A Simple Control Scheme for Near-optimal Operation of Parallel Heat Exchanger Systems

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Abstract

For heat exchanger networks with stream splits, we present a simple way of controlling the split ratio. We introduce the "Jäschke Temperature", which for a branch with one exchanger is defined as $T_J = \frac{(T-T_0)^2}{T_h-T_0}$, where T_0 and T are the inlet and outlet temperatures of the split stream (usually cold), and T_h is the inlet temperature of the other stream (usually hot). Assuming the heat transfer driving force is given by the arithmetic mean temperature difference, the Jäschke Temperatures of all branches must be equal to achieve maximum heat transfer. The optimal controlled variable is the difference between the Jäschke Temperatures of each branch, which should be controlled to zero. Heat capacity or heat transfer parameters are not needed, and no optimization is required to find the optimal setpoints for the controlled variables. Most importantly, our approach gives near-optimal operation for systems with logarithmic mean temperature difference as driving force.

Keywords: Heat exchanger networks, Parallel systems, Self-optimizing control, Optimal operation

1. Introduction

Global climate challenges and competition require efficient energy usage, and this typically implies re-using energy as much as possible. In the chemical and process industries, large amounts of energy can be saved by using heat recovery in heat exchanger networks, which transfer energy in form of heat

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from one set of (hot) streams to another set of (cold) streams. By optimizing layout and operation of these heat exchangers, the overall consumption of natural resources for heating and cooling can be reduced considerably. In addition, this often results in significantly reduced operating costs.

The potential of heat exchanger networks for saving energy and costs has led to a large body of research, and most of the literature falls into one of two categories. The first category is the design and synthesis of heat exchanger networks (see e.g. Saboo & Morari (1984); Saboo et al. (1985); D.Colberg & Morari (1990); Yee & Grossmann (1990); Gundersen et al. (1997); Furman & Sahinidis (2002); Laukkanen et al. (2010)). Most contributions fall into this category, where some likely conditions and scenarios are assumed, and the task is to find the optimal type, size and structure of interconnections of the heat exchangers. Generally this results in large mixed integer optimization problems, and most of the literature addresses the issue of finding optimal solutions in an efficient way.

The second category deals with optimal operation of heat exchanger networks (Aguilera & Marchetti, 1998; Glemmestad et al., 1999; Rodera et al., 2003; Lersbamrungsuk et al., 2008). Generally, the conditions in the real plant differ from those assumed during design, and the contributions from this category study how the available degrees of freedom, such as valves, bypasses and utility heaters, can be used to optimally match the real operating conditions. Even if the actual operating conditions are the same as assumed during plant design, Jensen & Skogestad (2008) showed that because of simplifying assumptions during design, like fixing ΔT_{min} to 10 K, the optimal design point is often not the same as the optimal operating point. Although there has been some research activity in this area, there is still a need to find methods for optimizing operation of heat exchanger networks, which are easy to implement in real plants.

When implementing optimal operation in a process, such as heat exchanger networks, there are two fundamental model-based approaches which can be taken: Online optimization and offline optimization. In online optimization (Grötschel et al., 2001), the model is used to formulate an optimization problem, which is repeatedly solved online in a fast optimization software. The optimal input values obtained from the software are then applied to the plant. In this approach, the plant measurements are primarily used for adjusting model parameters, such that the model and the plant match. This approach may be implemented using a steady state model (Marlin & Hrymak, 1997; Lid et al., 2001), or alternatively a dynamic model (Grötschel et al., 2001). Implementing online optimization is relatively expensive due to the high costs for obtaining and maintaining a good process model, which can be optimized in real-time. However, if a good model is available, this approach can yield results which are very close to the true optimum. Due to the high costs, it is mainly implemented in cases where the immediate economic benefits are very high, such as refineries.

The alternative offline optimization approach exploits the structure of the optimal solution. This results in simple operating schemes which do not require online solution of optimization problems. The basic idea was first conceived by Morari et al. (1980), who write that "we want to find a function c of the process variables [...] which when held constant, leads automatically to the optimal adjustment of the manipulated variables, and with it, the optimal operating conditions." This idea has been followed in the paradigm of self-optimizing control, where such variables are found in a systematic manner, and in NCO-tracking, where these variables are the necessary optimality conditions (NCO) (Mathisen et al., 1992; Skogestad, 2000; Srinivasan & Bonvin, 2004; Lersbamrungsuk et al., 2008; Jäschke & Skogestad, 2012a). Although typically some degree of sub-optimality will have to be tolerated, these approaches are attractive in practice, because they are simple and easy to implement.

Considering the structure of the optimal solution, the steady state optimal operating point of heat exchanger networks without stream splits and with only single bypasses and utilities as manipulated variables, is characterized by being at constraints (Aguilera & Marchetti, 1998; Lersbamrungsuk et al., 2008), and can be described by a linear programming problem. In this case all degrees of freedom are used to specify target temperatures or are kept at constraints (e.g. bypass valves are used to control a target temperature, or are either fully open or fully closed). The problem of optimal operation is then reduced to finding and tracking the set of active constraints (Lersbamrungsuk et al., 2008), which often can be done without online optimization.

In this paper we study heat exchanger networks with stream splits where the steady state optimal operating point is generally unconstrained. A simple example for such a system is shown in Figure 1. Here, the split fraction must be continuously adapted to match varying operating conditions such as changing inlet temperatures ($T_0, T_{h1,1}, T_{h1,2}$), flow rates ($F_0, F_{h1,1}, F_{h1,2}$), and heat transfer properties ($UA_{1,1}, UA_{1,2}$). In practice, these cases are either handled by an online optimization approach (Lid et al., 2001), when the potential savings are very high, or simply operated in an open-loop fashion,

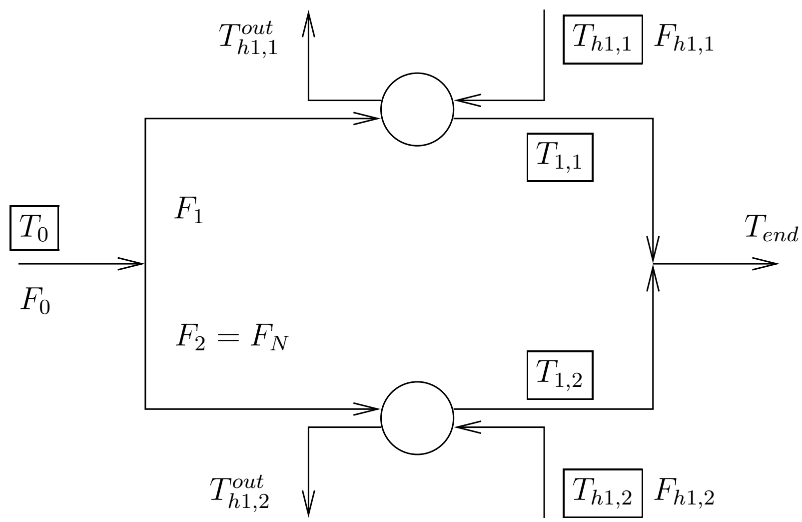


Figure 1: Simple heat exchanger network with one split. The boxed variables are needed for obtaining the Jäschke Temperatures.

where the split ratio is set to some constant. Other ad-hoc solutions include isothermal mixing and controlling some outlet temperatures to a setpoint. These solutions are suboptimal.

The contribution of this paper is to present a simple method for optimizing heat exchanger networks with stream splits. The results have been submitted for patenting (Jäschke & Skogestad, 2012c). Nevertheless, the derivation is of interest for the scientific community and deserves the separate discussion provided in this paper. To obtain our results, we follow the general approach described by Jäschke & Skogestad (2012b): We set up a simple model, formulate the optimality conditions, and then eliminate the unmeasured variables from the optimality conditions. The obtained expression is a function of measurements only, and controlling it is equivalent to controlling the optimality conditions.

Note that the results in this paper also are applicable when a hot stream is split into parallel streams which are cooled down individually. To simplify notation, however, we chose to present only the case, where the parallel streams are heated.

This paper is organized as follows: In the next section, we give a short motivation for eliminating unknown variables from the optimality conditions, while Section 3 presents some preliminary material on parallel systems. In Section 4 we describe the heat exchanger model used for obtaining the main results presented in Section 5. The derivation of the results is given in Section 6, and Section 7 contains some case to demonstrate the applicability of the results. Finally, the paper is closed with a discussion and conclusions in Sections 8 and 9.

2. General elimination

The necessary condition of optimality for an unconstrained optimum is that the gradient is zero ($J_u = \frac{\partial J}{\partial u} = 0$). However, the gradient generally depends on unknown variables (u, d). We include this motivational section here to demonstrate on a toy example, how this condition can be reformulated to derive an optimality condition where the unknown parameters are replaced by measured variables y. A similar approach will be applied later to the heat exchanger process. Let us start with a general smooth optimization problem. After the active constraints are satisfied (e.g. by control) we can describe optimal operation as an unconstrained optimization problem,

$$\min_u J(u, d).$$

Here $u \in \mathbb{R}^{n_u}$ denotes the unconstrained degrees of freedom, and $d \in \mathbb{R}^{n_d}$ denotes the vector of disturbances. To fully specify operation, we need as many controlled variables c as there are degrees of freedom u, $n_c = n_u$.

Under a suitable second order condition, the ideal controlled variable is the reduced gradient, which must be controlled to zero for optimality, (Halvorsen & Skogestad, 1997; Bonvin et al., 2001)

$$c = J_u(u, d) = \frac{\partial J(u, d)}{\partial u}.$$

However, this gradient is rarely known in practice, because of unmeasured disturbances (d) and measurement noise.

If the measurement noise is negligible, one approach is to use a model to formulate the optimality conditions and use a measurement model to eliminate all unmeasured variables. Alstad & Skogestad (2007) stated that this requires that the number of independent measurements is greater or equal to the sum of the number of degrees of freedom and the disturbances,

$n_y \geq n_u + n_d$. However, this requirement may be conservative sometimes. Consider the following toy example,

$$\min J = \frac{1}{2}(u-d)^2.$$

The optimal controlled variable is the gradient,

$$c = J_u = (u-d), \tag{1}$$

and the minimum number of measurements is $n_u = 1$. To need only a single measurement, we must measure the gradient itself, $y = J_u = (u-d)$, or some locally monotone scalar valued function of it.[1] However, if the gradient is not measured directly, we need two independent measurements, one for eliminating u, and one for eliminating d from the gradient expression (1).

Now consider another toy example, where the cost function is

$$\tilde{J} = \frac{1}{2}(u-1)^2 + d^2. \tag{2}$$

Here, the gradient $\tilde{J}_u = u - 1$ contains only one variable, and only one measurement is required. That is, we can measure u and set $u = 1$.

If however, new variables (disturbances) are present in the measurement equations, then additional measurements are required. Consider again (2). If the only measurement is $y_1 = G_1 u + G_{1d} d$, then it cannot be used to eliminate u, because a new variable d is introduced. In this case, a second measurement $y_2 = G_2 u + G_{2d} d$ necessary to eliminate d from the measurement equation for y_1, such that $d = \frac{1}{G_{2d}}(y_2 - G_2 u)$. Inserting into the first measurement equation gives $y_1 = G_1 u + G_{1d}\frac{1}{G_{2d}}(y_2 - G_2 u)$. Thus, the optimal controlled variable $\tilde{J}_u = u - 1$ can be expressed using the measurements as $c = \tilde{J}_u = u - 1 = \frac{y_1 - \frac{G_{1d}}{G_{2d}} y_2}{G_1 - \frac{G_{1d}}{G_{2s}} G_2} - 1$.

In conclusion, the minimum number of measurements depends on the process structure, and for finding a controlled variable which is equivalent to controlling the gradient, the minimum number of measurements required is n_u (when the gradient is measured) and at most $n_u + n_d$. This assumes that the measurements are independent, and n_d denotes all disturbances (unknown variables) which are present in the model and the objective function.

[1]The monotonicity around its optimal value is necessary for controllability.

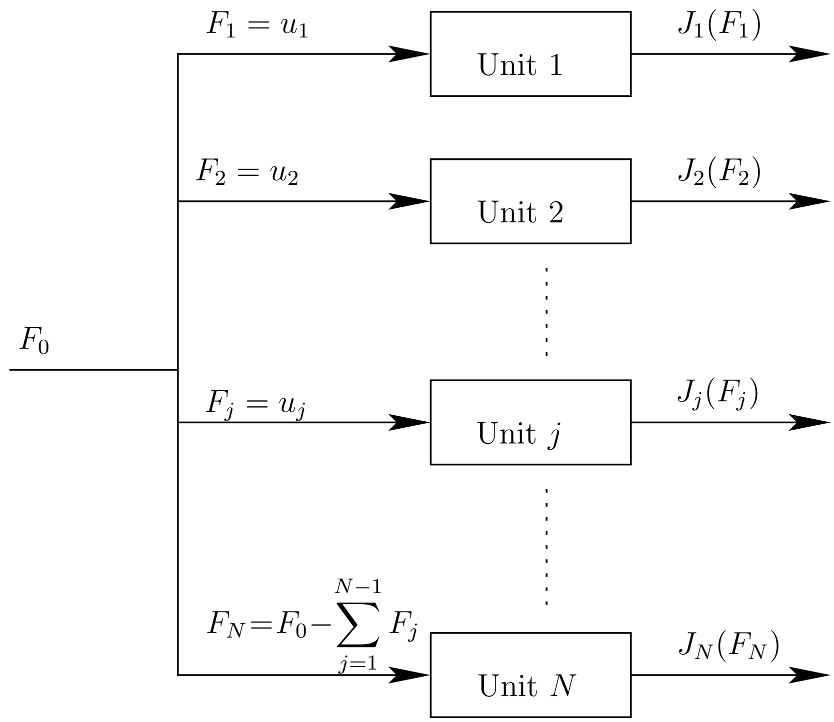


Figure 2: Parallel units connected to a common stream. Each unit j has an associated load-dependent operating cost $J_j(F_j)$.

As an alternative to directly eliminating the unknowns, one may also attempt to find linear approximations of the gradient, which do not contain any of the unmeasured variables. This approach is followed in Alstad et al. (2009).

3. Parallel systems

Consider a system with the topology given in Figure 2, with N parallel streams F_i which are branched off a given common feed stream F_0. The total operating cost J of the system is assumed to be the sum of the individual scalar costs J_j from each line j,

$$J = \sum_{j=1}^{N} J_j(F_j),$$

and the operational objective is to distribute the streams F_j such that the total operating cost J is minimized. Since all streams F_j are coming from one overall feed stream F_0, conservation of mass requires a common coupling constraint,

$$F_0 - \sum_{j=1}^{N} F_j = 0.$$

Because of this coupling constraint, only $N-1$ streams can be adjusted independently. The N-th flow rate is given by the mass balance, $F_N = F_0 - \sum_{j=1}^{N-1} F_j$.

We denote the adjustable degrees of freedom by the vector u. The degrees of freedom u may be difficult to set to a given constant value, e.g. $u = 5$, unless they are measured. For example, consider fixing the flow rate in a shower to 5l/m.

Now let

$$u = [F_1 \; F_2 \; \ldots, F_{N-1}]^T$$

denote the degrees of freedom. Adjusting one flow to decrease the cost in one branch will eventually cause the cost of another branch to become unacceptably high. Therefore, this class of systems exhibits an unconstrained optimum, where the necessary condition for optimality is that the gradient is zero:

$$J_u = \frac{\partial J}{\partial u} = 0.$$

The result summarized in the following theorem is an important component for obtaining simple controlled variables for parallel systems.

Theorem 1 (Downs & Skogestad (2011)). *For a parallel system as in Figure 2, the optimality condition can be written as*

$$\frac{\partial J_1}{\partial F_1} = \frac{\partial J_2}{\partial F_2} = \ldots \frac{\partial J_j}{\partial F_j} = \ldots \frac{\partial J_N}{\partial F_N}, \tag{3}$$

which leads to the optimal controlled variable

$$c = J_u = \begin{pmatrix} \frac{\partial J_1}{\partial F_1} - \frac{\partial J_N}{\partial F_N} \\ \frac{\partial J_2}{\partial F_2} - \frac{\partial J_N}{\partial F_N} \\ \vdots \\ \frac{\partial J_j}{\partial F_j} - \frac{\partial J_N}{\partial F_N} \\ \vdots \\ \frac{\partial J_{N-1}}{\partial F_{N-1}} - \frac{\partial J_N}{\partial F_N} \end{pmatrix}. \tag{4}$$

Proof. Assume the $N-1$ degrees of freedom are chosen as the flows in branches 1 to $N-1$. Then a flow change in any branch $j = 1 \dots N-1$ is compensated by a change of flow in branch N, so we have

$$dF_N = -dF_j \quad \text{for } j \neq N. \tag{5}$$

The change δJ in the cost for a variation in δF_j is

$$\frac{\delta J}{\delta F_j} = \frac{\delta(J_1 + J_2 + \cdots + J_j + J_N)}{\delta F_j} = \frac{\delta(J_j + J_N)}{\delta F_j},$$

using (5), this becomes

$$\frac{\partial J}{\partial F_j} = \frac{\partial J_j}{\partial F_j} - \frac{\partial J_N}{\partial F_N}.$$

The fact that this is required to hold for all units $j = 1 \dots N$, leads to (3), and the second statement of the theorem follows trivially.

□

Theorem 1 states that the *marginal costs* $\frac{\partial J_j}{\partial F_j}$ must be equal for all lines. Each marginal cost is associated with its own line j, and contains only variables from line j. This structure can be exploited for breaking down the large problem into smaller problems, where unknown variables can be eliminated from the gradient expression. Moreover, since the optimality condition can be written as a pair-wise condition (4), we can without loss of generality, consider a system with only two branches.

4. Parallel heat exchanger systems

4.1. Heat exchanger network model

We consider a heat exchanger network with N parallel lines. A line j is assumed to have M_j heat exchangers, as illustrated in Figure 3. For heat

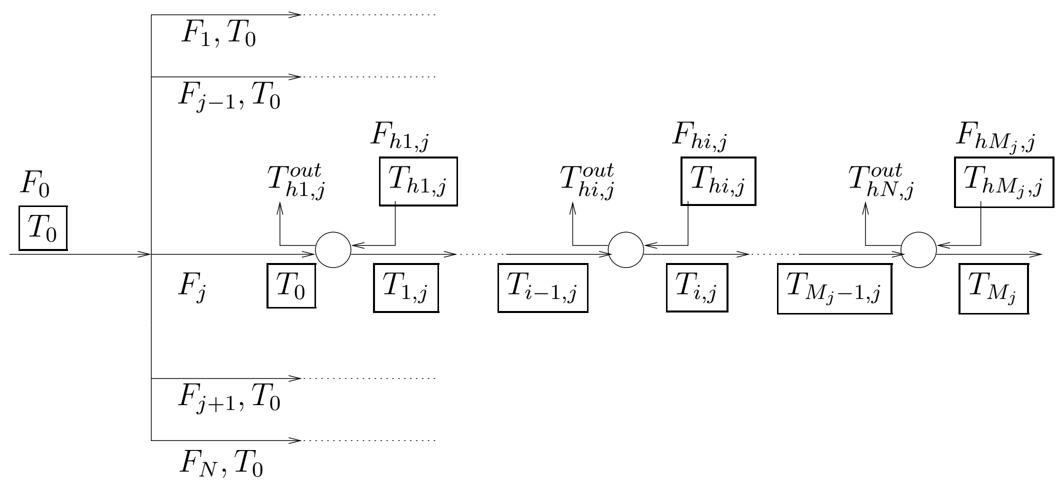


Figure 3: Detail of a heat exchanger network with N lines and with M_j heat exchangers on line j. The boxed temperatures are needed for obtaining the Jäschke Temperatures

exchanger i on line j, $T_{i,j}$, $T_{hi,j}$, $T_{hi,j}^{out}$ denote the cold stream temperature after heat exchanger, the hot inlet temperature, and the outlet temperature of the hot stream, respectively. Before we proceed with the model equations, we present some assumptions our model is based on.

Assumption 1 (Single phase)**.** *There is no phase change in the heat exchangers.*

Assumption 2 (Constant heat capacity)**.** *The specific heat capacity c_p of the fluids is constant.*

To simplify notation, we introduce the heat capacities w of the cold and hot streams,

$$w_j = F_j c_{p0}$$
$$w_{hi,j} = F_{hi,j} c_{phi,j}.$$

An energy balance around the hot and cold stream of heat exchanger i on line j yields

$$Q_{i,j} = w_i(T_{i,j} - T_{i-1,j}) \tag{6}$$
$$Q_{i,j} = w_{hi,j}(T_{hi,j} - T_{i-1,j}^{out}). \tag{7}$$

where $Q_{i,j}$ denotes the heat transferred in the heat exchanger i on line j. The amount of transferred heat is given by

$$Q_{i,j} = UA_{i,j}\Delta T_{Di,j},$$

where $UA_{i,j}$ denotes the product of heat transfer area and overall heat transfer coefficient, and $\Delta T_{Di,j}$ denotes the driving force. Typically, the driving force is modeled as the logarithmic mean temperature

$$\Delta T_{Di,j} = \Delta T_{log,i,j} = \frac{(T_{hi,j} - T_{i,j}) - (T_{hi,j}^{out} - T_{i-1,j})}{\log \frac{T_{hi,j} - T_{i,j}}{T_{hi,j}^{out} - T_{i-1,j}}}.$$

When the heat capacity of the hot and the cold streams have similar magnitude, the arithmetic mean temperature is a good approximation of the logarithmic mean temperature

$$\Delta T_{Ai,j} = \frac{(T_{hi,j} - T_{i,j}) + (T_{hi,j}^{out} - T_{i-1,j})}{2} \approx \Delta T_{log,i,j}.$$

This approximation has an error of less than 1 % (Skogestad, 2008) when the temperature changes of the hot and cold side are within ±40%, that is when

$$\frac{1}{\sqrt{2}} \leq \frac{T_{i,j} - T_{hi,j}}{T_{hi,j}^{in} - T_{i-1,j}} \leq \sqrt{2}.$$

To be able to derive simple results, we make the following additional assumption:

Assumption 3 (Arithmetic mean temperature driving force)**.** *The driving force for heat transfer is given by the arithmetic mean temperature difference.*

The stream splitter is described by a simple mass balance,

$$w_0 - \sum_{j=1}^{N} w_j = 0,$$

and the energy balance yields

$$T_{0,j} = T_0 \text{ for all } j.$$

In the case that the streams are merged again, using the energy balance, the end temperature out of the mixer can be calculated by the weighted sum of the temperatures of the individual lines,

$$T_{end} = \frac{1}{w_0} \sum_{j=1}^{N} w_j T_{M_j}.$$

4.2. Objective function

Our goal is to adjust the splits between the lines of the heat exchanger network such that the operating cost J is minimized. In a general form we may write the cost J as

$$J = -income + expenses.$$

We denote the price that has to be paid for transferring heat in heat exchanger i on line j as $p_{i,j}^{cost}$. Similarly, the price for the added value is denoted $p_{i,j}^{rev}$. The cost thus becomes

$$J = -\sum_{j=1}^{N}\sum_{i=1}^{M_j} p_{i,j}^{rev} Q_{i,j} + \sum_{j=1}^{N}\sum_{i=1}^{M_j} p_{i,j}^{cost} Q_{i,j} = -\sum_{j=1}^{N}\sum_{i=1}^{M_j} (p_{i,j}^{rev} - p_{i,j}^{cost}) Q_{i,j}.$$

In practice, the prices $p_{i,j}^{rev}$ will often be equal, $p_{i,j}^{rev} = p^{rev}$, while the prices for using different hot streams $p_{i,j}^{cost}$ may differ significantly. By defining a new price $p_{i,j} = p_{i,j}^{rev} - p_{i,j}^{cost}$ we may simplify the cost function to

$$J = -\sum_{j=1}^{N}\sum_{i=1}^{M_j} p_{i,j} Q_{i,j}. \tag{8}$$

When all prices for using the hot streams are equal, $p_{i,j}^{cost} = p^{cost}$, and the prices for the heated streams are equal, $p_{i,j}^{rev} = p^{rev}$, this is equivalent to maximizing the total heat transfer. Furthermore, if the branches are merged again, it corresponds to maximizing the end temperature T_{end}.

5. Main Results

When setting up the optimality conditions for a heat exchanger network, the expressions will generally contain all variables. Some variables may be easy to measure, such as temperatures, while others are more difficult to measure or estimate, such as heat capacities and heat transfer coefficients. As motivated in Section 2, our goal is to find controlled variables, which are functions of measurements that are easy to obtain, and are equivalent to controlling the gradient to zero.

We state the main results in this section, and prove them in Section 6. For convenience, we introduce the shifted temperature θ, which is formed by subtracting the feed temperature T_0,

$$\theta = T - T_0.$$

Theorem 2 (Maximize heat transfer)**.** *Under Assumptions 1-3, and equal prices $p_{i,j} = 1$ the marginal costs for each branch $j = 1 \dots N$ can be expressed as*

$$\frac{\partial J}{\partial F_j} = T_{J,j} \tag{9}$$

where $T_{J,j}$ is the Jäschke Temperature on branch j, defined as

$$T_{J,j} = \sum_{i=1}^{M_j} a_{i,j}, \tag{10}$$

with the parameter $a_{i,j}$ defined recursively as

$$a_{i,j} = \frac{(\theta_{i,j} - \theta_{i-1,j})(\theta_{i,j} + \theta_{i-1,j} - a_{i-1,j})}{\theta_{hi,j} - \theta_{i-1,j}}, \quad a_{0,j} = 0. \tag{11}$$

Theorem 2 is proved for $M_j = 1, 2, 3$ (see Section 6), and conjectured for $M_j \geq 4$. It implies that the optimal split that maximizes the total heat transfer can be obtained by simply controlling the Jäschke Temperatures in all branches to equal values. Note that the Jäschke Temperature on branch j only depends on the temperatures on this branch ($\theta_{i,j}$), and the hot inlet temperatures on this branch ($\theta_{hi,j}$). In Figures 1 and 3 the temperatures required to calculate the Jäschke Temperatures are highlighted in boxes. In particular, we do not need to know the heat capacities or the flow rates of the streams ($F_0, F_j, F_{hi,j}$), nor do we require information about the heat transfer properties $UA_{i,j}$ to calculate the Jäschke Temperatures.

Example 1 (Maximize heat transfer, $M_1 = M_2 = 1$)**.** *For the network depicted in Figure 1, the controlled variable is*

$$\begin{aligned} c &= T_{J,1} - T_{J,2} \\ &= \frac{\theta_{1,1}^2}{\theta_{h1,1}} - \frac{\theta_{1,2}^2}{\theta_{h1,2}}. \end{aligned}$$

Adjusting the split between the branches such that $c = 0$ results in optimal operation when the arithmetic mean temperature difference assumption is satisfied. Moreover, keeping c at zero is optimal in spite of varying operating conditions, such as changing stream temperatures or changing heat transfer properties due to fouling in the heat exchangers.

We can extend Theorem 2 to the case where the hot sources have different prices $p_{i,j}$, and where the objective is to minimize the economic cost of operating the heat exchanger network.

Theorem 3. *Under Assumptions 1-3, the marginal costs for each branch $j = 1 \dots N$ can be expressed as*

$$\frac{\partial J}{\partial F_j} = T^e_{J,j} \tag{12}$$

where the Economic Jäschke Temperature $T^e_{J,j}$ is defined as

$$T^e_{J,j} = \sum_{i=1}^{M_j} p_{i,j} a_{i,j} \tag{13}$$

with the parameter $a_{i,j}$ defined as in Theorem 2.

Theorem 3 is proved for $M_j = 1, 2, 3$ (see Section 6), and conjectured for $M \geq 4$. Strictly speaking, Theorem 2 is a special case of Theorem 3. It is obtained by setting all $p_{i,j}$ equal to 1. However, because of its practical importance, we chose to write Theorem 2 as a separate theorem.

Example 2 (Minimizing economic cost, $M_1 = 1, M_2 = 2$)**.** *For a system with 1 heat exchanger on the first line and 2 heat exchangers on the second line, and with prices $p_{1,1}, p_{12}$ and p_{22} the controlled variable becomes*

$$\begin{aligned} c &= T^e_{J,1} - T^e_{J,2} \\ &= p_{1,1}\frac{\theta_{11}^2}{\theta_{h11}} - \left(p_{1,2}\frac{\theta_{12}^2}{\theta_{h12}} + p_{2,2}\frac{(\theta_{2,2} - \theta_{1,2})\left(\theta_{2,2} + \theta_{1,2} - \frac{\theta_{12}^2}{\theta_{h12}}\right)}{\theta_{h2,2} - \theta_{1,2}} \right). \end{aligned} \tag{14}$$

Note that when there is no heat exchange in the second heat exchanger of line 2, i.e we have $\theta_{12} = \theta_{22}$, Equation (14) *reduces to the case where there there is only one heat exchanger per line, and c becomes $c = p_{11}\frac{\theta_{11}^2}{\theta_{h11}} - p_{12}\frac{\theta_{12}^2}{\theta_{h12}}$. Similarly, when there is no heat exchange in the first heat exchanger on line 2 ($\theta_{12} = 0$), the expression simplifies to $c = p_{11}\frac{\theta_{11}^2}{\theta_{h11}} - p_{22}\frac{\theta_{22}^2}{\theta_{h22}}$.*

6. Derivation of main results

In this section we show how we obtained the results presented in the previous section. The procedure is based on Jäschke & Skogestad (2012b), where we use a model to formulate the optimality conditions, and then use the model to eliminate unmeasured variables from the optimality conditions. We present the proof for the economic case (Theorem 3), because the case of maximizing heat transfer (Theorem 2) is obtained as a special case by setting $p_{i,j} = 1$ for all i, j.

6.1. Proof of Theorem 3 for $M_j = 1$

An example of this case is shown in Figure 1. In this section, for simplicity of notation, since every line contains only one heat exchanger, we omit the index i denoting the heat exchanger, such that e.g. $T_{1,2}$ is denoted T_2. Expressed in shifted temperatures θ, the energy balance (6)-(7) around the heat exchanger on line j becomes

$$Q_j = w_j \theta_j \tag{15}$$
$$Q_j = w_{hj}(\theta_{hj} - \theta_{hj}^{out}) \tag{16}$$

Combining (15) and (16), and solving for θ_{hj}^{out} yields

$$\theta_{hj}^{out} = \frac{w_{hj}\theta_{hj} - w_j\theta_j}{w_{hj}}. \tag{17}$$

Under Assumption 3, the transfered heat is

$$Q_j = \frac{UA_j}{2}\left(\theta_{hj} - \theta_j + \theta_{hj}^{out}\right).$$

Equating with (15) we have

$$\frac{UA_j}{2}\left(\theta_{hj} - \theta_j + \theta_{hj}^{out}\right) = w_j\theta_j,$$

and inserting (17) yields

$$\frac{UA_j}{2}\left(\theta_{hj} - \theta_j + \frac{w_{hj}\theta_{hj} - w_j\theta_j}{w_{hj}}\right) = w_j\theta_j.$$

Upon solving for θ_j, we obtain

$$\theta_j = \frac{\theta_{hj}}{1 - w_j\left(\frac{1}{w_{hj}} + \frac{2}{UA_j}\right)}. \tag{18}$$

This expression for θ_i can be used in the objective function. For a system with N lines, we have

$$J = \sum_{j=1}^{N} J_i = -\sum_{j=1}^{N} p_j Q_j = -\sum_{j=1}^{N} p_j w_j \theta_j.$$

And the corresponding marginal cost for each branch is

$$\begin{aligned}
\frac{\partial J}{\partial w_j} &= p_j \frac{\partial Q_j}{\partial w_j} = p_j \frac{\partial}{\partial w_j} w_j \theta_j \\
&= p_j \frac{\partial}{\partial w_j} \left(\frac{w_j \theta_{hj}}{1 - w_j\left(\frac{1}{w_{hj}} + \frac{2}{UA_j}\right)} \right) \\
&= p_j \frac{\theta_{hj}\left(1 - w_j\left(\frac{1}{w_{hj}} + \frac{2}{UA_j}\right)\right) + w_j \theta_{hj}\left(\frac{1}{w_{hj}} + \frac{2}{UA_j}\right)}{\left(1 - w_j\left(\frac{1}{w_{hn}} + \frac{2}{UA_j}\right)\right)^2} \\
&= p_j \frac{\theta_{hj}}{\left(1 - w_j\left(\frac{1}{w_{hj}} + \frac{2}{UA_j}\right)\right)^2}.
\end{aligned} \tag{19}$$

By noting that (18) implies

$$\frac{\theta_j}{\theta_{hj}} = \frac{1}{1 - w_j\left(\frac{1}{w_{hj}} + \frac{2}{UA_j}\right)}$$

we can write (19) as

$$\frac{\partial J_j}{\partial w_j} = p_j \frac{\theta_j^2}{\theta_{hj}},$$

which completes the proof for $M_j = 1$.

6.2. Proof of Theorem 3 for $M_j = 2, 3$ (see also Appendix A)

The situation becomes more complicated when there are more than one heat exchangers on a line, because the marginal cost contains variables from all heat exchangers on this line, and represents the accumulated effect of a change in the flow rate. For proving Theorem 3 for $M_j > 1$, we applied the method described in Jäschke & Skogestad (2012b), which is based on three main steps:

1. Model the system and formulate the optimality conditions.
2. Calculate the reduced gradient J_u.
3. Symbolically eliminate unmeasured parameters and states from the reduced gradient J_u using the sparse resultant (Cox et al., 2005).

We considered two cases, each with 2 branches $j = 1, 2$, which were modelled in Step 1. In the first case we considered a model with $M_1 = 3$ and $M_2 = 1$, and in the second case we set $M_1 = 2$ and $M_2 = 1$. The fact that $M_2 = 1$ implies no limitation on the generality of the proof, since the marginal costs for each line are independent of each other, see Theorem 1.

To reduce the number of variables, the first case with $M_1 = 3$ was modelled with constant hot stream temperatures (this corresponds to a very large heat capacity of the hot streams, or condensing steam). Since a constant temperature in the hot stream can be modelled as a very large heat capacity, a disturbance which is eliminated, we expect to get the same result as when we have a finite heat capacity.

The reduced gradient in Step 2 was calculated in Maple(TM)[2], and contains the temperatures, flow rates and heat transfer variables from all heat exchangers. Finally the elimination of UA_{ij} and the heat capacities $w_j, w_{hi,j}$ from the reduced gradient in Step 3 was performed using the Maple package `multires` (Busé & Mourrain, 2003).

The symbolic elimination is computationally very expensive. For the case with finite hot stream heat capacity (i.e. non-constant hot stream temperature), it was not possible to perform the elimination for $M_j > 2$. When using a constant hot temperature (corresponding to infinite hot heat capacity), it was possible to obtain results for the case with $M_j = 3$. However we were not able to obtain results for $M_j > 3$.

[2]Maple is a trademark of Waterloo Maple Inc.

The procedure above leads to a large algebraic expression, which is zero whenever the gradient is zero. For the heat exchanger network with $M_1 = 3$ and $M_2 = 1$, factoring out the prices $p_{1,1}$,$p_{2,1}$,$p_{3,1}$, and $p_{1,2}$ leads to the expression (A.1) in Appendix A. Re-writing the marginal costs as shown in Appendix A gives for a general line j with $M_j = 3$

$$\begin{aligned} T^e_{J,j} &= \frac{\partial J_j}{\partial w_j} \\ &= p_{1,j}\frac{\theta^2_{1,j}}{\theta_{h1,j}} \\ &+ p_{2,j}\frac{(\theta_{2,j}-\theta_{1,j})(\theta_{2,j}+\theta_{1,j}-\frac{\theta^2_{1,j}}{\theta_{h1,j}})}{\theta_{h2,j}-\theta_{1,j}} \\ &+ p_{3,j}\frac{(\theta_{3,j}-\theta_{2,j})(\theta_{3,j}+\theta_{2,j}-\frac{(\theta_{2,j}-\theta_{1,j})(\theta_{2,j}+\theta_{1,j}-\frac{\theta^2_{1,j}}{\theta_{h1,j}})}{\theta_{h2,j}-\theta_{1,j}})}{\theta_{h3,j}-\theta_{2,j}}. \end{aligned} \tag{20}$$

For the second case with finite hot heat capacity (non-constant hot stream temperature) with $M_1 = 2$ and $M_2 = 1$, we obtain for $M_j = 2$ exactly the same expression as in (20), with the only difference being that the term corresponding to $p_{3,j}$ is zero.

6.3. Conjecture for the general case ($M \geq 4$)

From the recurring pattern, that coefficients of the previous prices appear in a fixed structure in the coefficients of the next price, we conjecture the general formula for the case with more than three heat exchangers on a line as:

$$T^e_{J,j} = \sum_{i=1}^{M_j} p_{i,j}a_{i,j}$$

where the $a_{i,j}$ is defined as

$$a_{i,j} = \frac{(\theta_{i,j}-\theta_{i-1,j})\left[\theta_{i,j}+\theta_{i-1,j}-a_{i-1,j}\right]}{\theta_{hi,j}-\theta_{i-1,j}}$$

and

$$a_{0,j} = 0.$$

In Appendix B we provide numerical evidence to support our conjecture. There, we set up a network with four heat exchangers on a line ($M_1 = 4$), and one on the second line ($M_2 = 1$). The heat exchangers are modelled with the arithmetic mean temperature difference as driving force for the heat transfer. Then we compare the results obtained from numerical optimization with the results obtained by controlling the Jäschke Temperatures to equal values. Since controlling the Jäschke Temperatures to equal values is conjectured to be equivalent to controlling the gradient, we expect to see the same results in both cases. Indeed, the numerical results for the case study in Appendix B strongly suggest that the conjecture is true.

7. Simulation case studies

In this section, we apply our approach to heat exchangers networks which are modelled using the logarithmic mean temperature difference as driving force for the heat transfer. Although the assumption of the arithmetic mean temperature difference is no longer satisfied, the simulations show that our approach yields good performance.

First we present two cases where the objective is simply to maximize the total heat transfer. Then we show an example where the operating cost is minimized when the hot stream prices differ. We compare the results obtained from controlling the Jäschke Temperatures with the true optimum, and we also also present results for the case when the end temperatures are controlled to equal values (isothermal mixing).

7.1. Case Study 1: One heat exchanger per branch

We consider a simple case with one heat exchanger per line, where the streams are merged after being heated, Figure 1. Instead of manipulating the flows directly, we select the split

$$u = \frac{F_1}{F_0} = \frac{w_1}{w_0}$$

as manipulated variable. We assume that the prices are equal on both lines, $p_{11} = p_{12}$, so the objective is to maximize the end temperature T_{end}. The stream parameters are given in Table 1.

Table 2 shows the end temperature T_{end} for the true optimum (maximally achievable end temperature), the end temperature from controlling the Jäschke Temperatures to equal values, and the end temperature obtained

Table 1: Data for Case Study 1

Variable	value	unit	Description
T_0	60	°C	Cold feed temperature
w_0	100	kW/K	Cold feed heat capacity
$w_{h1,1}$	30	kW/K	Hot stream 11 heat capacity
$w_{h1,2}$	50	kW/K	Hot stream 12 heat capacity
$T_{h1,1}$	120	°C	Hot stream 11 temperature
$T_{h1,2}$	220	°C	Hot stream 12 temperature
$UA_{1,1}$	50	°C	Heat exchanger 11 area× overall heat transfer coefficient
$UA_{1,2}$	80	°C	Heat exchanger 12 area× overall heat transfer coefficient

Table 2: Results for Case Study 1

	Optimized	Equal Jäschke Temp.	Isothermal mixing
End Temp	124.8917°C	124.8045°C	119.9388°C
Split $u = \frac{F_1}{F_0}$	0.2704	0.2361	0.0600

from isothermal mixing, which is obtained from controlling $T_{11} = T_{12}$. The Jäschke Temperature approach gives an end temperature which is very close to optimal, while isothermal mixing results in a loss of almost 5°C.

Figure 4 shows how well the arithmetic mean temperature approximates the logarithmic mean temperature in the exchangers. Further, we included the bounds where the approximation of the logarithmic mean temperature is less than 1%. At the optimum the approximation error of heat exchanger 2 is larger than 1%. This is the reason why the Jäschke Temperature approach deviates slightly from the optimal split. However, due to the flatness of the unconstrained optimum, our approach still gives a close to optimal T_{end}

In Figure 5 we plotted the split ratio against the final temperature T_{end} and included the difference between Jäschke Temperatures of the two lines, $c = \frac{\theta_{11}^2}{\theta_{h11}} - \frac{\theta_{12}^2}{\theta_{h12}}$. We see that there is only one point where $c = 0$, i.e. $T_{J,11} = T_{J,12}$, and that c is a monotone function of the input u. The monotonicity around the optimal point $c = 0$ is important as it ensures controllability.

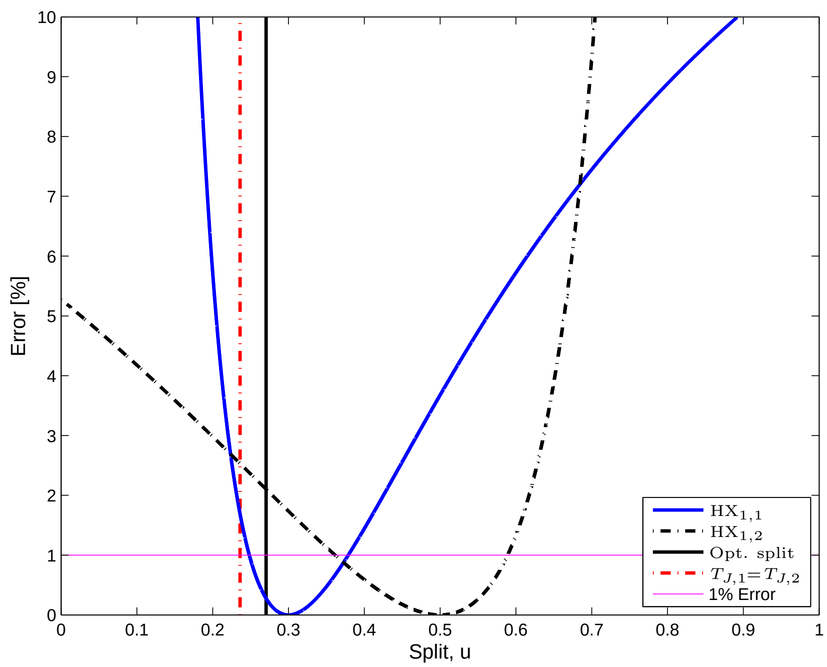


Figure 4: Case Study 1: Error between the logarithmic mean temperature difference in the heat exchangers and the arithmetic mean temperature as a function of the split $u = \frac{F_1}{F_0}$.

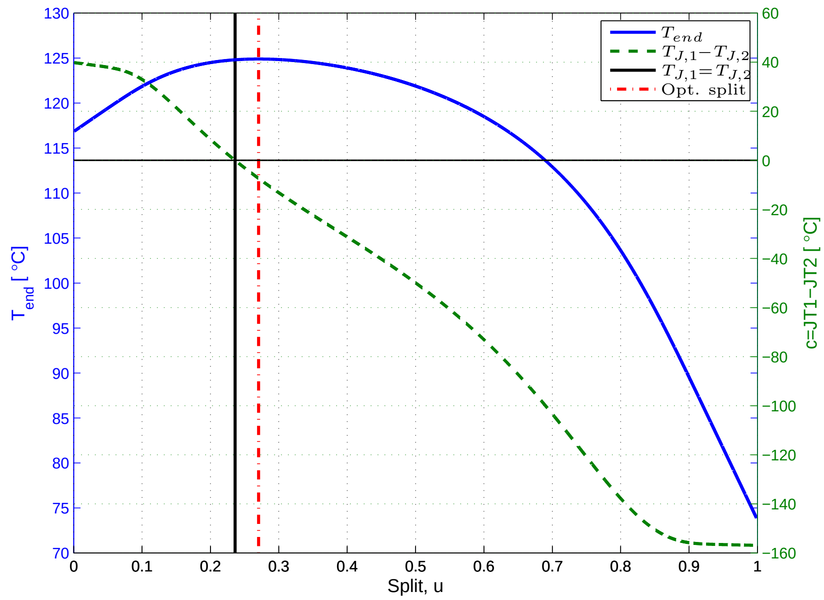


Figure 5: Case Study 1: End temperature and controlled variable $c = T_{J,1} - T_{J,2}$

7.2. Case Study 2: Two exchangers on first branch, one exchanger on second branch

Next, we give an example of a system with 2 heat exchangers in series on the first line, and one heat exchanger on the second line. Here too, the objective is to maximize the total heat transfer and to maximize T_{end}. The stream data are listed in Table 3, and the simulation results are summarized in Table 4, where the optimal end temperature is presented together with results that are obtained when using the our new approach and when implementing isothermal mixing. The difference between the optimal T_{end} and the T_{end} using the Jäschke Temperatures is very small, while the isothermal mixing strategy again results in a higher loss.

Figure 6 shows the approximation errors in the three heat exchangers as a function of the split u. Here, too, we find that even though the approximation error is about 23% for the exchanger 11 and larger than 2% for the other heat exchangers, the Jäschke Temperature approach gives good results.

In Figure 7 we plotted the cost function versus split ratio together with

Table 3: Data for Case Study 2

Variable	Value	Unit	Description
T_0	60	°C	Feed temperature
w_0	100	kW/K	Feed heat capacity
w_{h11}	50	kW/K	Hot stream 11 heat capacity
w_{h21}	30	kW/K	Hot stream 21 heat capacity
w_{h12}	40	kW/K	Hot stream 12 heat capacity
T_{h11in}	80	°C	Hot stream 11 inlet temperature
T_{h21in}	140	°C	Hot stream 21 inlet temperature
T_{h12in}	220	°C	Hot stream 12 inlet temperature
UA_{11}	80	kW/K	HX11 area × heat transf. coeff.
UA_{21}	50	kW/K	HX21 area × heat transf. coeff.
UA_{12}	65	kW/K	HX12 area × heat transf. coeff.

Table 4: Results for Case Study 2

	Optimized	Equal Jäschke Temp.	Isothermal mixing
End Temp.	122.7726°C	122.7549°C	120.8782°C
Split $u = \frac{F_1}{F_0}$	0.4326	0.4147	0.2559

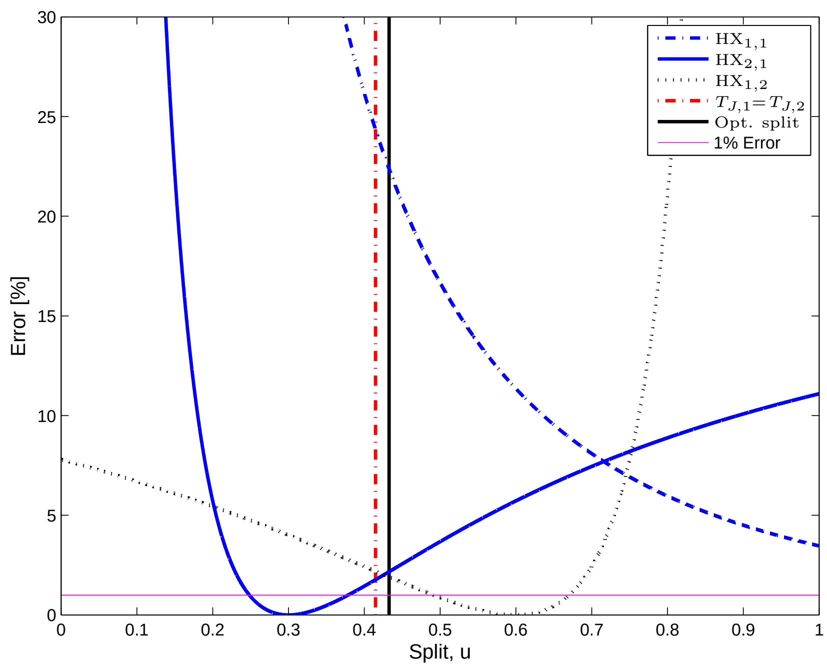


Figure 6: Case Study 2: Error between the logarithmic mean temperature difference in the heat exchangers and the arithmetic mean temperature as a function of the split $u = \frac{F_1}{F_0}$.

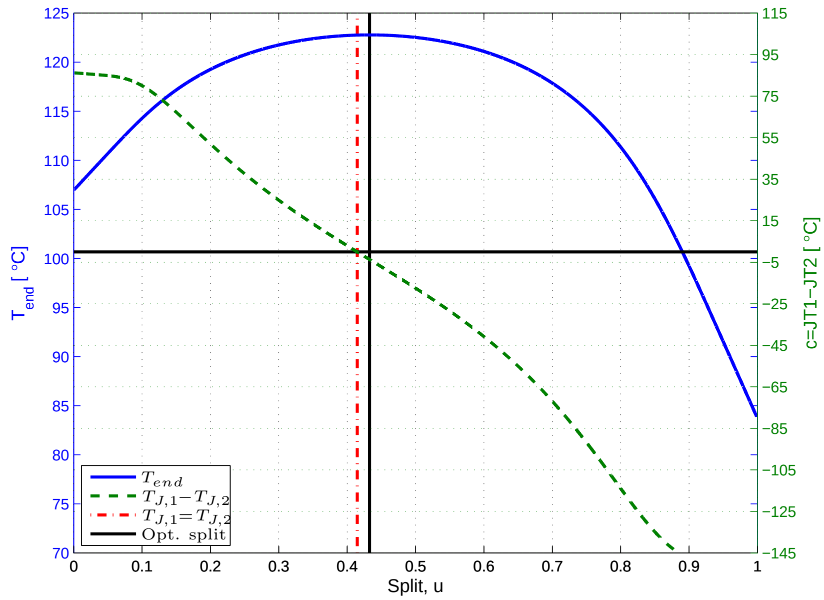


Figure 7: Case Study 2: End temperature and controlled variable $c = T_{J,1} - T_{J,2}$

the controlled variable

$$c = (T_{J,1,1} + T_{J,2,1}) - T_{J,1,2}. \tag{21}$$

We observe that here too, we the controlled variable is zero very close to the optimum, and crosses zero only once.

7.3. Case Study 3: One heat exchanger per branch – different prices

We consider the same heat exchanger network as in Case Study 1, but now with an economic objective involving different prices associated with heat transfer from the two heat exchangers. The stream temperatures and flow data are the same as before, given in Table 1. The economic cost is then

$$J = (p_{11}^{cost} - p_{11}^{rev})Q_{11} + (p_{12}^{cost} - p_{12}^{rev})Q_{12}. \tag{22}$$

Assuming that $p_{11}^{rev} = p_{12}^{rev} = 0.3\frac{cent}{\text{kJ}}$, and $p_{11}^{cost} = 0.1\frac{ct}{kJ}$, and $p_{12}^{cost} = 0.2\frac{ct}{kJ}$, the simplified cost (8) becomes

$$J = -0.2Q_{11} - 0.1Q_{12}, \tag{23}$$

Table 5: Results for Case Study 3

	Optimized	Equal Jäschke Temp	Isothermal mixing
Operating cost (\$/s)	-7.6256	-7.6250	-6.3536
Split $u = \frac{F_1}{F_0}$	0.3452\$	0.3364\$	0.0600\$

and this cost function encourages to prefer using heat exchanger 11 to heat exchanger 12. This will not result in the maximally possible end temperature, but it will optimize the process economics. Table 5 lists the optimal operating cost together with the cost from our approach and the isothermal mixing approach. The optimal cost, and the cost found by our approach are very close. As expected isothermal mixing gives the highest cost. This is further illustrated in Figure 8, where the profit and the controlled variables are shown for all possible splits u. As above, the controlled variable is zero only once, and very close to the optimal split.

8. Discussion

8.1. Flow configurations and network topologies

The assumption of the arithmetic mean temperature difference does not imply anything on the flow configuration directly. However, the assumption will generally be better satisfied in heat exchangers with counter-current flows, because the temperature profiles in the hot and the cold medium are closer to parallel.

Although the results in this paper were derived for a standard heat exchanger configuration, as shown in Figure 1 and 3, the results are more widely applicable than it may seem at first sight, because thy can be applied to related, equivalent topologies. In our approach, the heat capacities w, the heat transfer properties UA, and the outlet temperatures of the hot streams can be considered as disturbances, which have been eliminated from the optimality conditions. It is therefore possible to use our controlled variables for systems, which can be modeled as standard systems, by adjusting the heat transfer area and/or the heat capacities. For example, the system in Figure 9a may be modelled as a single counter-current heat exchanger, and the Jäschke Temperature can be calculated accordingly. Similarly, when more

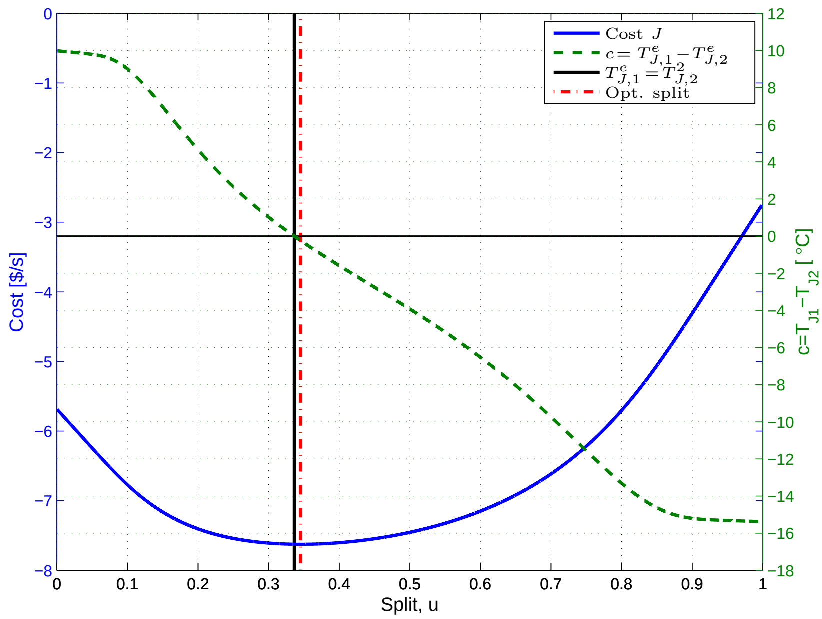
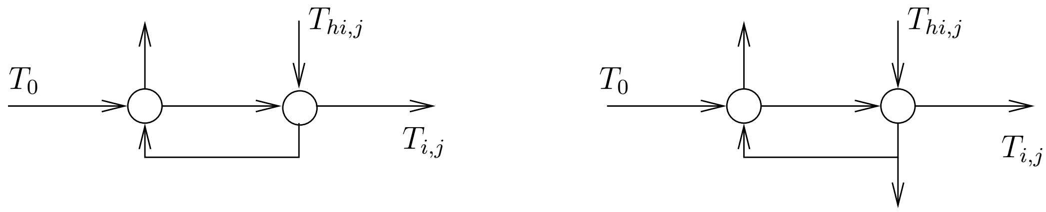


Figure 8: Case study 3: Economic cost and controlled variable $c = T^e_{J,1} - T^e_{J,2}$ for different values of the split ratio



$$T_{J,j} = \frac{(T_{i,j}-T_0)^2}{T_{hi,j}-T_0} = \frac{\theta_{i,j}^2}{\theta_{hi,j}} \qquad T_{J,j} = \frac{(T_{i,j}-T_0)^2}{T_{hi,j}-T_0} = \frac{\theta_{i,j}^2}{\theta_{hi,j}}$$

Figure 9: Examples of Topologies with corresponding Jäschke Temperature

than two heat exchangers on a branch are connected to a single hot stream, they may be modelled as a single heat exchanger. Also, the configuration in Figure 9b can be modelled as a single heat exchanger, because the effect of the reduced flow in the second heat exchanger on the hot path is equivalent to the effect of a reduced overall heat transfer coefficient. Of course, it is possible to combine all the above configurations, such that beside the examples given above, all parallel flow configurations which can be modelled as one or more heat exchangers on a line, may be operated close to optimal by our approach.

Our controlled variable may also be be used when the flow thorough one or more branches is not available as a degree of freedom. An example for this case is given in Figure 10, where there is a temperature constraint on the hot outflow of the first branch. In this case, the fraction through the first branch u_1 is not available as a degree of freedom for optimization, as it must be used to maintain the exit temperature of the hot stream. However, the second degree of freedom u_2 can be used control the difference between the Jäschke Temperatures in the second and third branch. This maximizes the heat transfer in the exchangers on these branches (and T_{end}).

8.2. Strengths and limits of our approach

The big advantage of our method is that it is very simple, and still gives close to optimal performance. Neither flow rates nor heat capacities need to be measured. The main underlying assumption in the derivation of the Jäschke Temperatures is the arithmetic mean temperature difference as the driving force for heat transfer. Even if this assumption is violated, we have shown that our controlled variables give good performance. However, for

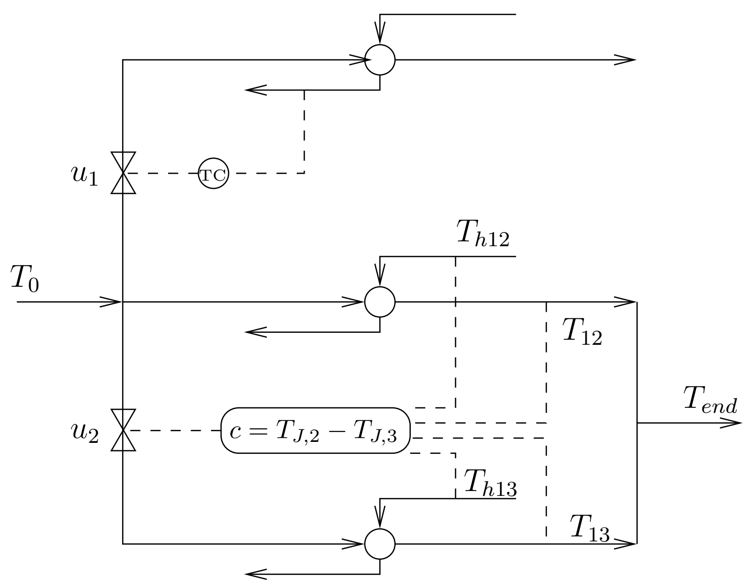


Figure 10: Control structure where one degree of freedom is used to control another variable

very poorly designed heat exchanger networks, which are operated with very unequal heat capacities on the hot and the cold side, the performance can deteriorate. For very extreme cases, that is when one of the hot streams has a much higher heat capacity than the cold stream, and a very much higher temperature than the other hot streams, the approximation by the arithmetic temperature difference may become very poor, and it may not be possible to control the differences between the Jäschke Temperatures to zero. However, when such a case is implemented using PI controllers with anti-windup, the controller will attempt to control the difference to zero, and this will result in fully opening the dominating branch and closing the others. This will not be optimal, but the loss will be relatively small, because most of the heat is transfered in the dominating branch, and the other branches contribute only marginally to the total transferred heat. If truly optimal performance is required for such extreme cases, a different approach has to be chosen, which essentially is based on a more accurate model. Such a method will, however, typically require more effort in implementing and maintaining.

In well designed heat exchanger networks, however, the hot and the cold heat capacities will be approximately equal, and this will make the approximation error small. Another factor which mitigates the effect of the approximation error is that the optima in such systems tend to be very flat, such that a split which is only close to optimal will give a good performance.

One of the limitations of this method is that it requires the marginal costs to be decoupled, i.e. it is not possible to have crossovers between the lines. If there are crossovers, it is not possible to find optimality conditions of the form $c = c_1 - c_2$, where c_j contains only variables from branch j, and we must use other approaches, such as described in e.g. Alstad et al. (2009).

8.3. Implementation issues

The formulas for the Jäschke Temperatures derived in this paper contain temperature differences in the denominators, $\theta_{hi,j} - \theta_{i-1,j}$ which under practical operation may cross zero and become negative. When this temperature difference becomes negative, the cold stream is cooled down (rather than heated) in the particular heat exchanger. This results in a negative contribution to the Jäschke Temperature.

When the temperatures of the hot inlet stream and the cold inlet stream of a heat exchanger are equal, $\theta_{hi,j} = \theta_{i-1,j}$, there is no heat transfer in that particular heat exchanger. Operation at such this point is unlikely in practice, as this is a zero-measure set. However during transients this may well happen, and then the Jäschke Temperatures become singular, which can result in unpredictable controller behavior.

To avoid this undesirable behavior, there are two possible approaches. The first one is to implement the controlled variable denominator-free by multiplying the controlled variable with the greatest common denominator. For a system with one heat exchanger per line, as in Example 1, the controlled variable would become

$$c = p_{1,1}\theta_{1,1}^2\theta_{h1,2} - p_{12}\theta_{1,2}^2\theta_{h1,1}.$$

In this case the controlled variable does never become singular. The topologies with more than one heat exchanger on a line can be treated similarly.

An alternative approach is to use a piecewise defined Jäschke Temperatures to patch the singular point. Here, we define the variable $a_{i,j}$ in (11) as

$$a_{ij} = \begin{cases} \frac{(\theta_{i,j}-\theta_{i-1,j})[\theta_{i,j}+\theta_{i-1,j}-a_{i-1,j}]}{\theta_{hi,j}-\theta_{i-1,j}} & \text{for } |\theta_{hi,j} - \theta_{i-1,j}| > \epsilon \\ 0 & \text{for } |\theta_{hi,j} - \theta_{i-1,j}| \leq \epsilon \end{cases}$$

where ϵ is a small tunable parameter, e.g. $\epsilon = 10^{-3}$.

9. Conclusions and future work

We have presented an approach for optimizing the split between the lines of a parallel heat exchanger system. Although the Jäschke Temperatures are developed for the arithmetic temperature difference, they give good results for the realistic case with logarithmic temperature difference. In particular, our approach gives good performance for well-designed heat exchangers, where the heat capacities flows on the hot and the cold side are approximately the same.

For the case with 1-3 heat exchangers per line, we have proven under the assumption of the arithmetic mean temperature difference as driving force for the heat transfer, that the Jäschke Temperatures are equal to the marginal costs. The more general case with more than 3 heat exchangers is conjectured.

We would like to mention again that the Jäschke Temperatures can be used also in the case where the hot stream is split into parallel streams, which are cooled down.

Controlling the Jäschke Temperatures for optimization has an additional practical advantage. If one of the flows F_i is set to manual (or is used to control some thing else) the rest of the control structure remains unaffected.

A limitation is that we cannot handle systems with crossovers, because the optimality conditions are no longer decoupled. Future work will consider possibilities to handle coupled optimality conditions, and to integrate our approach into a larger setting, where not only the heat transferred in the heat exchangers is maximized, but rather the economics of the whole plant.

Acknowledgements

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Appendix A. Derivation for 3 HX on a line

To derive the general expression, we consider the system shown in Figure A.11, where the objective is to minimize the cost

$$J = p_{1,1}Q_{1,1} + p_{2,1}Q_{2,1} + p_{3,1}Q_{3,1} + p_{1,2}Q_{1,2}$$

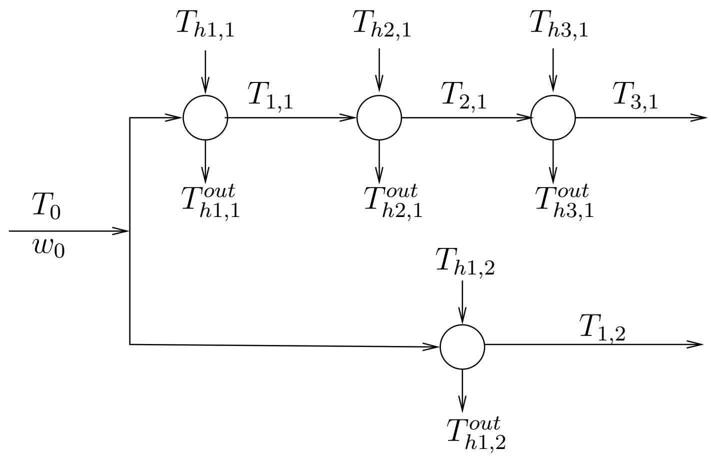

Figure A.11: HEN with 6 HX on the first line and one HX on the second line

where we assume $w_{hi,j} \to \infty$, or equivalently that $T_{hi,j}^{out} = T_{hi,j}$. Using Maple, the optimality conditions are calculated for the model described in Section 4, and all unknown variables are eliminated using the sparse resultant to give the controlled variable c, which is equivalent to the gradient. Upon collecting the coefficients of the different prices, we obtain

$$
\begin{aligned}
c = & \left(\frac{-\theta_{1,1}^2\theta_{2,1}\theta_{h2,1}+\theta_{1,1}^3\theta_{2,1}+\theta_{1,1}^2\theta_{h2,1}\theta_{h3,1}-\theta_{1,1}^3\theta_{h3,1}}{\theta_{h1,1}(-\theta_{2,1}\theta_{h2,1}+\theta_{h2,1}\theta_{h3,1}+\theta_{1,1}\theta_{2,1}-\theta_{1,1}\theta_{h3,1})}\right)p_{1,1} \\
& + \left(\frac{\begin{array}{r}\theta_{1,1}^2\theta_{2,1}\theta_{h1,1}+\theta_{1,1}^2\theta_{2,1}^2-\theta_{1,1}^3\theta_{2,1}-\theta_{1,1}^2\theta_{2,1}\theta_{h3,1}+\theta_{1,1}^3\theta_{h3,1} \\ -\theta_{1,1}^2\theta_{h1,1}\theta_{h3,1}-\theta_{2,1}^3\theta_{h1,1}+\theta_{2,1}^2\theta_{h1,1}\theta_{h3,1}\end{array}}{\theta_{h1,1}(-\theta_{2,1}\theta_{h2,1}+\theta_{h2,1}\theta_{h3,1}+\theta_{1,1}\theta_{2,1}-\theta_{1,1}\theta_{h3,1})}\right)p_{2,1} \\
& + \left(\frac{\begin{array}{r}-\theta_{1,1}^2\theta_{2,1}\theta_{h1,1}+\theta_{1,1}^2\theta_{2,1}\theta_{h2,1}+\theta_{1,1}^2\theta_{3,1}\theta_{h1,1}-\theta_{1,1}^2\theta_{3,1}\theta_{h2,1} \\ -\theta_{1,1}^2\theta_{2,1}^2+\theta_{3,1}^2\theta_{h1,1}\theta_{h2,1}+\theta_{1,1}\theta_{2,1}^2\theta_{h1,1}-\theta_{2,1}^2\theta_{h1,1}\theta_{h2,1} \\ -\theta_{1,1}\theta_{3,1}^2\theta_{h1,1}+\theta_{1,1}^2\theta_{2,1}\theta_{3,1}-\theta_{2,1}^2\theta_{3,1}\theta_{h1,1}+\theta_{2,1}^3\theta_{h1,1}\end{array}}{\theta_{h1,1}(-\theta_{2,1}\theta_{h2,1}+\theta_{h2,1}\theta_{h3,1}+\theta_{1,1}\theta_{2,1}-\theta_{1,1}\theta_{h3,1})}\right)p_{3,1} \\
& - \left(\frac{\theta_{1,2}^2}{\theta_{h1,2}}\right)p_{1,2}.
\end{aligned}
\tag{A.1}
$$

Already now we see Economic Jäschke Temperature $T_{J,2} = p_{1,2}\frac{\theta_{1,2}^2}{\theta_{h1,2}}$ for second line with heat exchanger 12. The three terms corresponding to $p_{1,1}, p_{2,1}, p_{3,1}$ correspond to the Economic Jäschke Temperature for the first line.

The coefficient of $p_{1,1}$ in (A.1) can be written as

$$
\begin{aligned}
a_{1,1} &= \frac{-\theta_{1,1}^2\theta_{2,1}\theta_{h2,1}+\theta_{1,1}^3\theta_{2,1}+\theta_{1,1}^2\theta_{h2,1}\theta_{h3,1}-\theta_{1,1}^3\theta_{h3,1}}{\theta_{h1,1}(-\theta_{2,1}\theta_{h2,1}+\theta_{h2,1}\theta_{h3,1}+\theta_{1,1}\theta_{2,1}-\theta_{1,1}\theta_{h3,1})} \\
&= \frac{\theta_{1,1}^2\left(\theta_{h21}-\theta_{1,1}\right)\left(\theta_{h3,1}-\theta_{2,1}\right)}{\theta_{h1,1}\left(\theta_{h21}-\theta_{1,1}\right)\left(\theta_{h3,1}-\theta_{2,1}\right)} \\
&= \frac{\theta_{1,1}^2}{\theta_{h1,1}},
\end{aligned}
$$

the second term can be written as

$$
\begin{aligned}
a_{2,1} &= \frac{\begin{array}{r}\theta_{1,1}^2\theta_{2,1}\theta_{h1,1} + \theta_{1,1}^2\theta_{2,1}^2 - \theta_{1,1}^3\theta_{2,1} - \theta_{1,1}^2\theta_{2,1}\theta_{h3,1} + \theta_{1,1}^3\theta_{h3,1} \\ - \theta_{1,1}^2\theta_{h1,1}\theta_{h3,1} - \theta_{2,1}^3\theta_{h1,1} + \theta_{2,1}^2\theta_{h1,1}\theta_{h3,1}\end{array}}{\theta_{h1,1}\left(\theta_{h2,1} - \theta_{1,1}\right)\left(\theta_{h3,1} - \theta_{2,1}\right)} \\
&= \frac{\left(\theta_{h3,1} - \theta_{2,1}\right)\left[\theta_{h1,1}(\theta_{2,1}^2 - \theta_{1,1}^2) - \theta_{1,1}^2(\theta_{2,1} - \theta_{1,1})\right]}{\theta_{h1,1}\left(\theta_{h2,1} - \theta_{1,1}\right)\left(\theta_{h3,1} - \theta_{2,1}\right)} \\
&= \frac{\theta_{2,1}^2 - \theta_{1,1}^2 - \frac{\theta_{1,1}^2}{\theta_{h1,1}}(\theta_{2,1} - \theta_{1,1})}{\theta_{h2,1} - \theta_{1,1}} \\
&= \frac{\left(\theta_{2,1} - \theta_{1,1}\right)\left(\theta_{2,1} + \theta_{1,1} - \frac{\theta_{1,1}^2}{\theta_{h1,1}}\right)}{\theta_{h2,1} - \theta_{1,1}} \\
&= \frac{\left(\theta_{2,1} - \theta_{1,1}\right)\left(\theta_{2,1} + \theta_{1,1} - a_{1,1}\right)}{\theta_{h2,1} - \theta_{1,1}},
\end{aligned}
$$

and finally, factorizing and simplifying the coefficient of $p_{3,1}$ gives

$$
\begin{aligned}
a_{3,1} &= \frac{\begin{array}{r}-\theta_{1,1}^2\theta_{2,1}\theta_{h1,1} + \theta_{1,1}^2\theta_{2,1}\theta_{h2,1} + \theta_{1,1}^2\theta_{3,1}\theta_{h1,1} - \theta_{1,1}^2\theta_{3,1}\theta_{h2,1} \\ - \theta_{1,1}^2\theta_{2,1}^2 + \theta_{3,1}^2\theta_{h1,1}\theta_{h2,1} + \theta_{1,1}\theta_{2,1}^2\theta_{h1,1} - \theta_{2,1}^2\theta_{h1,1}\theta_{h2,1} \\ - \theta_{1,1}\theta_{3,1}^2\theta_{h1,1} + \theta_{1,1}^2\theta_{2,1}\theta_{3,1} - \theta_{2,1}^2\theta_{3,1}\theta_{h1,1} + \theta_{2,1}^3\theta_{h1,1}\end{array}}{\theta_{h1,1}(-\theta_{2,1}\theta_{h2,1} + \theta_{h2,1}\theta_{h3,1} + \theta_{1,1}\theta_{2,1} - \theta_{1,1}\theta_{h3,1})} \\
&= \frac{\left(\theta_{3,1} - \theta_{2,1}\right)\left(\theta_{3,1} + \theta_{2,1} - \frac{\left(\theta_{2,1} - \theta_{1,1}\right)\left(\theta_{2,1} + \theta_{1,1} - \frac{\theta_{1,1}^2}{\theta_{h1,1}}\right)}{\theta_{h2,1} - \theta_{1,1}}\right)}{\theta_{h3,1} - \theta_{2,1}} \qquad \text{(A.2)} \\
&= \frac{\left(\theta_{3,1} - \theta_{2,1}\right)\left(\theta_{3,1} + \theta_{2,1} - a_{2,1}\right)}{\theta_{h3,1} - \theta_{2,1}}.
\end{aligned}
$$

Due to computational limitations the elimination procedure failed for systems with more than 3 heat exchangers on a line. However, the structure of the solutions for the case with 1-3 heat exchangers lets us conjecture that it is true for N heat exchangers.

Appendix B. Numerical validation of the case with more than 3 exchangers on a line (Validation Case Study, $M_1 = 4, M_2 = 1$)

In this section we give some numerical evidence that the general formula for the Jäschke Temperature is true for $M_j > 3$. We expect that the results when controlling the Jäschke Temperatures to equal values, will be the same as when we optimize a model which uses the arithmetic mean temperature as driving force. This will be the case when, as we conjecture, the controlled variable is equivalent to the gradient.

Consider the heat exchanger network given in Figure B.12, with the corresponding stream and price data given in Table B.6. The controlled variable is

$$
\begin{aligned}
c = {} & p_{1,1}\frac{\theta_{1,1}^2}{\theta_{h1,1}} + p_{2,1}\frac{(\theta_{2,1}-\theta_{1,1})(\theta_{2,1}+\theta_{1,1}-\frac{\theta_{1,1}^2}{\theta_{h1,1}})}{\theta_{h2,1}-\theta_{1,1}} \\
& + p_{3,1}\frac{(\theta_{3,1}-\theta_{2,1})(\theta_{3,1}+\theta_{2,1}-\frac{(\theta_{2,1}-\theta_{1,1})(\theta_{2,1}+\theta_{1,1}-\frac{\theta_{1,1}^2}{\theta_{h1,1}})}{\theta_{h2,1}-\theta_{1,1}})}{\theta_{h3,1}-\theta_{2,1}} \\
& + p_{4,1}\frac{(\theta_{4,1}-\theta_{3,1})(\theta_{4,1}+\theta_{3,1}-\frac{(\theta_{3,1}-\theta_{2,1})(\theta_{3,1}+\theta_{2,1}-\frac{(\theta_{2,1}-\theta_{1,1})(\theta_{2,1}+\theta_{1,1}-\frac{\theta_{1,1}^2}{\theta_{h1,1}})}{\theta_{h2,1}-\theta_{1,1}})}{\theta_{h3,1}-\theta_{2,1}})}{\theta_{h4,1}-\theta_{3,1}} \\
& - p_{1,2}\frac{\theta_{1,2}^2}{\theta_{h1,2}}.
\end{aligned}
\tag{B.1}
$$

The values from optimizing the split with fmincon (Matlab) and from using our approach are compared in Table B.7. The cost function for the optimized and the case using the Jäschke Temperature approach give the same values of the cost function. However, the end temperatures are not exactly the same. The reasons for this discrepancy are the flatness of the optimum and numerical round-off errors, which occur due to the temperature difference terms in the controlled variable.

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Table B.6: Stream and price data for validation case study, $M_1 = 4, M_2 = 1$

Variable	Value	Unit	Description
w_0	100	kW/°C	Heat capacity cold stream
$w_{h1,1}$	50	kW/°C	Heat capacity hot stream 11
$w_{h2,1}$	30	kW/°C	Heat capacity hot stream 21
$w_{h3,1}$	15	kW/°C	Heat capacity hot stream 31
$w_{h4,1}$	25	kW/°C	Heat capacity hot stream 41
$w_{h1,2}$	70	kW/°C	Heat capacity hot stream 12
T_0	130	°C	Cold stream temperature
$T_{h1,1}$	190	°C	Hot stream 11 temperature
$T_{h2,1}$	203	°C	Hot stream 21 temperature
$T_{h3,1}$	220	°C	Hot stream 31 temperature
$T_{h4,1}$	235	°C	Hot stream 41 temperature
$T_{h1,2}$	225	°C	Hot stream 12 temperature
$UA_{1,1}$	5	kW/°C	Heat transfer coefficient times area of Exchanger 11
$UA_{2,1}$	7	kW/°C	Heat transfer coefficient times area of Exchanger 21
$UA_{3,1}$	10	kW/°C	Heat transfer coefficient times area of Exchanger 31
$UA_{4,1}$	12	kW/°C	Heat transfer coefficient times area of Exchanger 41
$UA_{1,2}$	11	kW/°C	Heat transfer coefficient times area of Exchanger 12
$p_{1,1}$	1	USD/kW	Price for using heat from stream 11
$p_{2,1}$	1.2	USD/kW	Price for using heat from stream 21
$p_{3,1}$	1.3	USD/kW	Price for using heat from stream 31
$p_{4,1}$	1.5	USD/kW	Price for using heat from stream 41
$p_{1,2}$	1.5	USD/kW	Price for using heat from stream 12

Table B.7: Results for validation case study, $M_1 = 4, M_2 = 1$

	Optimized	Equal Jäschke Temp.
Cost J	-3.9388	-3.9388
End Temp. T_{end}	158.7738	158.7741
Split $u = \frac{F_1}{F_0}$	0.7141	0.7152

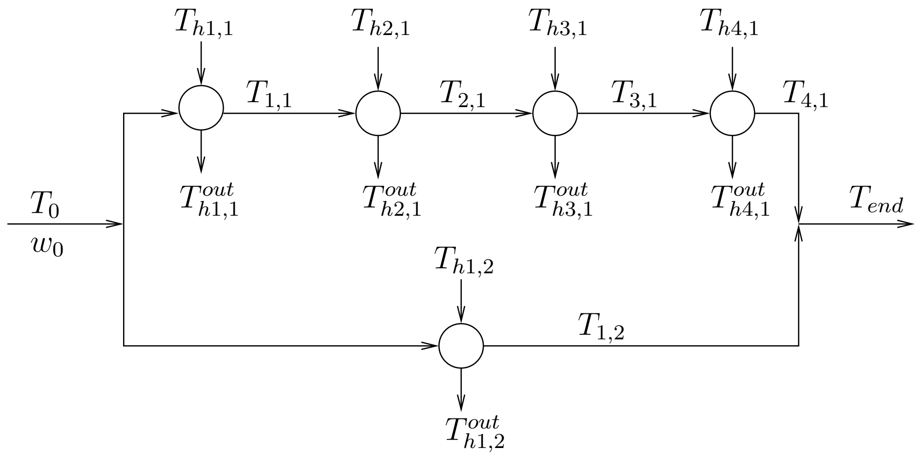

Figure B.12: Heat exchanger network with $M_1 = 4$ and $M_2 = 1$

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