x, t) \in \mathcal{T}\}$, does not vary with t and takes the form $S_c(t) = z^0(t) +$

S where S can be precomputed. For this problem it is possible to replace determination of $\kappa_N(x, t)$ by the simple affine control $u = v + K(x - z)$ where K defines a stabilizing feedback controller; for example, K is the optimal feedback for the infinite horizon unconstrained problem. It is then easy to determine how much the constraints $x \in \mathbb{X}$, $u \in \mathbb{U}$ should be tightened for the nominal control problem to ensure so that \mathbf{z}^0 and \mathbf{v}^0 satisfy $z^0(t) + S \in \mathbb{X}$ and $v^0(t) + KS \in \mathbb{U}$ for all $t \geq 0$ thereby ensuring $x(t) = x^{\kappa_N}(t; x) \in \mathbb{X}$ and $u(t) = \kappa_N(x(t)) \in \mathbb{U}$ for all $t \geq 0$.

In the nonlinear case, considered here, for a given initial state x , the cross section varies with $t \in \mathbb{I}_{0:N-1}$ and is constant for $t \in \mathbb{I}_{N:\infty}$ and cannot be easily pre-computed and stored. Indeed, the constant c that defines the tube cannot be easily computed. Instead the following, complex, off-line problem is solved. Let $\theta \in \Theta$ be a parameter vector that defines the tightened constraints for the nominal control problem $\mathbb{P}_N^{\text{nom}}(x)$ and let X_I denote the desired set of initial states. The offline problem \mathcal{P} is determination of a $\theta \in \Theta$ such that

$$x^{\kappa_N, \mathbf{w}}(t; x) \in \mathbb{X} \quad (30)$$

for all $x \in X_I$, all $t \geq 0$ and all $\mathbf{w} \in \mathbb{W}^N$; $\kappa_N(\cdot)$ depends implicitly on θ . The control constraint is automatically satisfied because of the definition of the optimal control problem $\mathbb{P}_N(x)$. Although the problem is complex, the dimension of the decision variable θ is relatively low. The constraint dimension is infinite, but scenario optimization may be employed if satisfaction of state constraints with a pre-specified probability is acceptable.

5.3 Stochastic model predictive control

In stochastic model predictive control, the disturbance is a random variable with a specified probability distribution and is not usually confined to lie in a compact set. A fair amount of attention is given in the literature to consideration of control of a stochastic linear system defined by

$$x^+ = f(x, u, w) = Ax + Bu + w \quad (31)$$

where $\{w(0), w(1), w(2), \dots\}$ is a sequence of independent, identically distributed random variables. The system is subject to the control constraint $u \in \mathbb{U}$ and the state constraint $x \in \mathbb{X}$. Sometimes the parameters A and B are also random but, for simplicity, that case is not considered here although much of the proposed procedure is relevant. The procedures proposed in the literature and accompanying analysis are complex even for linear systems. Consider the following naive procedure when $x^+ = Ax$ is asymptotically stable and there are no state constraints. For a given initial state x compute a control sequence $\{v(i)\}$ that steers the state of the nominal system $z^+ = Az + Bv$ asymptotically to the origin. Then, the state of the system $x^+ = Ax + Bv + w$ satisfies $x = z + e$ where e satisfies $e^+ = Ae + w$. Suppose the sequence $\{w(i)\}$ is Gaussian with zero mean and variance Σ . Then e is Gaussian with variance $\Sigma_e(i)$ satisfying $\Sigma_e(i+1) = A\Sigma_e(i)A' + \Sigma$ that converges to $\bar{\Sigma}_e$ satisfying $\bar{\Sigma}_e = A\bar{\Sigma}_eA' + \Sigma$ so that the long run average expected cost is $\text{trace}(\bar{\Sigma}_e Q)$. It is not obvious how to obtain a feedback controller analogous to the robust case discussed above where the disturbance is bounded.

As pointed out in (Chatterjee and Lygeros (2015)), state constraints are very difficult to handle in stochastic model predictive control and it does not seem possible to have a *local* stabilizing terminal condition similar to those employed in conventional model predictive control. The paper (Chatterjee and Lygeros (2015)) therefore employs a *global* stochastic Lyapunov function $V_f(\cdot)$ as the terminal cost and no terminal constraint. The terminal cost is assumed to have the following properties:

H1: There exists a terminal control law $\kappa_f : \mathbb{R}^n \rightarrow \mathbb{U}$, a constant $b > 0$ and a compact X_f such that:

$$E_{|x} V_f(x^+) + \ell(x, \kappa_f(x)) \leq V_f(x) \quad \forall x \notin X_f \quad (32)$$

$$E_{|x} V_f(x^+) + \ell(x, \kappa_f(x)) \leq V_f(x) + b \quad \forall x \in X_f \quad (33)$$

in which $x^+ \triangleq f(x, \kappa_f(x), w)$. This assumption is rather strong and seems to imply (because the control is bounded) that A is a stability matrix. Suppose that a central trajectory $(\mathbf{z}^*, \mathbf{v}^*)$ is generated as above and that subsequently the control u is determined by solving the optimal control problem $\mathbb{P}_N(x, t)$ defined by

$$V_N^0(x, t) \triangleq \min_{\mathbf{u}} E_{|x} \{V_N(x, t, \mathbf{u}, \mathbf{w}) \mid \mathbf{u} \in \mathbb{U}^N\} \quad (34)$$

in which $V_N(\cdot)$ is defined by

$$V_N(x, t, \mathbf{u}, \mathbf{w}) \triangleq V_f(x(N)) + \sum_{i=0}^{N-1} \ell(x(i) - z^*(t+i), u(i) - v^*(t+i)) \quad (35)$$

with \mathbf{w} now defined by $\mathbf{w} \triangleq \{w(0), 0, \dots, 0\}$. Let $\mathbf{u}^0(x, t)$ denote the optimizing sequence and $\kappa_N(x, t)$ the first element of this sequence. The procedure is simpler than that described in (Chatterjee and Lygeros (2015)) in that optimization is now over control sequences rather than policies and the expectation is with respect to a single disturbance $w(0)$ rather than with respect to a disturbance sequence $\{w(0), w(1), \dots, w(N-1)\}$; the optimization implicitly yields a control policy. However, the problem is still complex since an expected cost (36) is employed even though the expectation is over a single random variable rather than a sequence. Given this cost, it is conjectured that a result similar to Theorem 3 in (Chatterjee and Lygeros (2015)) may be obtained:

$$E_{|x} V_N^0(f(x, \kappa_N(x, t), w)) \leq V_N^0(x, t) - \ell(x, \kappa_N(x, t)) + b \quad (36)$$

and that $E_{|x} V_N^0(x, t)$ is bounded for all $t \geq 0$.

The usefulness of this conjecture, if true, is questionable since the terminal cost assumption appears to be equivalent to knowing, a priori, a global, stabilizing control law $\kappa_f(\cdot)$, an assumption that model predictive control was intended to avoid!

The difficulties encountered in stochastic model predictive control remind us of an early warning on optimal stochastic control (Wonham (1969)):

Since the mathematical model is usually greatly complicated by explicitly including stochastic features, it is always to be asked whether the extra effort is worthwhile, i.e. whether it leads to a control markedly superior to one designed on the assumption that stochastic disturbances are absent. In the case of feedback controls the general conclusion is that only marginal improvements can be obtained unless the disturbance level is very high in which case the

fractional improvement from stochastic optimization may be large but the system is useless anyway.

It is to be hoped that this forecast is wrong but it does suggest that the performance of proposed stochastic model predictive controllers should be compared with the performance of conventional model predictive control.

6. CONCLUSIONS

In robust model predictive control, the control $\kappa_N(x)$ for state x is often obtained by solving an optimal control problem that requires optimization subject to satisfaction by the model of constraints for all possible disturbance sequences; in stochastic model predictive control the requirement is minimization of an expected cost where the expectation is with respect to disturbance sequences. In both cases the computational expense for solving the optimal control problem is high, possibly too high for situations routinely encountered in the process control industry, especially if the system being controlled is nonlinear. It is argued that a profitable research direction is to search for much simpler on-line strategies that can be designed employing possibly extensive off-line optimization. For example, for robust model predictive control, a possible strategy is to employ conventional model predictive control with tightened constraints together with extensive off-line optimization to determine the tightened constraints. Examples of using simple algorithms for complex control problems occur elsewhere. In adaptive control nobody attempts to solve on-line the dual optimal control problem; instead a relative simple problem is solved on-line, and conditions that ensure its properties are satisfactory are separately determined. The examples given in this paper should be regarded merely as ‘existence’ results that show it is possible to employ simpler on-line strategies. The research community is surely able to improve considerably on these few proposals.

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