Appendix A

The vectorization procedure of convex optimization problem in decision matrix **H**

$$\begin{aligned} &\min_{\mathbf{H}} \|\mathbf{HY}\|_F \\ &\text{s.t. } \mathbf{HG}^y = \mathbf{J}_{uu}^{1/2} \end{aligned}$$

to convex optimization problem in $\mathbf{h}_\delta$ is described (Alstad et al., 2009). We write

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1n_y} \\ h_{21} & h_{22} & \dots & h_{2n_y} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n_u1} & h_{n_u2} & \dots & h_{n_un_y} \end{bmatrix} = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \dots & \mathbf{h}_{n_y} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{h}}_1^T \\ \tilde{\mathbf{h}}_2^T \\ \vdots \\ \tilde{\mathbf{h}}_{n_u}^T \end{bmatrix}$$

where

$$\begin{aligned} \mathbf{h}_j &= j^{th}\text{column of } \mathbf{H},\ \mathbf{h}_j \in \mathbb{R}^{n_u \times 1} \\ \tilde{\mathbf{h}}_j &= j^{th}\text{row of } \mathbf{H},\ \tilde{\mathbf{h}}_j \in \mathbb{R}^{n_y \times 1} \end{aligned}$$

The transpose must be included because all vectors including $\tilde{\mathbf{h}}_i$ are column vectors.

Similarly, let $\mathbf{J}_{uu}^{1/2} = [\mathbf{j}_1 \quad \mathbf{j}_2 \quad \dots \quad \mathbf{j}_{n_u}]$.

We further introduce the long vectors $\mathbf{h}_\delta$ and $\mathbf{j}_\delta$,

$$\mathbf{h}_\delta = \begin{bmatrix} \tilde{\mathbf{h}}_1 \\ \tilde{\mathbf{h}}_2 \\ \dots \\ \tilde{\mathbf{h}}_{n_u} \end{bmatrix} = \begin{bmatrix} h_{11} \\ h_{12} \\ \vdots \\ h_{1n_y} \\ h_{21} \\ h_{22} \\ \vdots \\ h_{2n_y} \\ h_{n_u 1} \\ h_{n_u 2} \\ \vdots \\ h_{n_u n_y} \end{bmatrix} \in \mathbb{R}^{n_u n_y \times 1}$$

$\mathbf{j}_\delta^T = \begin{bmatrix} \mathbf{j}_1^T & \mathbf{j}_2^T & \dots & \mathbf{j}_{n_u}^T \end{bmatrix} \in \mathbb{R}^{n_u n_u \times 1}$
and the large matrices

$$\mathbf{G}_\delta^T = \begin{bmatrix} \mathbf{G}^{y^T} & 0 & 0 & \cdots \\ 0 & \mathbf{G}^{y^T} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & \dots & \mathbf{G}^{y^T} \end{bmatrix}, \mathbf{Y}_\delta = \begin{bmatrix} \mathbf{Y} & 0 & 0 & \cdots \\ 0 & \mathbf{Y} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & \dots & \mathbf{Y} \end{bmatrix}$$

Then, $\mathbf{HY} = \begin{bmatrix} \tilde{\mathbf{h}}_1^T\mathbf{Y} \\ \tilde{\mathbf{h}}_2^T\mathbf{Y} \\ \dots \\ \tilde{\mathbf{h}}_{n_u}^T\mathbf{Y} \end{bmatrix}$ and for the frobenius norm the following equalities apply.

$$\|\mathbf{HY}\|_F^2 = \| \begin{matrix} \tilde{\mathbf{h}}_1^T\mathbf{Y} \\ \tilde{\mathbf{h}}_2^T\mathbf{Y} \\ \vdots \\ \tilde{\mathbf{h}}_{n_u}^T\mathbf{Y} \end{matrix} \|_F = \| \begin{matrix} \tilde{\mathbf{h}}_1^T\mathbf{Y} & \tilde{\mathbf{h}}_2^T\mathbf{Y} & \dots & \tilde{\mathbf{h}}_{n_u}^T\mathbf{Y} \end{matrix} \|_F$$
$$= \|\mathbf{h}_\delta^T\mathbf{Y}_\delta\|_F = \|\mathbf{h}_\delta\mathbf{Y}_\delta^T\|_F = \mathbf{h}_\delta^T \underbrace{\mathbf{Y}_\delta\mathbf{Y}_\delta^T}_{\mathbf{F}_\delta} \mathbf{h}_\delta = \mathbf{h}_\delta^T\mathbf{F}_\delta\mathbf{h}_\delta$$

Because $\mathbf{HG}^y = \mathbf{J}_{uu}^{1/2}$ where $\mathbf{J}_{uu}^{1/2}$ is symmetric matrix, we have $\mathbf{HG}^y = \mathbf{G}^{y^T}\mathbf{H}^T = \mathbf{J}_{uu}^{1/2}$ and

$$\begin{bmatrix} \mathbf{G}^{y^T}\tilde{\mathbf{h}}_1 & \mathbf{G}^{y^T}\tilde{\mathbf{h}}_2 & \dots & \mathbf{G}^{y^T}\tilde{\mathbf{h}}_{n_u} \end{bmatrix} = [\mathbf{j}_1 \quad \mathbf{j}_2 \dots \mathbf{j}_{n_u}] \implies \mathbf{G}_\delta^T\mathbf{h}_\delta = \mathbf{j}_\delta$$