Preprints of the 18th IFAC World Congress
Milano (Italy) August 28 - September 2, 2011

Optimal Controlled Variable Selection with Structural Constraints Using MIQP Formulations ⋆

Ramprasad Yelchuru **Sigurd Skogestad**

Department of Chemical Engineering, Norwegian University of Science and Technology, NTNU, Trondheim, Norway (e-mail: {skoge}@chemeng.ntnu.no)

Abstract: Optimal control structure selection is vital to operate the process plants optimally in the presence of disturbances. In this paper we review the controlled variable selection, $\mathbf{c} = \mathbf{Hy}$, where $\mathbf{y}$ includes all the measurements. The objective is to find the matrix $\mathbf{H}$ such that steady-state operation is optimized while controlled variables $\mathbf{c}'s$ are kept constant using inputs, when there are disturbances. Several cases are studied such as the optimal individual measurements, the optimal combinations of fewer/all measurements and the optimal combinations with structural constraints. The proposed methods are evaluated on a distillation column case study with 41 trays.

Keywords: Optimal operation, selection of controlled variables, measurement combination, plantwide control, structural constraints, Mixed Integer Quadratic Programming.

1. INTRODUCTION

Optimal operation of process plants aids in improved profitability. Apporipriate control structure selection facilitates optimal operation. The decision on which variables should be controlled, which variables to be measured, which inputs to be manipulated and which links should be made between them are called the controlled structure selection. Generally, the control structure selection decisions are done based on heuristic methods or on the intuition of process engineers. These methods cannot guarantee optimality and makes the analysis difficult to analyze and improve the proposals.

In this paper we consider the selection of controlled variables (CVs) associated with the unconstrained degrees of freedom. We assume that the CVs $\mathbf{c}'s$ are selected as individual measurements or combinations of fewer/all available measurements $\mathbf{y}$. This can be written as

$\mathbf{c} = \mathbf{Hy}$ where $n_y = n_c$;

n_y: number of measurements; n_c: number of CVs = number of unconstrained MVs = n_u ; where the objective is to find a good choice for the matrix $\mathbf{H}$. In general, we also include the inputs (MVs) in the available measurements set $\mathbf{y}$.

Assuming that the plant economics are primarily determined by the pseudo/steady state behavior, Skogestad and coworkers (Skogestad, 2000) have proposed to use the steady state process model to find "self-optimizing" controlled variable as combinations of measurements. The objective is to find $\mathbf{H}$ such that when the CVs are kept at constant set points, the operation gives acceptable steady state loss from the optimal operation in the presence of disturbances. The theory for self-optimizing control (SOC) is well developed for quadratic optimization problems with linear models. This may seem restrictive, but any unconstrained optimization problem may locally be approximated suitably by this method. The "exact local method" (Halvorsen et al., 2003) handles both disturbances and implementation errors. Here after we call "exact local method" as "minimum loss method". The problem of finding CVs as optimal variable combinations ($\mathbf{c} = \mathbf{Hy}$, where $\mathbf{H}$ is a "full" matrix) was originally believed to be non-convex and thus difficult to solve numerically (Halvorsen et al., 2003), but later it has been shown that this problem may be reformulated as a quadratic optimization problem with linear constraints (Alstad et al., 2009). The problem of selectingthe controlled variables as the individual measurements, the combinations of best measurement subsets is more difficult because of the combinatorial nature of the problem. As the number of alternatives increase rapidly with the process dimensions, resorting to exhaustive search methods to find the optimal solution is computationally intractable. Kariwala and Cao (Kariwala and Cao, 2009) have derived effective branch and bound methods that make use of the monotonicity property in the objective function for these cases, but these cannot handle the structural constraints. This motivates the need to develop simple but still efficient methods to find the optimal solution $\mathbf{H}$ with structural constraints.

Structural constraints are needed to improve dynamic controllability (i.e. fast response, control loop localization), to reduce the time delay between the MVs to CVs. In this paper, we consider the case where the $\mathbf{c}'s$ are obtained as combinations of specified structures and the cases with structural constraints. Unfortunately for these cases we do not have a convex problem formulation, but we derive

⋆ This work was supported by the Faculty of Natural Sciences, NTNU.

upper bounds to SOC problems with structural constraints by formulating them as convex QP problems at each node in MIQP formulation in 3 different approaches. We explore one of the 3 approaches in this paper. As a precursor to these approaches, we briefly review the methods of finding the globally optimal $\mathbf{H}$, when $\mathbf{c}'s$ are obtained as optimal individual measurements, optimal combinations of fewer/all measurements of the process plant in Mixed Integer Quadratic Programming (MIQP) framework (Yelchuru et al., 2010; Yelchuru and Skogestad, 2010).

In summary, we consider two interesting problems related to finding $\mathbf{H}$:

(1) Selection of CVs as combination of measurements with specified structures
(2) Selection of CVs as combination of measurement subsets with specified structures using n measurements and also obey few additional structural constraints. Where $n \in [n_u, n_y]$

We consider the solution of these problems when applied to the minimum loss method formulation of (Halvorsen et al., 2003). Heldt (Heldt, 2009) has reported an iterative approach to solve problem 1 with a unitary matrix constraint method, but it is still non-convex and does not guarantee global optimum. In this study we propose an MIQP based approach to problem 1 and 2. Even though the proposed methods cannot give a globally optimal $\mathbf{H}$ to obtain optimal $\mathbf{c}'s$ as combinations of measurements with specified structures, the bounds obtained in the proposed method are of significant value from a practical point of view. The developed methods are evaluated on a binary distillation column with 41 trays, where $\mathbf{c}'s$ are combinations of measurements with specified structures. The developed MIQP methods for SOC are generic and can easily be evaluated for any process plant.

2. MINIMUM LOSS METHOD

We here review the "minimum loss method" formulation from Halvorsen et al. (2003) and its optimal solution from Alstad et al. (2009) and present some new results (Theorems 4,6). We then provide some new ideas for dealing with the nonconvex case with structural constraints on $\mathbf{H}$. We denote measurements, inputs or manipulated variables, disturbances by $\mathbf{y}$,$\mathbf{u}$ and $\mathbf{d}$ respectively. The economic cost function for the steady state operation is denoted by $\mathbf{J}(\mathbf{u}, \mathbf{d})$. In order to keep the operation optimal in the presence of varying disturbances, the inputs $\mathbf{u}$ are updated according to $\mathbf{d}$ using online optimization (real-time optimization). We denote the optimal cost as $\mathbf{J}_{opt}(\mathbf{u}_{opt}(\mathbf{d}), \mathbf{d})$.

A simple and effective alternative is to update $\mathbf{u}$ using a feedback controller, which manipulates $\mathbf{u}$ to keep the CVs $\mathbf{c}'s$ at their specified set points $\mathbf{c}_s$.

$$\mathbf{c} = \mathbf{H}\mathbf{y} \tag{1}$$

where $\mathbf{c}_s = \mathbf{H}\mathbf{y}_{opt}(\mathbf{d}^*)$, $\mathbf{H}$ is the combination matrix and $\mathbf{y}$ are measurements.

Note that feedback introduces implementation error (noise) n^c. In the presence of integral action in feedback control the implementation error $n^c = \mathbf{H}n^y$. The difference between the cost functions of these two strategies is defined as the loss (Skogestad and Postlethwaite, 2005).

$$\mathbf{L} = \mathbf{J}(\mathbf{u}, \mathbf{d}) - \mathbf{J}_{opt}(\mathbf{u}_{opt}(\mathbf{d}), \mathbf{d}) \tag{2}$$

Here "Self optimizing control" can be viewed as the selection of optimal $\mathbf{H}$ in $\mathbf{c} = \mathbf{H}\mathbf{y}$ and by keeping these $\mathbf{c}'s$ at constant set point $\mathbf{c}_s$ results in the minimal loss or that gives acceptable loss from the optimal operation. The set point $\mathbf{c}_s$ are obtained from the optimal solution for the nominal disturbance $\mathbf{d}$.

In order to express the loss ($\mathbf{L}$) as a function of disturbances, implementation errors locally, the loss is approximated using a second order Taylors series expansion around the "moving" optimal $\mathbf{u}_{opt}(\mathbf{d})$. We assume that the set of active constraints for the process does not change with $\mathbf{d}$ and n^c. The linearized (local) model in terms of the deviation variables is written as

$$\Delta\mathbf{y} = \mathbf{G}^y\Delta\mathbf{u} + \mathbf{G}^y_d\Delta\mathbf{d} \tag{3}$$

$$\Delta\mathbf{c} = \mathbf{G}\Delta\mathbf{u} + \mathbf{G}_d\Delta\mathbf{d} \tag{4}$$

where $\mathbf{G} = \mathbf{H}\mathbf{G}^y$and $\mathbf{G}_d = \mathbf{H}\mathbf{G}^y_d$. For a constant set point policy ($\mathbf{c}_s = 0$) (Halvorsen et al., 2003).

It is assumed that the number of $\mathbf{c}'s$ is the same as the number of unconstrained degrees of freedom $\mathbf{u}$ and that $\mathbf{G} = \mathbf{H}\mathbf{G}^y$ is invertible. This assumption is needed to guarantee that the CVs are controlled at the specified set points using a controller with integral action.

Theorem 1. (Alstad et al., 2009; Halvorsen et al., 2003; Kariwala et al., 2008) Minimum loss method : To minimize the average and worst case loss for expected noise and disturbances, $\left\|\begin{bmatrix}\mathbf{d}' \\ n^{y'}\end{bmatrix}\right\|_2 \leq 1$, find the $\mathbf{H}$ that solves the problem

$$\min_{\mathbf{H}} \left\|\mathbf{J}_{uu}^{1/2}(\mathbf{H}\mathbf{G}^y)^{-1}\mathbf{H}\mathbf{Y}\right\|_{2,F} \tag{5}$$

where $\mathbf{Y} = [\mathbf{F}\mathbf{W}_d\mathbf{W}_n)]$; $\mathbf{F} = \frac{\partial\mathbf{y}_{opt}}{\partial\mathbf{d}} = \mathbf{G}^y\mathbf{J}_{uu}^{-1}\mathbf{J}_{ud} - \mathbf{G}^y_d$; the 2-norm (maximum singular value) is for worst case loss, frobenius norm (F) is for average loss.

In many cases it is easier to find the optimal disturbance sensitivity matrix $\mathbf{F}$ numerically by reoptimizing for various disturbances. Kariwala et al. (Kariwala et al., 2008) prove that the combination matrix $\mathbf{H}$ that minimizes the average loss in (5) is super optimal and in the sense that the same $\mathbf{H}$ minimizes the worst case loss in (5). Hence, only optimization problem (5) involving the frobenius norm (F) is considered in the rest of the paper.

2.1 *Finding full* $\mathbf{H}$ *without structural constraints*

Theorem 2. (Reformulation as a convex problem). The problem in equation (5) may seem non-convex (Alstad et al., 2009), but for the standard case where $\mathbf{H}$ is a full matrix (with no structural constraints), it can be reformulated as a constrained quadratic programming problem (Alstad et al., 2009)

$$\begin{aligned} &\min_{\mathbf{H}} ||\mathbf{H}\mathbf{Y}||_F \\ &\text{s.t. } \mathbf{H}\mathbf{G}^y = \mathbf{J}_{uu}^{1/2} \end{aligned} \tag{6}$$

Proof: From the original problem in equation (5) the optimal solution $\mathbf{H}$ is non-unique. If $\mathbf{H}$ is a solution then $\mathbf{H}_1 = \mathbf{D}\mathbf{H}$ is also a solution as $(\mathbf{J}_{uu}^{1/2}(\mathbf{H}_1\mathbf{G}^y)^{-1}\mathbf{H}_1\mathbf{F}) = (\mathbf{J}_{uu}^{1/2}(\mathbf{H}\mathbf{G}^y)^{-1}\mathbf{H}\mathbf{F})$ for any non-singular matrix $\mathbf{D}$ of size

$n_u \times n_u$. This means the objective function is unaffected by the choice of $\mathbf{D}$. One implication is that $\mathbf{HG}^y$ can be chosen freely. We can thus make $\mathbf{H}$ unique by adding a constraint, for example $\mathbf{HG}^y = \mathbf{J}_{uu}^{1/2}$. More importantly this simplifies the optimization problem in equation (5) to optimization problem shown in equation (6). *End Proof*

Theorem 3. (Alstad et al. (2009)). An analytical solution to (5) in Theorem 1 using Theorem 2 is
$$\mathbf{H}^T = \left(\mathbf{YY}^T\right)^{-1}\mathbf{G}^y\left(\mathbf{G}^{y^T}\left(\mathbf{YY}^T\right)^{-1}\mathbf{G}^y\right)^{-1}\mathbf{J}_{uu}^{1/2}.$$

Theorem 4. (Simplified analytical solution). Another analytical solution for the problem in (5) is
$$\mathbf{H}^T = \left(\mathbf{YY}^T\right)^{-1}\mathbf{G}^y\mathbf{Q} \tag{7}$$
where $\mathbf{Q}$ is any non-singular matrix of $n_c \times n_c$.

Proof. This follows trivially from Theorem 3, since if $\mathbf{H}^T$ is a solution then so is $\mathbf{H}_1^T = \mathbf{H}^T\mathbf{D}^T$ and we simply select $\mathbf{D}^T = (\mathbf{Q}^{-1}(\mathbf{G}^{y^T}(\mathbf{YY}^T)^{-1}\mathbf{G}^y)^{-1}\mathbf{J}_{uu}^{1/2})^{-1} = \mathbf{J}_{uu}^{-1/2}\mathbf{G}^{y^T}(\mathbf{YY}^T)^{-1}\mathbf{G}^y\mathbf{Q}$ which is a $n_c \times n_c$ matrix.*End Proof.*

Corollary 5. (Important insight). Theorem 4 gives the very important insight that $\mathbf{J}_{uu}$ is not needed for finding the optimal $\mathbf{H}$, provided we have the standard case where $\mathbf{H}$ can be any $n_c \times n_y$ matrix.

This means that in (6) we can replace $\mathbf{J}_{uu}^{1/2}$ by any non-singular matrix, and still get an optimal $\mathbf{H}$. This can greatly simplify practical calculations, because $\mathbf{J}_{uu}$ may be difficult to obtain numerically because it involves the second derivative. On the other hand, we have that $\mathbf{F}$, which enters in $\mathbf{Y}$, is relatively straightforward to obtain numerically. Although $\mathbf{J}_{uu}$ is not needed for finding the optimal $\mathbf{H}$, it would be required for finding a numerical value for the loss.

Theorem 6. (Generalized convex formulation). An optimal $\mathbf{H}$ for the problem (5) can be written as in (8) using Theorem 4, where $\mathbf{Q}$ is any non-singular matrix of $n_c \times n_c$.
$$\begin{aligned}&\min_{\mathbf{H}} \|\mathbf{HY}\|_F \\ &\text{s.t. } \mathbf{HG}^y = \mathbf{Q}\end{aligned} \tag{8}$$

Proof. The result follows from Corollary 5, but can more generally be derived as follows. The problem in (6) is to minimize $\left\|\underbrace{(\mathbf{J}_{uu}^{1/2}(\mathbf{HG}^y)^{-1}}_{\mathbf{X}}\mathbf{HY})\right\|_F$. The reason why we can omit the $n_c \times n_c$ matrix $\mathbf{X}$, is that if $\mathbf{H}$ is an optimal solution then so is $\mathbf{H}_1 = \mathbf{DH}$ where $\mathbf{D}$ is any nonsingular $n_c \times n_c$ (see proof of Theorem 2). However, note that the matrix $\mathbf{X}$, or equivalently the matrix $\mathbf{Q}$, must be fixed during the optimization, so it needs to be added as a constraint. *End Proof.*

Numerical evidence show that replacing the equality constraint in (8) with inequality constraint in the QP also give the solution to (5). This is also a convex formulation as
$$\begin{aligned}&\min_{\mathbf{H}} \|\mathbf{HY}\|_F \\ &\text{s.t. } \mathbf{HG}^y \leq \mathbf{Q}\end{aligned} \tag{9}$$
Here $\mathbf{Q}$ can be any non-singular matrix, but to find a non-trivial solution; $\mathbf{Q}$ is chosen to have non-negative elements in each row. Note that (6),(8),(9) are convex reformulations of the SOC problem in (5) only for a given measurement set. But we could not find any mathematical proof for the equivalence of the formulations in (5) to (9).

2.2 Dealing with structural constraints on $\mathbf{H}$

For practical reasons, it may be interesting to obtain the $\mathbf{c}'s$ as combinations of measurements with a specified structure.
$$\begin{aligned}&\min_{\mathbf{H}} \left\|\mathbf{J}_{uu}^{1/2}(\mathbf{HG}^y)^{-1}\mathbf{HY}\right\|_F^2 \\ &\text{s.t.}\mathbf{H} = [\text{specified structure}]\end{aligned} \tag{10}$$
We will consider the following special cases:

Case 1. Selecting subset of measurements (some columns in $\mathbf{H}$ are zero)

(a) **Fixed subset**. For example, $\mathbf{H} = \begin{bmatrix} 0 & \mathbf{h}_{12} & 0 & \mathbf{h}_{14} & \mathbf{h}_{15} \\ 0 & \mathbf{h}_{22} & 0 & \mathbf{h}_{24} & \mathbf{h}_{25}\end{bmatrix}$. In such cases, both Theorem 2 and 6 hold. This implies $\mathbf{J}_{uu}$ is *not* needed. This is quite obvious since it corresponds to deleting some measurements.

(b) **Optimal subset**. where the objective is to select measurements (e.g. 3 out of 5). In this case, only Theorem 2 hold and we need $\mathbf{J}_{uu}$. This is because in Theorem 2, $\mathbf{HG}^y = \mathbf{J}_{uu}^{1/2}$ and the ordering of the loss in (5) and $\|\mathbf{HF}\|_F$ is the same for all possible subsets.

Case 2. Specified structure (specified elements are zero in addition to some columns in $\mathbf{H}$ are zero)

(I) **Decentralized structure**. For example, If a process has 2 inputs and 5 measurements with 2 disjoint measurement sets {1,2,3},{4,5}; then the structure is $\mathbf{H}_I = \begin{bmatrix} \mathbf{h}_{11} & \mathbf{h}_{12} & \mathbf{h}_{13} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{h}_{24} & \mathbf{h}_{25}\end{bmatrix}$

(II) **Triangular structure**. For example, If a process has 2 inputs and 5 measurements with partially disjoint measurement sets as $\{1,2,3,4,5\}$ for one CV and $\{4,5\}$ for another CV, then the structure is $\mathbf{H}_{II} = \begin{bmatrix} \mathbf{h}_{11} & \mathbf{h}_{12} & \mathbf{h}_{13} & \mathbf{h}_{14} & \mathbf{h}_{15} \\ 0 & 0 & 0 & \mathbf{h}_{34} & \mathbf{h}_{35}\end{bmatrix}$

Theorem 2 do *not* hold in case 2. The reason is that to have same structure as $\mathbf{H}$ in $\mathbf{H}_1 = \mathbf{DH}$, $\mathbf{D}$ must have a structure $\mathbf{D}_I = \begin{bmatrix} \mathbf{d}_{11} & 0 \\ 0 & \mathbf{d}_{22}\end{bmatrix}$, $\mathbf{D}_{II} = \begin{bmatrix} \mathbf{d}_{11} & \mathbf{d}_{12} \\ 0 & \mathbf{d}_{22}\end{bmatrix}$ respectively so $\mathbf{D}$ is not a full matrix as assumed when deriving Theorem 2.

Case 3. Selecting the best individual measurements for decentralized control, for example, $\mathbf{H} = \begin{bmatrix} \mathbf{h}_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{h}_{24} & 0\end{bmatrix}$. This is a special case of case 2 (I), but Theorem 2 holds as it can also be viewed as case 1(b) as the selection of the best n_u measurements. Then the non-zero part of $\mathbf{H}$ is a square matrix and later we can choose $\mathbf{D}$ as inverse of this square full matrix to arrive at a decentralized diagonal $\mathbf{H}$.

2.3 Dealing with specified structures

Controlled variables $\mathbf{c}'s$ as combinations of measurements with specified structures. This is Case 2 for $\mathbf{H}$ (section 2.2), Here we consider 2 specified structures;

(1) to have separate control of individual process units, the structure of $\mathbf{H}$ "disjoint" will be

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{H}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{H}_{n_{iu}} \end{bmatrix} \tag{11}$$

where each $\mathbf{H}_i$ corresponds to measurements and inputs of process unit i. Where n_{iu} is number of individual process units in the plant.

(2) certain controlled variable $\mathbf{c}'s$ can be combinations of all measurements, but other $\mathbf{c}'s$ should be combinations of only a measurement subset ($\mathbf{H}$ "triangular") as in

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} & \cdots & \mathbf{H}_{1n_{iu}} \\ 0 & \mathbf{H}_{22} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{H}_{n_{iu}n_{iu}} \end{bmatrix} \tag{12}$$

For these specified structures (case 2 in section 2.3), Theorem 2, 6 does not apply. Note that, as opposed to cases 1 (a), $\mathbf{J}_{uu}$ is needed to find the optimal solution for case 2. So we do not have a convex problem formulation, that is, we need to solve the nonconvex problem in (5) (with additional constraints on the structure of $\mathbf{H}$ as in (10)). Nevertheless, using the ideas from Theorems 2 and 6, with additional constraints on the structure of $\mathbf{H}$, give convex optimization problems that provide *upper bounds* on the optimal $\mathbf{H}$ for case 2. In particular, in Theorem 6 we may make use of the extra degree of freedom provided by the matrix $\mathbf{Q}$ (Yelchuru and Skogestad, 2010).

The idea is to exclude the matrix $\mathbf{J}_{uu}^{1/2}(\mathbf{HG}^y)^{-1}$ in front of $\mathbf{HF}$ in (10). However, when $\mathbf{H}$ has a specified structure, we do not generally have enough degrees of freedom to make $\mathbf{J}_{uu}^{1/2}(\mathbf{HG}^y)^{-1} = I$. To proceed, we have considered the following 3 options :

(1) Use the non-zero (n_{nz}) elements in $\mathbf{D}$ to match any n_{nz} number of elements in $\mathbf{HG}^y$ to $\mathbf{J}_{uu}^{1/2}$ (Yelchuru and Skogestad, 2011).
(2) Introduce a constraint $\mathbf{HG}^y \leq \mathbf{Q}$ as in (9), this provides extra freedom to choose optimal structured $\mathbf{H}$. $\mathbf{Q}$ must be chosen to have negative elements in each row to obviate the trivial solution.
(3) Use a constraint to let $\mathbf{J}_{uu}^{1/2}(\mathbf{HG}^y)^{-1}$ have a structure similar to the $\mathbf{D}$ that preserves the structure in $\mathbf{H}, \mathbf{DH}$ and minimize $||\mathbf{HF}||_F$.

Numerical evidence shows that option 1 (Yelchuru and Skogestad, 2011), option 2 (current work) and option 3 provide good upper bounds to the problem in (10). We present details of option 2 in this paper.

The inequality constraint in option 2 is the halfspace of the affine constraint $\mathbf{HG}^y = \mathbf{Q}$ and this results in a simplified convex formulation (10)

$$\begin{aligned} &\min_{\mathbf{H}} ||\mathbf{HY}||_F \\ &\text{s.t. } \mathbf{HG}^y \leq \mathbf{Q} \\ &\quad\ \mathbf{H} = [\text{specified structure}] \end{aligned} \tag{13}$$

and (13) is vectorized (Yelchuru et al., 2010) to result in a formulation in equation (14). Solving equation (14) results in controlled variables $\mathbf{c}'s$ as combinations of measurements with specified structures. This provides the upper bound for problem in equation (10).

$$\begin{aligned} &\min_{\mathbf{x}_\delta} \mathbf{x}_\delta^T \mathbf{F}_\delta \mathbf{x}_\delta \\ &\text{s.t. } \mathbf{G}_\delta^{y^T} \mathbf{x}_\delta \leq \mathbf{Q}_\delta \\ &\text{set of eqns } \mathbf{x}_\delta(\text{ind}) = 0 \\ &\text{ind is associated to 0 elements in } \mathbf{H} \end{aligned} \tag{14}$$

where $\mathbf{X}_\delta^T = \begin{bmatrix} \mathbf{x}_1^T \mathbf{x}_2^T \ldots \mathbf{x}_{n_u}^T \end{bmatrix}$; $\mathbf{Q}_\delta^T = \begin{bmatrix} \mathbf{Q}_1^T \mathbf{Q}_2^T \ldots \mathbf{Q}_{n_u}^T \end{bmatrix}$ and large matrices $\mathbf{G}_\delta^T = \begin{bmatrix} \mathbf{G}^{y^T} & 0 & 0 & \cdots \\ 0 & \mathbf{G}^{y^T} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & \ldots & \mathbf{G}^{y^T} \end{bmatrix}$ $\mathbf{Y}_\delta = \begin{bmatrix} \mathbf{Y} & 0 & 0 & \cdots \\ 0 & \mathbf{Y} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & \ldots & \mathbf{Y} \end{bmatrix}$ $\mathbf{F}_\delta = \mathbf{Y}_\delta \mathbf{Y}_\delta$. where $\mathbf{X}_\delta, \mathbf{Q}_\delta, \mathbf{G}_\delta^T, \mathbf{F}_\delta$ are of size $(n_u n_y) \times 1, (n_u n_u) \times 1, (n_u n_u) \times (n_y n_u), (n_u n_y) \times (n_u n_y)$ respectively.

3. MIQP FORMULATIONS

3.1 CVs as combinations of the best measurement sets

The best measurement subset selection problem is to find $\mathbf{c}'s$ as best combinations of measurement subsets. This is Case 1 for $\mathbf{H}$ for which Lemma 1 holds. Some solution approaches are

- partial branch and bound methods (Kariwala and Cao, 2010)
- generalized singular value decomposition methods (Heldt, 2009)
- MIQP based formulations (Yelchuru et al., 2010; Yelchuru and Skogestad, 2010).

We discuss only the MIQP formulations here. Starting from (6), the best measurement subset selection problem can be formulated in (15) as a Mixed Integer Quadratic Programming (MIQP) problem where the non-singular $\mathbf{Q}$ matrix and the "big M" parameter are used as extra degrees of freedom to reduce the computational time in solving the MIQP problem (Yelchuru and Skogestad, 2010).

$$\begin{aligned} \min_{\mathbf{x}_{aug}} \quad & \mathbf{x}_{aug}^T \mathbf{F}_{aug} \mathbf{x}_{aug} \\ st. \quad & \mathbf{G}_{new}^{y^T} \mathbf{x}_{aug} = \mathbf{Q}_\delta \\ & \mathbf{P}\mathbf{x}_{aug} = n \end{aligned} \tag{15}$$

$$\begin{bmatrix} -\mathbf{M} & 0 & 0 & \cdots \\ 0 & -\mathbf{M} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & \ldots & -\mathbf{M} \end{bmatrix} \sigma_i \leq \begin{bmatrix} \mathbf{x}_i \\ \mathbf{x}_{n_y+i} \\ \vdots \\ \mathbf{x}_{(n_u-1)n_u+i} \end{bmatrix} \leq \begin{bmatrix} \mathbf{M} & 0 & 0 & \cdots \\ 0 & \mathbf{M} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & \ldots & \mathbf{M} \end{bmatrix} \sigma_i$$

$$\sigma_i \in 0, 1 \quad \forall i = 1, 2, \cdots, n_y$$

where $\mathbf{X}_\delta^T = \begin{bmatrix} \mathbf{x}_\delta^T & \sigma_1 & \sigma_2 & \ldots & \sigma_{n_y} \end{bmatrix}$; and $\mathbf{F}_{aug} = [\mathbf{F}_\delta 0]$; $\mathbf{G}_{new}^{y^T} = \begin{bmatrix} \mathbf{G}_\delta^T 0_{n_u n_y \times n_y} \end{bmatrix}$; $\mathbf{P} = [01]$ where n is the measurement subset size, $\mathbf{X}_\delta, \mathbf{F}_{aug}, \mathbf{P}$ are of size $(n_u n_y + n_y) \times 1, (n_u n_y + n_y) \times (n_u n_y + n_y), 1 \times (n_u n_y + n_y)$ respectively.

3.2 CVs as combinations of all measurements with specified structure

We considered two specified structures as "disjoint" and "triangular" structures as in Case 2 in section 2.2. The problem in equation (10) with these structures (Case 2 for $\mathbf{H}$) is non-convex, and, unfortunately, Theorem 2 cannot be used to get a convex QP because of the structural constraints in $\mathbf{H}$. But we derive a convex QP that provides a good upper bound (section 2.3) to find $\mathbf{H}$ with specified structures.

3.3 CVs as combinations of fewer measurements sets with the specified structure

It is easy to extend the problem formulation in (14) to find CVs as best combinations of fewer measurements with the specified structure by introducing n_y new binary variables $\sigma_1, \sigma_2, \cdots, \sigma_{n_y} \in \{0, 1\}$. The MIQP problem is the same as (15) with a change in equality constraint $\mathbf{G}^{y^T}_{new}\mathbf{x}_{aug} = \mathbf{Q}_\delta$ to $\mathbf{G}^{y^T}_{new}\mathbf{x}_{aug} \leq \mathbf{Q}_\delta$ and a few additional constraints associated to the decentralized structure.

$$
\begin{aligned}
&\text{set of eqns} \sum_{l=1}^{n_{y_k}} \sigma_{(n_{y_{(k-1)}}(k-1)+l)} = n_{u_k} \\
&\sum_{m=1}^{n_{y_k}} \sigma_{(n_{y_{(k-1)}}(k-1)+m)} = n_k \qquad (16)\\
&\forall k = 1, 2, \ldots, \text{number of blocks} \\
&\text{set of eqns } \mathbf{x}_{\text{aug}}(\text{ind}) = 0 \\
&\text{ind is associated to 0 elements in } \mathbf{H}
\end{aligned}
$$

where n_{u_k}, n_{y_k} and n_k are the numbers of inputs, measurements and measurements to be selected in each set k as per the specified structure.

4. RESULTS

The MIQP formulations for obtaining CVs with specified structures are evaluated on binary distillation column case study (Skogestad, 1997), where reflux $\mathbf{L}$ and boilup $\mathbf{V}$ are the remaining steady-state degrees of freedom ($\mathbf{u}$). The 41 stage temperatures are taken as candidate measurements. Note that we do not include the inputs in the candidate measurements for this case study. The economic objective $\mathbf{J}$ for the indirect composition control problem is

$$
\mathbf{J} = \left(\frac{\mathbf{x}^H_{top} - \mathbf{x}^H_{top,s}}{\mathbf{x}^H_{top,s}}\right)^2 + \left(\frac{\mathbf{x}^L_{btm} - \mathbf{x}^L_{btm,s}}{\mathbf{x}^L_{btm,s}}\right)^2 \qquad (17)
$$

where $\mathbf{J}$ is the relative steady state composition deviation. $\mathbf{x}^H_{top}, \mathbf{x}^L_{btm}, \mathbf{x}^H_{top,s}, \mathbf{x}^L_{btm,s}, L,$and H denote the heavy component composition in top tray, light component composition in bottom tray, specification of heavy component composition in top tray, specification of light component composition in bottom tray, light and heavy key components respectively. The MIQP is implemented (section 3.1) for the distillation column with 41 trays to find the 2 CVs as the combinations of 41 tray temperatures. An MIQP is set up for this distillation column with the choice $M = 1$ for the big-M constraints in equation (15). We solved the MIQP to find the CVs as the combinations of best measurement subset size from 2 to 41. The CPLX solver in IBM ILOG Optimizer was used in YALMIP toolbox (Lofberg, 2004) to solve the MIQP problem (IBM, 2010). We also study the cases with specified structures (11),(12) of $\mathbf{H}$; with the "disjoint" $\mathbf{H}$ (11), one $\mathbf{c}$ for the top and one $\mathbf{c}$ for the bottom part of the distillation column. This structure is desirable mainly for dynamic reasons; to select one combined measurement $\mathbf{c}_1$ from the top section (trays 21 to 41) and one combined measurement $\mathbf{c}_2$ from the bottom section (trays 1 to 20). The "triangular" structure (12) is desirable as controlling the combinations of top section measurements $\mathbf{c}_1$ with reflux ($\mathbf{L}$) and using the combinations of all measurements (trays 1 to 41) $\mathbf{c}_2$ with boilup ($\mathbf{V}$) is better dynamically. As including the bottom tray temperatures in $\mathbf{c}_1$ results in large delays between $\mathbf{c}_1$ and $\mathbf{L}$. For this distillation column case study

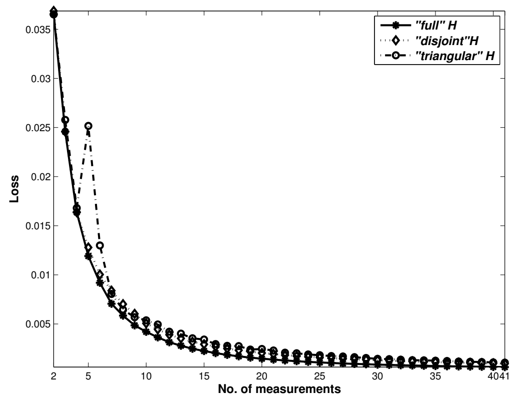

Fig. 1. The loss vs. the number of included measurements where the $\mathbf{c}'s$ are combinations of (i) all measurements (solid), (ii) disjoint measurements sets (11) (top and bottom of column), (iii) specified structure of $\mathbf{H}$ in (12)

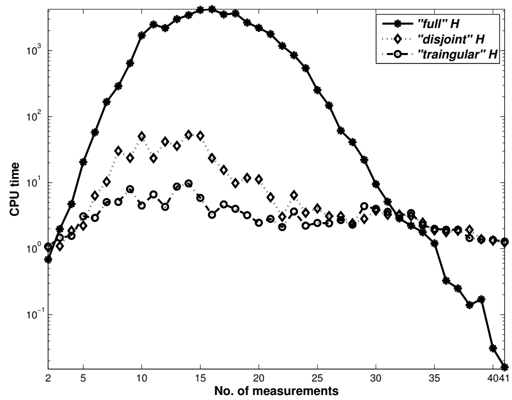

Fig. 2. CPU time requirement for computations in Fig. 1

Table 1. The self optimizing variables $\mathbf{c}'s$ as (i) combinations of measurements (ii) specified structure of $\mathbf{H}$ (12) (iii) combinations of disjoint measurement subsets (11) with their associated losses

Meas	Structure	Controlled variables ($\mathbf{c}'s$)	Loss
2	Full $\mathbf{H}$	$\mathbf{c}_1 = T_{12}$ $\mathbf{c}_2 = T_{30}$	0.036517
2	Triangular $\mathbf{H}$††	$\mathbf{c}_1 = T_{12}$ $\mathbf{c}_2 = T_{30}$	0.036517
2	Disjoint $\mathbf{H}$†	$\mathbf{c}_1 = T_{12}$ $\mathbf{c}_2 = T_{29}$	0.036868**
3	Full $\mathbf{H}$	$\mathbf{c}_1 = T_{11}$ $\mathbf{c}_2 = T_{30} + 0.0446T_{31}$	0.024583
3	Triangular $\mathbf{H}$††	$\mathbf{c}_1 = T_{30} + 0.9887T_{31}$ $\mathbf{c}_2 = T_{11} + 0.7365T_{30} + 0.7812T_{31}$	0.025775*
3	Disjoint $\mathbf{H}$†	$\mathbf{c}_1 = T_{30} + 0.9887T_{31}$ $\mathbf{c}_2 = T_{12}$	0.024593**
4	Full $\mathbf{H}$	$\mathbf{c}_1 = T_{11} + 11.2295T_{30} + 11.5251T_{31}$ $\mathbf{c}_2 = T_{12} - 11.5844T_{30} + 11.5251T_{31}$	0.016365
4	Triangular $\mathbf{H}$††	$\mathbf{c}_1 = T_{30} + 0.9887T_{31}$ $\mathbf{c}_2 = T_{10} + 1.0060T_{11} + 0.6901T_{30} + 0.7739T_{31}$	0.016803*
4	Disjoint $\mathbf{H}$†	$\mathbf{c}_1 = T_{30} + 0.9887T_{31}$ $\mathbf{c}_2 = T_{11} + 0.992T_{12}$	0.016385**

†"disjoint" $\mathbf{H}$,††"triangular" $\mathbf{H}$ (Case 2 for $\mathbf{H}$); $*$ clearly not optimal because optimal solution with "triangular" $\mathbf{H}$ is at least as good as "disjoint" $\mathbf{H}$; $**$ clearly not optimal because this is Case 3 for $\mathbf{H}$ and all structures must give same solution;

with 41 trays, in addition to the structures mentioned in (11), (12), the following structural constraints are also incorporated. To select n number of measurements, $\lfloor n/2 \rfloor$ number of measurements should be selected from top trays $\lfloor ny/2+1:ny \rfloor$, and rest of the measurements should be selected from $\{1:\lfloor ny/2 \rfloor\}$;

(1) to select 2 measurements, $\lfloor n/2 \rfloor = 1$ measurement should be selected from top trays 21 to 41 temperature measurements and other 1 measurement from bottom trays 1 to 20 temperature measurements
(2) to select 9 measurements, $\lfloor ny/2 \rfloor = 4$ measurements should be selected from top trays 21 to 41 temperatures, and rest of the measurements from bottom trays 1 to 20 temperatures.

The loss associated to these (11), (12) and these structural constraints is also shown in Fig. 1. Figure 1 show that the loss in terms of the relative composition deviation (17), decreases as the number of included measurements increases from 2 to 41. For each number of measurements, the actual measurements set is determined as part of the MIQP solution. The actual optimal controlled variables (measurement combination $\mathbf{H}$) for the cases with 2, 3 and 4 measurements are shown in Table 4. For the case with 2 measurements, we just give the measurement, and not the combination, because we can always choose the $\mathbf{D}$ matrix to make $\mathbf{H} = I$ (identity).

From Figure 1, we see that the losses with the specified structures are very close to the loss with $\mathbf{c}'s$ as combinations of all the included measurements. For "triangular" $\mathbf{H}$, the optimal solution should at least be as good as "disjoint" $\mathbf{H}$, but in Table 4 the loss with "triangular" is higher than "disjoint" case. The reason is that we are only minimizing the in convex formulation (13) for the given $\mathbf{H}$ structure and at the optimal solution, $||\mathbf{HF}||_F^2$ is smaller for "triangular" $\mathbf{H}$, but when we evaluate the original loss $\left|\left|\mathbf{J}_{uu}^{1/2}(\mathbf{HG}^y)^{-1}\mathbf{HF}\right|\right|_F^2$ the "disjoint" $\mathbf{H}$ has smaller loss. For "triangular" $\mathbf{H}$ there is sudden increase in loss from 4 to 5. This may be due to numerical issues associated to the cplex algorithm. The computational time required to find the optimal $\mathbf{H}$ with "full", "disjoint" (11) and "triangular" (12) structures are shown in Figure 2. From Figure 2, we see that computational time taken to obtain the $\mathbf{c}'s$ as combinations $\mathbf{H}$ with "disjoint" (11), "traingular" (12) structures with additional structural constraints are 1.8, 2.6 orders faster than "full" $\mathbf{H}$ case respectively.

For the case with 2 measurements, the optimal measurement set is $\{T_{12}, T_{29}\}$. However, for the disjoint measurement case, the convex formulation in (13) only gives an upper bound and it gives a non-optimal set $\{T_{12}, T_{30}\}$ and the loss is increased slightly from 0.0365 to 0.0369. Whereas the measurements and loss are same with special structure on $\mathbf{H}$ as in (12).

Interestingly, the optimal measurement sets are same for both "full"$\mathbf{H}$ and "disjoint"$\mathbf{H}$ cases when the measurement subset size is 3 and 4 (Table 4). However, since we are restricted in how we can combine measurements in the "disjoint" case, there is a small difference in the associated losses. Thus, although the method (16) developed for obtaining $\mathbf{c}'s$ with structural constraints are not exact, it serves as a tight upper bound for the true optimal solution for the problem in (10).

5. CONCLUSION

The minimum loss method of self optimizing control for optimal control structure selection with economic cost function as criterion is addressed. The MIQP based formulations to find controlled variables as best individual measurements, as best combinations of fewer/all measurements are reviewed. To improve dynamic controllability the controlled variables are only allowed to be combinations of measurements with specified structures. The proposed methods are not exact, but provides very close upper bound to the exact solution of $\mathbf{c}'s$ as combinations of measurements with specified structures, which is of significant value from a practical point of view. For the distillation column case study the loss increases only by a small value for the two specified structures considered.

REFERENCES

Alstad, V., Skogestad, S., and Hori, E. (2009). Optimal measurement combinations as controlled variables. *Journal of Process Control*, 19(1), 138–148.
Halvorsen, I.J., Skogestad, S., Morud, J.C., and Alstad, V. (2003). Optimal selection of controlled variables. *Ind. Eng. Chem. Res.*, 42.
Heldt, S. (2009). On a new approach for self-optimizing control structure design. In *ADCHEM 2009, July 12-15, Istanbul*, 807–811.
IBM (2010). Ibm ilog optimizer,ibm academic initiative.
Kariwala, V. and Cao, Y. (2009). Bidirectional branch and bound for controlled vaiable selection. part ii: Exact local method for self-optimizing control. *Computers and Chemical Engineering*, 33, 1402–1414.
Kariwala, V. and Cao, Y. (2010). Bidirectional branch and bound for controlled vaiable selection. part iii: Local average loss minimization. *IEEE Transactions on Industrial Informatics*, 6, 54–61.
Kariwala, V., Cao, Y., and Janardhanan, S. (2008). Local self-optimizing control with average loss minimization. *Ind. Eng. Chem. Res.*, 47, 1150–1158.
Lofberg, J. (2004). Yalmip : a toolbox for modeling and optimization in matlab. In *Computer Aided Control Systems Design, 2004 IEEE International Symposium on*, 284 –289. doi:10.1109/CACSD.2004.1393890.
Skogestad, S. (1997). Dynamics and control of distillation columns: A tutorial introduction. *Chemical Engineering Research and Design*, 75(6), 539 – 562. Distillation.
Skogestad, S. (2000). Plantwide control: The search for the self-optimizing control structure. *Journal of Process Control*, 10, 487–507.
Skogestad, S. and Postlethwaite, I. (2005). *Multivariable Feedback Control, 2. Edition.* Wiley.
Yelchuru, R. and Skogestad, S. (2010). Miqp formulation for optimal controlled variable selection in self optimizing control. In *PSE Asia, 2010. July 25-28, Singapore.*
Yelchuru, R. and Skogestad, S. (2011). Optimal controlled variable selection for individual process units in self optimizing control with miqp formulation. In *accepted for American Control Conference, San Francisco, USA.*
Yelchuru, R., Skogestad, S., and Manum, H. (2010). Miqp formulation for controlled variable selection in self optimizing control. In *DYCOPS, 2010. July 7-9, Brussels.*