IMC-like Analytical $\mathcal{H}_\infty$ design with S/SP mixed sensitivity consideration: Utility in PID tuning guidance☆

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Abstract

This article presents an $\mathcal{H}_\infty$ design that alleviates some difficulties with standard Internal Model Control (IMC), while still obeying the same spirit of simplicity. The controller derivation is carried out analytically based on a *weighted sensitivity* formulation. The corresponding frequency weight, chosen systematically, involves two tuning parameters with clear meaning in terms of common design specifications: one adjusts the *robustness/performance* trade-off as in the IMC procedure; the other one balances the *servo* and *regulatory* performance. For illustration purposes, the method is applied to analytical tuning of PI compensators. Due to its simplicity and effectiveness, the presented methodology is also suitable for teaching purposes.

Key words: $\mathcal{H}_\infty$ control, Weighted Sensitivity, IMC, PID tuning

1. Introduction

Simplicity is a desired feature of a control algorithm: we would like it to be widely applicable and easy to understand, involving as few tuning parameters as possible. Ideally, these parameters should possess a clear engineering meaning, making the tuning a systematic task according to the given specifications. As for implementation, low-order controllers are preferable.

In this line, the Proportional-Integrative-Derivative (PID) controller is recognized to be the bread and butter of automatic control, being by far the most dominating form of feedback in a wide range of industrial applications [20, 3]; the PID strategy is particularly effective in process control, where a combination of benign process dynamics and modest performance requirements finds its place. The ideal PID law is based on the present (P), past (I) and estimated future (D) error

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information. In accordance with this original conception, there are only three tuning parameters. Even for such a simple strategy, it is not easy to find good settings without a systematic procedure [17, 21, 15].

During the last twenty years, there has been a revived interest in PID control, motivated by the advent of model predictive control, which requires well-tuned PID compensators at the bottom level, and the emergence of auto-tuning tools [2]. As a result, numerical (optimization-based) techniques have been suggested in the literature [29, 26, 3, 24]. In the same vein, analytically-derived tuning rules have appeared [9, 13, 19, 25]. Another reason for the PID revival has been the lack of results regarding stabilization of delayed systems [20, 10, 23, 16]. These research efforts, specially the trend for analytical design, has incorporated into the PID arena the control theory mainstream developments, leaving aside more specific techniques.

Among the analytical methods, IMC [14] has gained remarkable industrial acceptance due to its simple yet effective procedure [21, 6]. Internal Model Control theory was first applied to PID control of stable plants in [18], solving the robustness problems associated with some early tunings like [30]. Although the IMC-PID settings [18] are robust and yield good set-point responses, they result in poor load disturbance rejection for integrating/lag-dominant plants [5, 11]. Alternative PID tuning rules aimed at good regulatory performance can be consulted in [11, 19]. In [21], remarkably simple tuning rules which provide balanced servo/regulator performance are proposed based on a modification of the settings in [18]. It is important to realize that the problems with the original IMC-based tunings come indeed from inherent shortcomings of the IMC procedure, thoroughly revised in [6].

The purpose of this article is to present an $\mathcal{H}_\infty$ design which avoids some of the limitations of the IMC method, while retaining its simplicity as much as possible. In particular, the method is devised to work well for plants of modest complexity, for which analytical PID tuning is plausible.

Roughly speaking, the design procedure associated with modern $\mathcal{H}_\infty$ control theory involves the selection of frequency weights which are used to shape prescribed closed-loop transfer functions. Many practitioners are reluctant to use this methodology because it is generally difficult to design the frequency weights properly. At the end of the day, it is quite typical to obtain high-order controllers, which may require the use of model order reduction techniques. Apart from the cumbersome design procedure, control engineers usually find the general theory difficult to master as well. To alleviate the above difficulties, we rely here on the plain $\mathcal{H}_\infty$ weighted sensitivity problem. By investigating its analytical solution, the involved frequency weight is chosen systematically in such a way that a good design in terms of basic conflicting trade-offs can be attained. The main

contributions of the proposed procedure are:

(a) The selection of the weight is *systematic* (this is not common in $\mathcal{H}_\infty$ control) and *simple*, only depending on two types of parameters:

- One adjusts the robustness/performance trade-off in the line of the IMC approach.
- The other one allows to balance the performance between the servo and regulator modes. As it will be explained, this can be interpreted in terms of a mixed S/SP sensitivity design.

(b) The method is *general*: both stable and unstable plants are dealt with in the same way. This differs from other analytical $\mathcal{H}_\infty$ procedures.

(c) The controller is derived *analytically*. For simple models, this leads to well-motivated PID tuning rules which consider the stable/unstable plant cases simultaneously.

The rest of the article is organized as follows: Section 2 revisits IMC and $\mathcal{H}_\infty$ control. Section 3 presents the proposed design method, based on the $\mathcal{H}_\infty$ weighted sensitivity problem, while Section 4 deals with its application to analytical tuning of PI controllers. Simulation examples are given in Section 5 to emphasize the new features of the proposed approach. Finally, Section 6 contains the conclusions of this work.

2. Background: an overview of IMC and $\mathcal{H}_\infty$ paradigms

This section outlines the basic principles of IMC [14] and the $\mathcal{H}_\infty$ control problem [22]. The pros and cons of each method are stressed so as to motivate the proposed design of Section 3. We base our discussion on the unity feedback, LTI and SISO system in Figure 1.

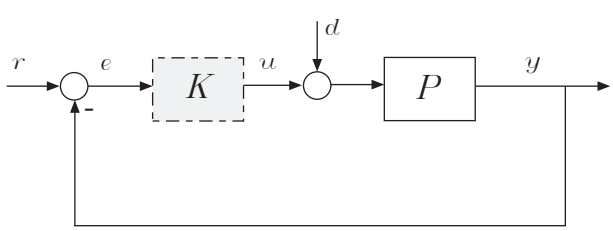


Figure 1: Conventional feedback configuration.

Two exogenous inputs to the system are considered: d and r. Here, d represents a disturbance affecting the plant input, and the term *regulator mode* refers to the case when this is the main

exogenous input. The term *servo mode* refers to the case when the set-point change r is the main concern. As mentioned, an important contribution in this paper is the possibility of making a trade-off between the regulator and servo modes. Although the reference tracking can be improved by using a two-degree-of-freedom controller, there will always be some unmeasured disturbance directly affecting the plant output, which may be represented as an unmeasured signal r (in this case, $-e$ will represent the plant output). In summary, there is a fundamental trade-off between the regulator (input disturbance) and servo (output disturbance) modes. The closed-loop mapping for the system in Figure 1 is given by

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} T & SP \\ KS & S \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix} \doteq H(P,K) \begin{bmatrix} r \\ d \end{bmatrix} \tag{2.1}$$

where $S \doteq \frac{1}{1+PK}$ and $T \doteq \frac{PK}{1+PK}$ denote the sensitivity and complementary sensitivity functions [22], respectively. In terms of the performance for the regulator and servo modes, note that the closed-loop effect of disturbance and set-point changes on the output error is given by

$$y - r = -e = Sr + SPd \tag{2.2}$$

The most basic requirement for the controller K is *internal stability*, which means that all the relations in $H(P,K)$ are stable. The set of all internally stabilizing feedback controllers will be hereafter denoted by $\mathcal{C}$. At this point, it is also convenient to introduce a special notation for the set of stable transfer functions, or $\mathcal{RH}_\infty$ for short.

2.1. Internal Model Control

Let us start factoring the plant as $P = P_a P_m$, where $P_a \in \mathcal{RH}_\infty$ is all-pass and P_m is minimum-phase. As reported in [22, 6], the broad objective of the IMC procedure is to specify the closed-loop relation $T_{yr} = T = P_a f$, where f is the so-called IMC filter. Assuming that P has k unstable poles, the filter is chosen as follows [14]:

$$f(s) = \frac{\sum_{i=1}^{k} a_i s^i + 1}{(\lambda s + 1)^{n+k}} \tag{2.3}$$

The purpose of f is twofold: first, to ensure the properness of the controller and the internal stability requirement (to this double aim, n must be equal or greater than the relative degree of P, whereas the $a_1 \cdots a_k$ coefficients impose $S = 0$ at the k unstable poles of P). Second, the λ parameter is used to find a compromise between robustness and performance. The main drawbacks of the IMC design are:

- For stable plants ($k = 0$), the poles of P are cancelled by the zeros of the controller K. This allows to place the closed-loop poles at $s = -1/\lambda$ but results into sluggish disturbance attenuation when P has slow/integrating poles [5, 11, 21, 19].

- For unstable plants, the pole-zero pattern of (2.3) can lead to large peaks on the sensitivity functions, which in turn means poor robustness and large overshoots in the transient response [4].

- In general, poor servo/regulator performance compromise is obtained [21].

2.2. $\mathcal{H}_\infty$ Control

Modern $\mathcal{H}_\infty$ control theory is based on the general feedback setup depicted in Figure 2, composed of the generalized plant G and the feedback controller K. Once the problem has been posed in this form, the optimization process aims at finding a controller K which makes the feedback system in Figure 2 stable, and minimizes the $\mathcal{H}_\infty$-norm of the closed-loop relation from w to z.

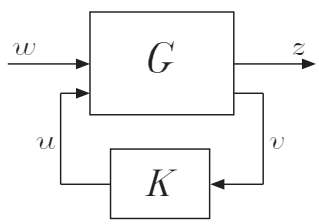


Figure 2: Generalized control setup.

Mathematically, the synthesis problem can be expressed as

$$\min_{K \in \mathcal{C}} \|\mathcal{N}\|_\infty = \min_{K \in \mathcal{C}} \|\mathcal{F}_l(G, K)\|_\infty \tag{2.4}$$

where

$$\mathcal{N} = \mathcal{F}_l(G, K) \doteq G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} = T_{zw} \tag{2.5}$$

An important feature of the $\mathcal{H}_\infty$-norm is that allows to consider both performance and robustness specifications simultaneously by means of mixed sensitivity problems [22]. As these problems consider several closed-loop transfer functions (and not only T as IMC does), a more sensible design can be obtained. The main difficulty with the $\mathcal{H}_\infty$ methodology is that the designer has to select suitable frequency weights (included in G), which may require considerable trial and error. In [6], a systematic $\mathcal{H}_\infty$ procedure to generalize IMC is presented. Due to its relevance in the

present work, this proposal is briefly sketched here. Consider the following problem [6]:

$$\begin{aligned}
\rho &= \min_{K\in\mathcal{C}} \|\mathcal{N}\|_\infty \\
&= \min_{K\in\mathcal{C}} \left\| \mathcal{F}_l\left(\left[\begin{array}{cc:c} -P_a f & \epsilon_2 P & P \\ 0 & \epsilon_1\epsilon_2 & \epsilon_1 \\ \hdashline 1 & -\epsilon_2 P & -P \end{array}\right], K \right) \right\|_\infty \\
&= \min_{K\in\mathcal{C}} \left\| \begin{array}{cc} T - P_a f & \epsilon_2 PS \\ \epsilon_1 KS & \epsilon_1\epsilon_2 S \end{array} \right\|_\infty
\end{aligned} \tag{2.6}$$

where ϵ_1 and ϵ_2 are stable, minimum-phase and proper weighting functions. The basic philosophy is to minimize the closeness between the input-to-output relation and a specified reference model, which is set as $P_a f$ in the line (but with more flexibility) of the standard IMC. At the same time, the (1,2) term of (2.6) limits the size of $PS = T_{yd}$, whereas the (2,1) term limits the size of $KS = T_{ur}$. The index in (2.6) automatically guarantees that

$$|T(j\omega) - P_a f(j\omega)| \leqslant \rho \ \forall\omega, \tag{2.7}$$

$$|PS(j\omega)| \leqslant \rho/|\epsilon_2(j\omega)| \ \forall\omega, \tag{2.8}$$

$$|KS(j\omega)| \leqslant \rho/|\epsilon_1(j\omega)| \ \forall\omega \tag{2.9}$$

Now, if the design specifications are written as $\|T - P_a f\|_\infty \leqslant \alpha$, $|PS(j\omega)| \leqslant \beta_p^i \ \forall\omega \in [w_1^i, w_2^i]$, and $|KS(j\omega)| \leqslant \beta_k^i \ \forall\omega \in [\omega_3^i, \omega_4^i]$ where $\alpha, \beta_p^i, \beta_k^i, w_1^i, w_2^i, w_3^i, w_4^i$ are positive real numbers representing the closed-loop objectives, ϵ_1 and ϵ_2 can be chosen as

$$|\epsilon_1(j\omega)| \geqslant \alpha/\beta_k^i \ \forall\omega \in [\omega_3^i, \omega_4^i] \quad \text{and} \quad |\epsilon_2(j\omega)| \geqslant \alpha/\beta_p^i \ \forall\omega \in [\omega_1^i, \omega_2^i] \tag{2.10}$$

Then, if $\rho \leqslant \alpha$, the design specifications are certainly met. Although the revised design method has a great versatility, blending $\mathcal{H}_\infty$ and IMC ideas elegantly, the resulting procedure is considerably more involved than IMC, even if $f, \epsilon_1, \epsilon_2$ can be chosen in a systematic way.

Generally speaking, we summarize here the most common disadvantages of $\mathcal{H}_\infty$ design methods [22]:

- The controller is found numerically (in contrast with the analytical perspective of IMC). Moreover, the inclusion of weights increments the complexity of the controller.
- For stacked problems involving three or more closed-loop transfer functions, the shaping becomes considerably difficult for the designer.

3. Proposed design procedure

The proposed approach stems from considering the Weighted Sensitivity Problem [28, 22]:

$$\begin{aligned} |\rho| &= \min_{K\in\mathcal{C}} \|\mathcal{N}\|_\infty \\ &= \min_{K\in\mathcal{C}} \left\| \mathcal{F}_l\left(\left[\begin{array}{c:c} W & -WP \\ \hdashline 1 & -P \end{array}\right], K \right) \right\|_\infty \\ &= \min_{K\in\mathcal{C}} \|WS\|_\infty \end{aligned} \tag{3.1}$$

3.1. Analytical solution

Before selecting W to shape S, we will look for an analytical solution of (3.1). The classical design found in [8, 7] consists of transforming (3.1) into a Model Matching Problem[1] using the Youla-Kucera parameterization [27]. From an analytical point of view, the problem with this parameterization is the need of computing a coprime factorization when P is unstable. In order to deal with stable and unstable plants in a unified way, it would be desirable to avoid any notion of coprime factorization. Towards this objective, the key point is to use a possibly unstable weight:

Theorem 3.1. *Assume that P is purely rational (i.e., there is no time delay in P) and has at least one Right Half-Plane (RHP) zero. Take W as a MP weight including the unstable poles of P. Then, the optimal weighted sensitivity in problem (3.1) is given by*

$$\mathcal{N}^o = \rho \frac{q(-s)}{q(s)} \tag{3.2}$$

where ρ and $q = 1 + q_1 s + \cdots + q_{\nu-1}s^{\nu-1}$ (Hurwitz) are uniquely determined by the interpolation constraints:

$$W(z_i) = \mathcal{N}^o(z_i) \qquad i = 1 \ldots \nu, \tag{3.3}$$

being $z_1 \ldots z_\nu$ ($\nu \geq 1$) the RHP zeros of P.

Proof. Consult the Appendix. □

Once the optimal weighted sensitivity has been determined, the following corollary of Theorem 3.1 gives the corresponding (complementary) sensitivity function and feedback controller:

[1] A detailed statement of the Model Matching Problem can be consulted in the Appendix and the references therein.

Corollary 3.1. *Consider the following factorizations:*

$$P = \frac{n_p}{d_p} = \frac{n_p^+ n_p^-}{d_p^+ d_p^-} \qquad W = \frac{n_w}{d_w} = \frac{n_w}{d_w' d_p^+} \tag{3.4}$$

where n_p^+, d_p^+ contain the unstable zeros of n_p, d_p, respectively. Similarly, n_p^-, d_p^- contain the stable zeros of n_p, d_p. Then,

$$S = \mathcal{N}^o W^{-1} = \rho \frac{q(-s) d_w}{q(s) n_w} \tag{3.5}$$

$$T = 1 - \mathcal{N}^o W^{-1} = \frac{n_p^+ \chi}{q(s) n_w} \tag{3.6}$$

$$K = \left(\frac{1 - \mathcal{N}^o W^{-1}}{\mathcal{N}^o W^{-1}} \right) P^{-1} = \frac{d_p^- \chi}{n_p^- q(-s) d_w'} \tag{3.7}$$

where χ is a polynomial such that

$$q(s) n_w - \rho q(-s) d_w = n_p^+ \chi \tag{3.8}$$

Proof. Consult the Appendix. □

Remark 3.1. *It is noteworthy that the feedback controller (3.7) is realizable only if P is biproper. Hence, in practice, it may be necessary to add fictitious high-frequency zeros to the initial model to meet this requirement.*

3.2. Selection of W

Let us denote by $\tau_1, \ldots, \tau_k$ the time constants of the unstable or slow poles of P. Equation (3.5) reveals that, except by the factor ρ, $|S|$ is determined by $|W^{-1}|$ ($\mathcal{N}^o$ is allpass). Based on (3.5) and (3.6), the following structure for the weight is proposed

$$W(s) = \frac{(\lambda s + 1)(\gamma_1 s + 1) \cdots (\gamma_k s + 1)}{s(\tau_1 s + 1) \cdots (\tau_k s + 1)} \tag{3.9}$$

where $\lambda > 0$, and

$$\gamma_i \in [\lambda, |\tau_i|] \tag{3.10}$$

The rationale behind the choice of W in (3.9) is further explained below:

- Let us start assuming that $k = 0$ (i.e., $W = \frac{\lambda s + 1}{s}$). The integrator in W forces $S(0) = 0$ for integral action. From (3.6), the term $(\lambda s + 1)$ in the numerator of W appears in the denominator of the input-to-output transfer function. Consequently, the closed-loop will have a pole $s = -1/\lambda$. The idea is to use λ to determine the speed of response, as in standard IMC.

- If P has slow stable poles, it is necessary that S cancels them if disturbance rejection is the main concern. Otherwise, they will appear in the transfer function $T_{yd} = PS$, making the response sluggish. This is why W also contains these poles. As a result, slow (stable) and unstable poles are treated basically in the same way. This unified treatment ensures internal stability in terms of the generalized $\mathcal{D}$-stability region of Figure 3.

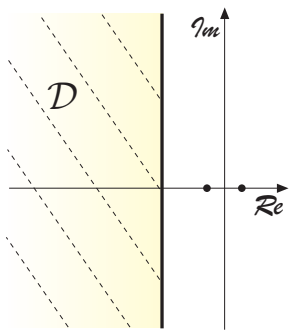


Figure 3: General stability region: slow and unstable poles are $\mathcal{D}$-unstable.

- As it has been said, producing $S(-1/\tau_i) = 0,\ i = 1, \ldots, k$ is necessary for internal stability and disturbance rejection. Notice, however, that these constraints mean decreasing $|S|$ at low frequencies. By a waterbed effect argument [22], recall the Bode's Sensitivity Integral:

$$\int_0^\infty |S(j\omega)| d\omega = \pi \sum_{i,\tau_i<0}^{k} |\tau_i|^{-1}, \tag{3.11}$$

  this will augment $|S|$ at high frequencies, maybe yielding an undesirable peak (M_S) on it. This, in turn, will probably augment the peak of $|T|$ (M_T) and the overshoot in the set-point response. In order to alleviate these negative effects, for each slow/unstable pole of P, we introduce a factor $(\gamma_i s + 1)$ in the numerator of W: as $\gamma_i \to |\tau_i|$, $\left|\frac{\tau_i j\omega+1}{\gamma_i j\omega+1}\right| \to 1^+$; the resulting flatter frequency response will reduce the overshoot (improving the robustness properties, see Section 3.3) at the expense of settling time.

- We have supposed that $\lambda < |\tau_i|\ \forall k = 1 \ldots k$. In other words, we are considering relatively slow plants: for stable plants without slow poles, the standard IMC procedure will provide good results in terms of tracking and disturbance rejection; there is no conflict between T_{yr} and T_{yd}. Note, in addition, that forcing $S = 0$ ($T = 1$) at high frequency is undesirable from a robustness point of view. This is why we discard rapid stable poles from the denominator

of W. If the plant is unstable, there is no option and one has to force $S = 0$ ($T = 1$) at the rapid unstable poles, which imposes a minimum closed-loop bandwidth.

Essentially, there are two tuning parameters in W: λ is intended to tune the robustness/performance compromise. The set of numbers γ_i allow us to balance the performance between the servo and regulator modes. The latter point can be interpreted in terms of a mixed S/SP sensitivity design: let us assume that $\lambda \approx 0$. Then, when $\gamma_i = |\tau_i|$ (servo tuning), we have that $|WS| \approx |S/s|$ and we are minimizing the peak of $|S|$ ($= |T_{er}|$) subject to integral action. In the other extreme, if $\gamma_i = \lambda$ (regulator tuning), the poles of P appear in W. If the zeros of P are sufficiently far from the origin, we have that $|WS| \approx |SP/s|$ in the low-middle frequencies. Heuristically, we are minimizing the peak of $|SP|$ ($= |T_{yd}|$) subject to integral action.

Remark 3.2. *Let us consider that P has a RHP pole at $s = -1/\tau_i$ ($\tau_i < 0$) and a RHP zero at $s = z_i$. Then, from (3.3) and (3.9), it follows that*

$$\left|\frac{1}{\tau_i z_i + 1}\right| \left|\frac{(\lambda z_j + 1)\prod_{j=1}^{k}(\gamma_j z_j + 1)}{z_j \prod_{j=1, j\neq i}^{k}(\tau_j z_j + 1)}\right| = |\rho| \left|\frac{q(-z_i)}{q(z_i)}\right| \tag{3.12}$$

As the RHP pole $-1/\tau_i$ and the RHP zero z_i get closer to each other, $\tau_i z_i \to -1$, which makes the left hand side grow unbounded. Since $\left|\frac{q(-z_i)}{q(z_i)}\right| \leqslant 1$, $|\rho| \to \infty$. Note that this happens regardless the values of λ and the γ_j's, and obeys the fact that plants with unstable poles and zeros close to each other are intrinsically difficult to control [14].

3.3. Stability and Robustness

Because of the assumptions in Theorem 3.1, the possible delay of the plant must be approximated by a non-minimum phase rational term. This approximation creates a mismatch between P (the purely rational model used for design) and the nominal model containing the time delay, let us call it P_o. The following sufficient condition for Nominal Stability can be derived from the conventional Nyquist stability criterion [22]:

Proposition 3.1. *Assume that P is internally stabilized by K, and that P and P_o have the same RHP poles. Then, K internally stabilizes P_o if*

$$\left|\frac{L_o - L}{1 + L}\right| < 1 \ \forall \omega \in \Omega_{pc} \tag{3.13}$$

where $L = PK, L_o = P_o K$, and $\Omega_{pc} = \left\{\omega : \angle\left(\frac{L_o - L}{1+L}\right) = -\pi + 2\pi n,\ n \in \mathbb{Z}\right\}$ is the set of phase crossover frequencies of $\frac{L_o - L}{1+L}$.

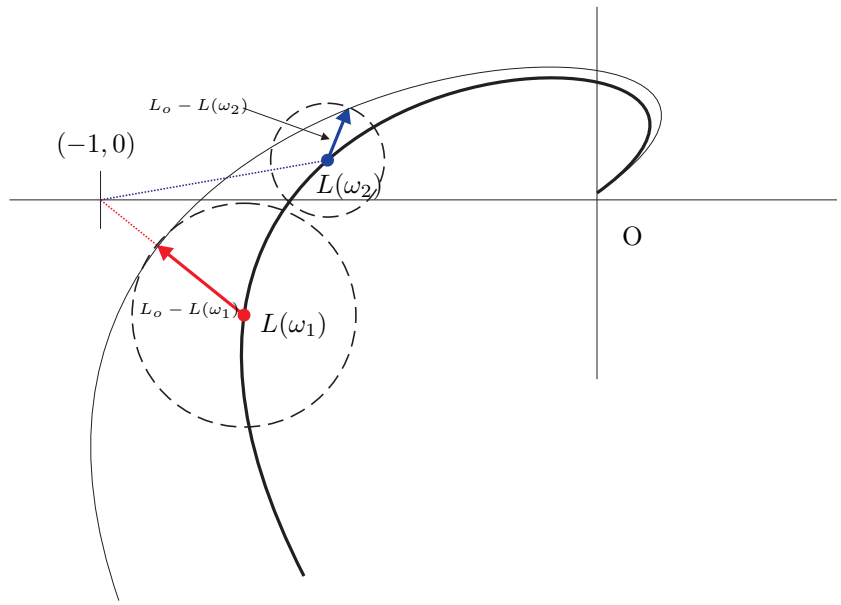


Figure 4: Stability condition for P_o in terms of P. The magnitude condition $|1 + L| > |L_o - L|$ must be true for ω_1 (which is a phase crossover frequency, i.e., $\omega_1 \in \Omega_{pc}$), but not for ω_2.

Figure 4 illustrates the situation graphically for a stable plant: the distance from L to the point $(-1, 0)$ must exceed $|L_o - L|$ when the *vectors* $L_o - L$ and $-1 - PK$ are aligned. Rather than using Proposition 3.1, a more practical approach is to check Robust Stability with respect to P_o [22, 14], including P in the uncertain set under consideration [25]. Generally, the way in which λ and γ_i influence robustness is:

- Augmenting λ decreases the closed-loop bandwidth, making the system more robust and less sensitive to noise.
- Decreasing γ_i improves the disturbance rejection, but increases the overshoot in the set-point response to the detriment of robustness.

These robustness implications can be understood in terms of the Robust Stability condition $\|\Delta T\|_\infty < 1$ (equivalently $|T| < 1/|\Delta|\ \forall\omega$), where Δ models the multiplicative plant uncertainty [22]. Augmenting λ makes the system slower, which favours Robust Stability. On the other hand, decreasing γ_i increments the peak of $|T|$ (responsible for the overshoot increment), which limits the amount of multiplicative uncertainty.

4. Application to PI tuning

This section deals with the application of the presented design method to the tuning of PI compensators.

4.1. Stable/unstable plants

Let us consider the First Order Plus Time Delay (FOPTD) model given by $P_o = K_g \frac{e^{-sh}}{\tau s+1}$, where K_g, h, τ are, respectively, the gain, the (apparent) delay, and the time constant — negative in the unstable case — of the process. For design purposes, we take

$$P = K_g \frac{-sh+1}{\tau s+1} \tag{4.1}$$

where a first order Taylor expansion has been used to approximate the time delay. From (3.9) and (4.1), with $k = 1$, the following weight results

$$W = \frac{(\lambda s+1)(\gamma s+1)}{s(\tau s+1)} \tag{4.2}$$

where $\lambda > 0, \gamma \in [\lambda, |\tau|]$. The optimal weighted sensitivity is determined from (3.3). In this case, P has a single RHP zero ($\nu = 1$), and $\mathcal{N}^o$ becomes

$$\mathcal{N}^o = \rho = \frac{(\lambda+h)(\gamma+h)}{\tau+h} \tag{4.3}$$

From (3.7), the controller is finally given by

$$K = \frac{1}{K_g} \frac{(\tau s+1)(\zeta s+1)}{(\lambda\gamma + h\zeta)s^2 + (\lambda+\gamma+h-\zeta)s} \tag{4.4}$$

where

$$\zeta = \frac{\tau(h+\lambda+\gamma) - \lambda\gamma}{\tau+h} \tag{4.5}$$

The feedback controller (4.4) can be cast into the PI structure:

$$K = K_c \left(1 + \frac{1}{T_i s}\right) \tag{4.6}$$

according to the tuning rule in the first row of Table 1.

Table 1: Proposed PI tuning rules.

Model	K_c	T_i	
$K_g \frac{e^{-sh}}{\tau s+1}$	$\frac{1}{K_g} \frac{T_i}{\lambda+\gamma+h-T_i}$	$\frac{\tau(h+\lambda+\gamma)-\lambda\gamma}{\tau+h}$	$\lambda > 0, \gamma \in [\lambda, \lvert\tau\rvert]$
$K_g \frac{e^{-sh}}{s}$	$\frac{1}{K_g} \frac{T_i}{\lambda\gamma+hT_i}$	$h+\lambda+\gamma$	$\lambda > 0, \gamma \in [\lambda, \infty)$

Table 2: PI tuning rules for the extreme values of γ.

$\gamma = \lambda$		$\gamma = \tau$	
K_c	T_i	K_c	T_i
$\frac{1}{K_g}\frac{\tau}{\lambda+h}\left(\frac{h+2\lambda-\lambda^2/\tau}{h+\lambda}\right)$	$\frac{\tau(h+2\lambda)-\lambda^2}{\tau+h}$	$\frac{1}{K_g}\frac{\tau}{\lambda+h}$	τ

Essentially, the trade-off between disturbance rejection and set-point tracking is controlled by T_i. This can be verified by considering the proposed PI settings for the extreme values of γ. This has been done in Table 2 for the stable plant case ($\tau > 0$). Certainly, T_i is the parameter which varies more with γ: K_c varies from $\frac{1}{K_g}\frac{\tau}{\lambda+h}$ to $\frac{1}{K_g}\frac{\tau}{\lambda+h}\left(\frac{h+2\lambda-\lambda^2/\tau}{h+\lambda}\right)$ as γ is decreased from τ to λ. This way, as we improve disturbance rejection, the controller gain increases. The multiplicative factor $\frac{h+2\lambda-\lambda^2/\tau}{h+\lambda}$ equals one when $\lambda = \tau$. If $\tau \gg h, \lambda$, then $\frac{h+2\lambda-\lambda^2/\tau}{h+\lambda} \approx \frac{h+2\lambda}{h+\lambda} < 2$, which shows that K_c augments moderately in the transition to the regulator mode. Based on these facts, it is reasonable to select $K_c = \frac{1}{K_g}\frac{\tau}{\lambda+h}$, and fix T_i for good servo/regulation trade-off. This strategy is the essence of the SIMC tuning rule for stable plants [21].

Next, we will compare the input-to-output transfer functions achieved for the extreme values of γ. In magnitude, it can be seen that

$$|T(j\omega)| \approx \left|\frac{1}{\lambda j\omega + 1}\right| \left|\frac{\zeta j\omega + 1}{\gamma j\omega + 1}\right| \tag{4.7}$$

For a lag-dominant plant, the following approximations are valid:

- When $\gamma = \lambda$, the closed-loop magnitude is

$$|T(j\omega)| \approx \left|\frac{1}{\lambda j\omega + 1}\right| \left|\frac{\left(\frac{\tau(h+2\lambda)-\lambda^2}{\tau+h}\right) j\omega + 1}{\lambda j\omega + 1}\right| \approx \left|\frac{1}{\lambda j\omega + 1}\right| \left|\frac{(h+2\lambda) j\omega + 1}{\lambda j\omega + 1}\right| \tag{4.8}$$

- When $\gamma = |\tau|$, we have that

$$|T(j\omega)| \approx \left|\frac{1}{\lambda j\omega + 1}\right| \tag{4.9}$$

  for the stable plant case ($\tau > 0$). If P is unstable ($\tau < 0$), T is such that

$$|T(j\omega)| \approx \left|\frac{1}{\lambda j\omega + 1}\right| \left|\frac{\left(\frac{\tau(h+\lambda+|\tau|)-\lambda|\tau|}{\tau+h}\right) j\omega + 1}{|\tau| j\omega + 1}\right| \approx \left|\frac{1}{\lambda j\omega + 1}\right| \left|\frac{(h+2\lambda+|\tau|) j\omega + 1}{|\tau| j\omega + 1}\right| \tag{4.10}$$

Therefore, as the value of γ is increased, the pole and the zero of $\frac{\zeta s+1}{\gamma s+1}$ in (4.7) get closer to each other, reducing the overshoot and providing flatter frequency response.

4.2. Integrating plant case ($\tau \to \infty$)

If the plant under control is integrating, it can be modelled by an Integrator Plus Time Delay (IPTD) model: $P_o = \frac{K_g e^{-sh}}{s}$. For this case, we take

$$P = K_g \frac{-sh+1}{s} \tag{4.11}$$

The corresponding weight is chosen as

$$W = \frac{(\lambda s+1)(\gamma s+1)}{s^2} \tag{4.12}$$

where $\lambda > 0, \gamma \in [\lambda, \infty)$. The optimal weighted sensitivity becomes

$$\mathcal{N}^o = \rho = (\lambda + h)(\gamma + h) \tag{4.13}$$

From (3.7),

$$K = \frac{1}{K_g} \frac{\zeta' s + 1}{(\lambda\gamma + h\zeta')s} \tag{4.14}$$

where

$$\zeta' = h + \lambda + \gamma \tag{4.15}$$

The associated PI tuning rule can be consulted in the second row of Table 1. Alternatively, the tuning rules for the IPTD model could have been derived by taking the limit $\tau \to \infty$ in the FOPTD settings, considering the approximation $K_g \frac{e^{-sh}}{\tau s+1} = \frac{K_g}{\tau} \frac{e^{-sh}}{s+1/\tau} \approx \frac{K_g}{\tau} \frac{e^{-sh}}{s}$.

5. Simulation Examples

This section evaluates the tuning rules given in Table 1 through four simulation examples. Examples 1–3 emphasize that the design presented in Section 3 generalizes standard IMC. The purpose of the fourth example is to illustrate that, for simple plants and modest specifications, the presented design overcomes basic limitations of IMC, thus not being advisable to embark on more complex strategies. A summary of the controller settings for Examples 1–4 can be consulted in Table 3.

5.1. Example 1

The IMC-based PI tuning rule for stable FOPTD processes is given by [14]:

$$K_c = \frac{1}{K_g} \frac{\tau}{\lambda + h} \qquad T_i = \tau \tag{5.1}$$

In this example, the following concrete process $\frac{e^{-0.073s}}{1.073s+1}$ is considered. Regarding the λ parameter, two different values are chosen in order to achieve *smooth* ($\lambda = 0.10731$) and *tight* ($\lambda = 0.05402$)

control [1], resulting into: $K_c^{sm} = 5.88, T_i^{sm} = 1.073$, and $K_c^{ti} = 8.38, T_i^{ti} = 1.073$. In the smooth control case, $M_S = 1.38$, whereas in the tight control case, $M_S = 1.71$. The associated disturbance responses are shown in Figure 5. As it can be seen, it is possible to reduce the magnitude of

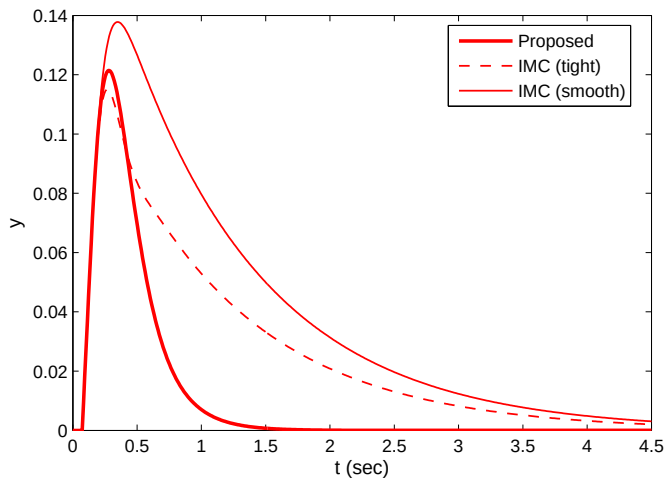


Figure 5: Load disturbance response for Example 1.

the disturbance rejection response by decreasing λ. However, the conventional IMC-based tuning continues to exhibit poor disturbance attenuation even for the tight case. To the detriment of robustness, decreasing further the value of λ would improve the regulatory performance a little, but the response would continue to be sluggish. Accordingly, it is not possible to get both good regulatory performance and good robustness for the process under examination.

In the design of Section 3, setting $\gamma = \lambda$ produces an improvement of the regulation performance. Consequently, the problem reduces now to finding a value for λ providing the prescribed robustness level. This is achieved for $\lambda = 0.1752$, which yields $M_S = 1.6551$. The corresponding time response is depicted in Figure 5.

It should be noted that the poor disturbance attenuation obtained through conventional IMC can be remedied in several (more *ad hoc*) ways. For example, approximating the process at hand by an integrating one [5]. Then, conventional IMC design gives satisfactory disturbance rejection. A limitation of this approach is that it does not consider the servo/regulator trade-off. Other IMC-based approaches for improved regulatory performance can be found in [11, 19]. However, even for the simple FOPTD model, these approaches require a more complicated control structure (PID or PID plus filter). Overall, the presented tuning rules are simpler and more instructive.

Table 3: Tuning of λ and γ, and the corresponding PI settings, for Examples 1–4.

Example	Plant model	λ	γ	K_c	T_i	Design type
1	$\frac{e^{-0.073s}}{1.073s+1}$	0.1752	0.1752	6.8765	0.3696	Regulator
2	$\frac{e^{-0.073s}}{1.073s+1}$	0.146	1.073	4.8995	1.0730	Servo (=IMC)
		0.146	0.4	5.8481	0.5286	Servo/Regulator
		0.146	0.146	7.7215	0.3231	Regulator
3	$\frac{e^{-s}}{-20s+1}$	2	2	-11.56	5.4737	Regulator (≈IMC)
		0.9	9	-11.9	11.9	Servo/Regulator
4	$\frac{-1}{-s+1}\left(\approx \frac{-e^{-0.01s}}{-s+1}\right)$	0.1	0.1	18.2	0.22	Regulator (≈IMC)
		0.1	1	10.9	1.22	Servo
		0.1	14	10.0642	15.667	Servo ($K \approx 10$)

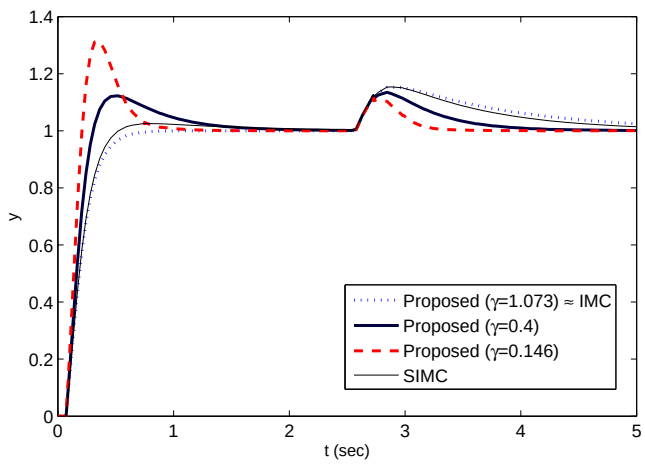


Figure 6: Tracking and disturbance responses for Example 2.

5.2. Example 2

Generally speaking, the γ parameter allows to balance the performance between set-point tracking and disturbance rejection. To clarify this, we will continue Example 1, selecting $\lambda = 2h = 0.146$ and considering three different values for γ. The first value is $\gamma = \tau = 1.073$ (servo tuning). The resulting design is identical to the conventional IMC one. The second value is $\gamma = \lambda = 0.146$ (regulator tuning). Finally, we set $\gamma = 0.4$ for balanced servo/regulator performance. Figure 6 shows the three time responses. We have also included the SIMC tuning rule [21]:

$$K_c = \frac{1}{K_g}\frac{\tau}{\lambda + h} \qquad T_i = \min\{\tau, 4(\lambda + h)\}, \tag{5.2}$$

which was presented as a modification of the original settings (5.1) for good servo/regulator performance. Note, however, that in the *edge case* $\tau \approx 4(\lambda + h)$, there is no difference between (5.2)

and (5.1). This is the situation in this example: $\tau = 1.073$ is close to $4(\lambda + h) = 0.876$. Looking at Figure 6, it is confirmed that the SIMC tuning gives approximately the same responses as conventional IMC. Lacking a rigorous analysis (this is not the intention here), the proposed PI tuning rule with $\gamma = 0.4$ seems to offer a better overall compromise. Finally, it is remarkable that, whereas the SIMC rule was derived only considering stable plants, the proposed tuning rule unifies the stable/unstable cases.

5.3. Example 3

As it has been revised in Section 2, for unstable plants, the IMC filter may cause large overshoot and poor robustness due to the large peak in the filter frequency response [4, 6]. The search of new filters to alleviate these shortcomings has resulted in more complicated (and application-specific) procedures [4]. In this example we deal with an unstable plant, analyzing how the proposed method, albeit simple, can mitigate these negative effects. Let us consider the unstable process $\frac{e^{-s}}{-20s+1}$. Following the discussion of Section 2.1, the IMC controller is such that $T = e^{-s}f$, where $f = \frac{a_1 s+1}{(\lambda s+1)^2}$ and $a_1 = 20\left(e^{1/20}(\lambda/20+1)^2 - 1\right)$. Suppose that $\lambda = 2$ produces the desired closed-loop bandwidth, then $a_1 = 5.4408$. The feedback controller is $K = (-20s+1)\frac{f}{1-e^{-s}f}$, which is not purely rational. Approximating $e^{-s} \approx -sh + 1$, we finally obtain

$$K_{imc} = \frac{-11.53s^2 - 1.542s + 0.1059}{s^2 - 0.04669s} \tag{5.3}$$

As for the proposed method, we start considering the initial tuning $\lambda = 2, \gamma = \lambda$. Figure 7 (Nominal Case) shows that this design is almost identical to the IMC one. Both K_{imc} and the proposed PI provide excellent disturbance rejection. However, it could be desirable to reduce the overshoot in the set-point response or improve the robustness properties. Within the IMC procedure, the only way to it is to roll-off the controller (increasing λ), making the system slower. Contrary to this, if we take $\lambda = 0.9, \gamma = 9 \in [0.9, 20] = [\lambda, |\tau|]$, it can be seen from Figure 7 (Nominal Case) that it is possible to reduce the overshoot (at the expense of disturbance attenuation and settling time) without slowing down the system. Figure 8 depicts the frequency response of $|S|$ and $|T|$. Recalling Section 3, the reduction of M_S and M_T confers more robustness and smoother control, as confirmed in Figure 7 (Uncertain Case), where the real plant delay is assumed to be $h = 1.6$ instead of one. Certainly, the new settings provide the best responses in both set-point tracking and disturbance attenuation.

5.4. Example 4

Finally, we revisit the design method in [6], briefly summarized in Section 2.2. This $\mathcal{H}_\infty$ procedure was devised to generalize IMC: in particular, for unstable plants, it allows to use a

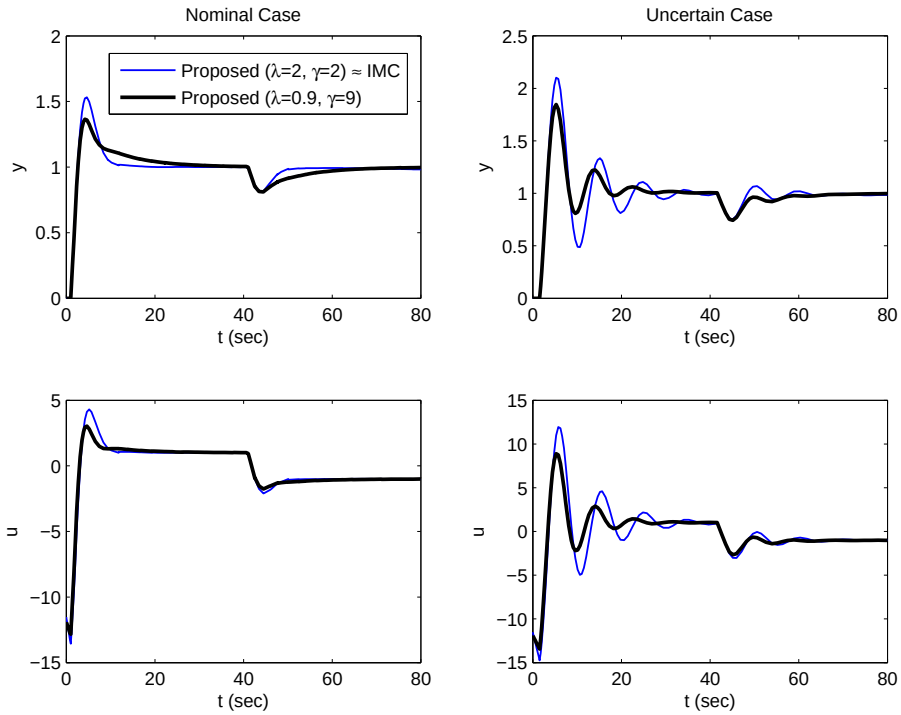


Figure 7: Tracking and disturbance responses for Example 3.

different filter from that in (2.3), hence proving more flexible. The following design example, taken from [6], makes it clear: given the unstable plant $\frac{-1}{-s+1}$ ($P_a = 1, P_m = \frac{-1}{-s+1}$), the controller is designed in order to achieve a closed-loop response similar to $\frac{1}{0.1s+1}$, that corresponds to $f = \frac{1}{0.1s+1}$ in problem (2.6). This specification is coherent, in the sense that the desired closed-loop bandwidth is considerably beyond the unstable pole frequency [6]. Note that $P_a f|_{s=1,0} \approx 1$, taking into account internal stability constraints and zero steady state error (unity low frequency gain). The desired closeness between T and $P_a f = \frac{1}{0.1s+1}$ is specified by the inequality $\|T - P_a f\|_\infty < \alpha$, with $\alpha = 0.1$. In addition, it is assumed that the actuators can pump up a maximum gain of 10 ($\beta_c = 10$). According to Section 2.2, the frequency cost ϵ_1 is chosen to gradually reach the maximum gain $\alpha/10$ as the plant model loses its bandwidth to the controller. Finally, $\epsilon_2 = 0$. Solving (2.6) leads to the $\mathcal{H}_\infty$ controller

$$K_\infty = \frac{1.099 \times 10^6 (s + 18.34)(s^2 + 6s + 9)}{(s + 1.15 \times 10^{15})(s + 17.14)(s^2 + 5.94s + 8.85)} \tag{5.4}$$

and the flag $\rho = 0.1 \leqslant \alpha$. This, supported by the discussion in Section 2.2, means that the desired objectives have been achieved. Figure 9 depicts the results both in the frequency and the time

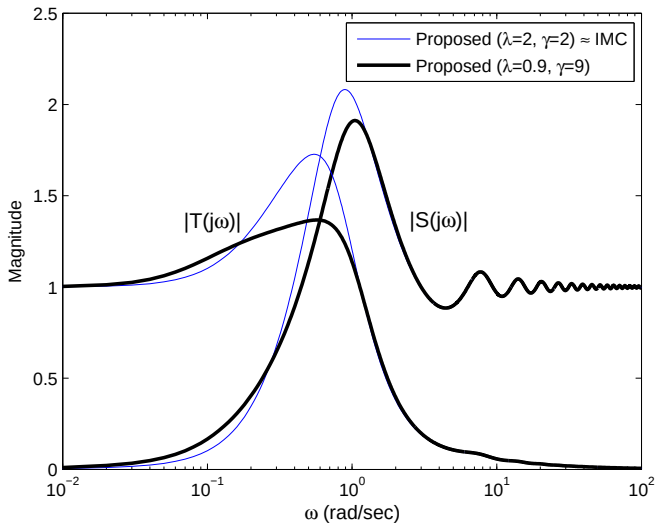
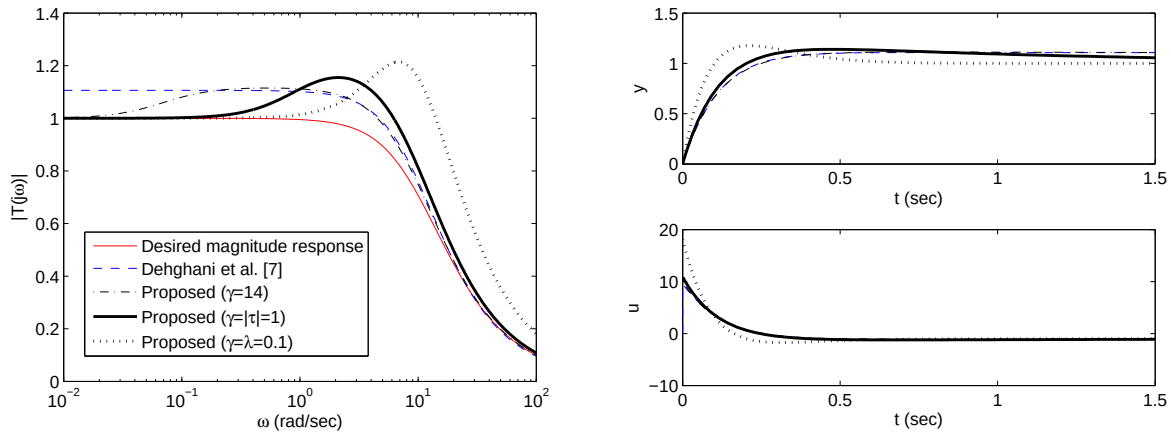


Figure 8: Magnitude frequency responses of S and T for Example 3. For $\lambda = 0.9, \gamma = 9$, the peaks of $|S|$ and $|T|$ are decreased without reducing the closed-loop bandwidth.

domain[2]. In view of Figure 9, it is clear that K_∞ does not provide integral action, even if $f|_{s=0} = 1$. As claimed in [12], where this and other pitfalls in applying the design in [6] are highlighted, there are two possible sources of difficulty: first, the fact that $f|_{s=1}$ is not exactly one, as required by the unstable plant pole at $s = 1$. Second, the fact that $\epsilon_1 \neq 0$ or $\epsilon_2 \neq 0$, as it is also the case in this example.

In what follows, we will inspect the results obtained with the proposed method, leaving the λ parameter fixed at $\lambda = 0.1$. Let us approximate $\frac{-1}{-s+1} \approx -\frac{e^{-0.01}}{-s+1}$ in order to apply the tuning rules of Table 1. We start by selecting $\lambda = 0.1, \gamma = \lambda$, but the actuator limits are violated. In order to adhere to the given specifications, we take $\gamma = 1$, which almost verifies the actuator restriction. As a matter of fact, we can make the closed-loop closer to $f = \frac{1}{0.1s+1}$ by increasing further the value of γ (the additional value $\gamma = 14$ has been considered). From Figure 9, it is evident that the proposed method always provides integral action. When $\gamma \to \infty$, a proportional controller $K = 10$ is obtained, for which the closed-loop is $\frac{1}{0.1s+0.9} \approx \frac{1}{0.1s+1}$. It is remarkable that K_∞ can be handcrafted into such a plain gain too, yielding the same results as the original fourth-order controller. However, in [6], the application of a model reduction algorithm only lowered the order of K_∞ to three. This point stresses that care has to be taken when using/implementing numerical designs. For the particular case at hand, $\gamma = 1$ gives a compromise between the desired

[2]These plots are absent in [6].



(a) Frequency responses.

(b) Set-point responses.

Figure 9: Frequency and time responses for Example 4.

magnitude response, control effort, controller complexity, and the inclusion of integral action in the loop. Obviously, the proposed design may be insufficient for more stringent specifications. In these cases, the more flexible procedure in [6] reveals advantageous.

6. Conclusions

This article has presented an analytical $\mathcal{H}_\infty$ design method based on minimizing the weighted sensitivity function. The proposed weight, chosen in a systematic way, guarantees internal stability. This point helps unifying the treatment of stable/unstable plants, avoiding the notion of coprime factorization. Another important feature of the proposed procedure is that it allows to balance the performance between the *servo* and *regulator* modes, and not only the *robustness/performance* compromise as in the original IMC procedure. Both for stable and unstable plants, it has been shown that this extra degree of freedom circumvents basic shortcomings of IMC reported in the literature.

For illustration purposes, the application to analytical tuning of PI controllers has been considered based on FOPTD and IPTD models. The suggested methodology allows to tune the controller in terms of two intuitive parameters (λ and γ), therefore guiding the tuning process. Truly-PID rules (including derivative action) could be derived similarly for the most common first and second order models. These and other extensions, as providing $\lambda\gamma$-based auto-tuning, will be published elsewhere.

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A. Appendix

This appendix contains the proofs of Theorem 3.1 and Corollary 3.1. First, the following result is necessary [8, 7, 25]:

Lemma A.1. *Consider the Model Matching Problem:*

$$\min_{Q\in\mathcal{RH}_\infty} \|\mathcal{E}\|_\infty = \min_{Q\in\mathcal{RH}_\infty} \|T_1 - T_2 Q\|_\infty \tag{A.1}$$

where $T_1, T_2 \in \mathcal{RH}_\infty$. The optimal matching error minimizing (A.1) is all-pass:

$$\mathcal{E}^o(s) = \begin{cases} 0 & \text{if} \quad \nu = 0 \\ \tilde{\rho}\frac{\tilde{q}(-s)}{\tilde{q}(s)} & \text{if} \quad \nu \geqslant 1 \end{cases} \tag{A.2}$$

where $\tilde{\rho} \in \mathbb{R}$ and $\tilde{q}(s) = 1 + \tilde{q}_1 s + \cdots + \tilde{q}_{\nu-1}s^{\nu-1}$ (strictly hurwitz) are uniquely determined by the interpolation constraints:

$$T_1(z_i) = \mathcal{E}^o(z_i) \qquad i = 1 \ldots \nu, \tag{A.3}$$

being $z_1 \ldots z_\nu$ the RHP zeros[3] of T_2.

Proof of Theorem 3.1. The following *change of variable* (or IMC parameterization [14])

$$K = \frac{Q}{1 - PQ} \tag{A.4}$$

puts $H(P, K)$ in the simpler form

$$H(P, K) = \begin{bmatrix} PQ & (1 - PQ)P \\ Q & 1 - PQ \end{bmatrix} \tag{A.5}$$

As shown in [14], internal stability is then equivalent to

- $Q \in \mathcal{RH}_\infty$
- $S = 1 - PQ$ has zeros at the unstable poles of P

The weighted sensitivity $WS = W(1 - PQ) = \mathcal{N}^o$ in (3.2) is achieved by

$$Q_0 = P^{-1}(1 - \mathcal{N}^o W^{-1}) \tag{A.6}$$

First, we must verify that Q_0 is internally stabilizing. That $Q_0 \in \mathcal{RH}_\infty$ follows from the interpolation constraints (3.3). On the other hand, $S = 1 - PQ_0 = \mathcal{N}^o W^{-1}$ is such that $S = 0$ at the

[3] For simplicity, we restrict ourselves to zeros with multiplicity one.

unstable poles of P (because W contains them by assumption). Now that internal stability has been verified, it remains to be proved that Q_0 (equivalently $\mathcal{N}^o$) is optimal. For this purpose, we use the result, proved in [14], that the set of internally stabilizing Q's can be expressed as

$$\mathcal{Q} = \{Q : Q = Q_0 + \Upsilon Q_1\} \tag{A.7}$$

where $Q_1 \in \mathcal{RH}_\infty$ is any stable transfer function, and $\Upsilon \in \mathcal{RH}_\infty$ has (exclusively) two zeros at each closed RHP pole of P (the exact shape of Υ is not necessary for the proof). Hence, any admissible weighted sensitivity has the form

$$\begin{aligned} W(1-PQ) &= W(1-P[Q_0+\Upsilon Q_1]) \\ &= W(1-PQ_0) - WP\Upsilon Q_1 \\ &= \mathcal{N}^o - WP\Upsilon Q_1 \end{aligned}$$

Minimizing $\|\mathcal{N}^o - WP\Upsilon Q_1\|_\infty$ is a standard Model Matching Problem in terms of Q_1, with $T_1 = \mathcal{N}^o \in \mathcal{RH}_\infty, T_2 = WP\Upsilon \in \mathcal{RH}_\infty$. From Lemma A.1, the optimal error $\mathcal{E}^o = T_1 - T_2 Q_1$ is all-pass and completely determined by the RHP zeros of T_2, which are those of P. More concretely, for each RHP zero of P, we have the interpolation constraint $\mathcal{E}^o(z_i) = \mathcal{N}^o(z_i)$. Obviously, this implies that $\mathcal{E}^o = \mathcal{N}^o$. Equivalently, the optimal solution is achieved for $Q_1 = 0$, showing that Q_0 is indeed optimal. □

Proof of Corollary 3.1. The optimal Weighted Sensitivity $\mathcal{N}^o$ corresponds to

$$S = \mathcal{N}^o W^{-1} \quad \text{and} \quad T = 1 - \mathcal{N}^o W^{-1} \tag{A.8}$$

From the definitions of S and T, the feedback controller can be expressed as

$$K = \frac{T}{S} P^{-1} = \frac{1 - \mathcal{N}^o W^{-1}}{\mathcal{N}^o W^{-1}} P^{-1} \tag{A.9}$$

Furthermore, the interpolation constraints (3.3) guarantee that $Q_0 \in \mathcal{RH}_\infty$. Thus, there exists a polynomial χ such that (A.6) can be rewritten as

$$Q_0 = \frac{d_p}{n_p^+ n_p^-} \left(\frac{q(s) n_w - \rho q(-s) d_w}{q(s) n_w} \right) = \frac{d_p \chi}{n_p^- q(s) n_w} \tag{A.10}$$

where the factorizations in (3.4) have been used. In terms of Q_0, we have that $S = 1 - PQ_0, T = PQ_0$ and $K = \frac{Q_0}{1-PQ_0}$. Finally, straightforward algebra yields the polynomial structure of equations (3.5)–(3.7). □