A convex formulation of fixed-order linear quadratic control with and without noise

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***Abstract*— By using a newly established link between self-optimizing control and linear-quadratic optimal control [1], [2], we show in this paper how to derive fixed-order linear quadratic optimal controllers (no noise) and fixed-order H2 optimal controllers (with noisy measurements) by solving a convex quadratic program. The method may be applied, for example, to find optimal SISO and MIMO PID controllers with and without noise. In the literature, these problems has previously been assumed to be non-convex [3]. The validity of the approach, and in particular of the noise assumptions, has been verified on a small-scale laboratory experiment.**

***Index Terms*— linear quadratic control, fixed-order control**

I. Introduction

A key result, which is a basis for this paper, is the *nullspace theorem* [4] (noise-free case, see Theorem 2):

For a quadratic static optimization problem there exists (infinitely many) linear measurement combinations $c = Hy$ that are optimally invariant to disturbances d, provided $n_y \geq n_u + n_d$.

Consider a LQ problem of the form

$$\begin{aligned} \min_u J(u, x(0)) &= x_N^{\mathrm{T}} P x_N + \\ &\quad + \sum_{k=0}^{N-1} \left[x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k \right] \\ \text{subject to } x_0 &= x(0) \\ x_{k+1} &= A x_k + B u_k, \quad k \geq 0 \\ y_k &= C x_k \end{aligned} \tag{1}$$

Here the initial states are the disturbances ($n_d = n_x$).

One sees immediately that there may be some link to linear quadratic optimal control (LQ), because the discrete LQ problem can be written as a static optimization problem. The link is: If we let the "measurements" y contain the inputs u plus the states x, then the invariant variable combination $c = Hy$ is the same as the LQ feedback law, i.e. $c = u - Kx$.

The measurements can in theory include previous and future outputs (states). However, for feedback control is all measurements need to be at the same time to avoid problems with causality. To have sufficient number of measurements ($n_y \geq n_u + n_d$) at the present time, we need information about all the present states x_k.

However, in general x is not measured directly. For the noise-free case one may use a Luenberger observer of order $n_x - n_y$ to estimate the remaining states and use the output from the observer as the input to the controller [5]. As noted in [5], "another approach is to differentiate the available outputs a number of times and the combine these derivatives appropriately to obtain the state vector." Is it further noticed that "in this case, the estimate responds instantaneously to disturbances, but it is severely degraded by a small quantity of additive noise in the measurements" In this work we use this approach where the derivatives give "state information". In addition, we provide a convex problem formulation to get fixed-order controllers for cases where of the derivatives are not available.

Importantly, results are further extended to the case with noise, that is we find combinations $c = Hy$ that yield minimum loss when held constant (Theorem 2).

Consider (1), but with *noisy* measurements

$$y_m = y + n^y. \tag{2}$$

As above, the initial state $x(0)$ is treated as the disturbance. We can now use a generalization of the nullspace theorem that handles noise as "measurements" y we include the output, a selected number of derivatives of the outputs plus the inputs, $(y_k, \frac{\partial y_k}{\partial t}, \ldots, u_k)$, and we derive a feedback law that minimizes the deterministic objective function in (1) *subject to* using noisy measurements. As for the noise-free case, we have a *convex* formulation of the fixed-order control problem.

The rest of the paper is organized as follows: In section II we review two theorems from self-optimizing control. In section III we will see that these theorems gives a nice link to LQ control, and several examples will be given.

A. Notation

In previous works on self-optimizing control and in particular the nullspace method, candidate variables are denoted y and the $n_c = n_u$ variable combinations (controlled variables) $c = Hy$. These candidate variables can be process outputs, and also inputs. On the other hand, in process control literature y is referred to as measurements or process output, but usually not inputs. In this paper we work most of the time with discrete models, and then y_k is a process output, whilst y is a vector of candidate variables, for example $y = (x_k, u_k)$.

Figure 1 shows the candidate variables y that are combined to $c = Hy$ and control them using a feedback controller. In this work we will show that the feedback controller can be obtained from $c = Hy$ itself, if we include the inputs u_k in the candidate variables y. Further, in this work, $c_s = 0$

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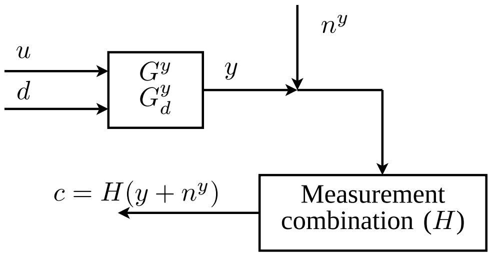

Fig. 1. Summary of important notation.

for all candidate variable combinations, as we will typically consider regulation problems in deviation variables.

The most important notation is also summarized in figure 1. Typically, $u = (u_k, u_{k+1}, \ldots, u_{k+N-1})$, $d = x_0$ and $y = (x_k, u_k)$ or $y = (y_k, \frac{\partial y_k}{\partial t}, \ldots, u_k, u_{k+1}, \ldots, u_{k+N-1})$.

II. Results from self-optimizing control

From [4] we have the following two theorems:

Theorem 1: (Nullspace theorem = Linear invariants for quadratic optimization problem) Consider an unconstrained quadratic optimization problem in the variables u (input vector of length n_u) and d (disturbance vector of length n_d)

$$\min_u J(u,d) = \begin{bmatrix} u \\ d \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} J_{uu} & J_{ud} \\ J_{ud}^{\mathrm{T}} & J_{dd} \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix}. \tag{3}$$

In addition, there are "measurement" variables $y = G^y u + G_d^y d$. If there exists $n_y \geq n_u + n_d$ independent measurements (where "independent" means that the matrix $\tilde{G}^y = \begin{bmatrix} G^y & G_d^y \end{bmatrix}$ has full rank), then the optimal solution to (3) has the property that there exists $n_c = n_u$ linear variable combinations (constraints) $c = Hy$ that are invariant to the disturbances d. The optimal measurement combination matrix H is found by:

First, let $F = \frac{\partial y^{\mathrm{opt}}}{\partial d}$, where the sensitivity matrix F can be obtained from

$$F = -(G^y J_{uu}^{-1} J_{ud} - G_d^y), \tag{4}$$

and select H such that

$$HF = 0. \tag{5}$$

That is, H is in the left nullspace of F.

A generalization of Theorem 1 is the following:

Theorem 2: (Loss by introducing linear constraint for noisy quadratic optimization problem) Consider the unconstrained optimization problem in Theorem 1,

$$\min_u J(u,d) = \begin{bmatrix} u \\ d \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} J_{uu} & J_{ud} \\ J_{ud}^{\mathrm{T}} & J_{dd} \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix},$$

and a set of noisy measurements $y_m = y + n^y$, where $y = G^y u + G_d^y d$. Assume that $n_c = n_u$ constraints $c = Hy_m = c_s$ are added to the problem, which will result in a non-optimal solution with a loss $L = J(u,d) - J_{\mathrm{opt}}(d)$. Consider disturbances d and noise n^y with magnitudes

$$d = W_d d'; \quad n^y = W_{n^y} n^{y}; \quad \left\| \begin{bmatrix} d' \\ n^{y'} \end{bmatrix} \right\|_2 \leq 1.$$

Then for a given H, the worst-case loss introduced by adding the constraint $c = Hy$ is $L_{wc} = \bar{\sigma}(M)/2$, where M is

$$\begin{aligned} M &\triangleq \begin{bmatrix} M_d & N_{n^y} \end{bmatrix} \\ M_d &= -J_{uu}^{1/2}(HG^y)^{-1}HFW_d \\ M_n &= -J_{uu}^{1/2}(HG^y)^{-1}HW_{n^y}. \end{aligned} \tag{6}$$

The optimal H that minimizes the loss can be found by solving the *convex* optimization problem

$$\begin{aligned} &\min_H \ \|H\tilde{F}\|_F \\ &\text{subject to } HG^y = J_{uu}^{1/2} \end{aligned} \tag{7}$$

Here $\tilde{F} = [FW_d \ \ W_{n^y}]$.

The reason for using the Frobenius norm is that minimization of this norm also minimizes $\bar{\sigma}(M)$ [6].

Remark 1: If $\tilde{F}\tilde{F}^{\mathrm{T}}$ is non-singular we have an explicit expression for the optimal H [4]:

$$H^{\mathrm{T}} = (\tilde{F}\tilde{F}^{\mathrm{T}})^{-1} G^y \left(G^{y\mathrm{T}} (\tilde{F}\tilde{F}^{\mathrm{T}})^{-1} G^y \right)^{-1} J_{uu}^{1/2}. \tag{8}$$

Remark 2: Since, in this particular case, the matrix H that minimizes the Forbenius norm also minimizes the maximum singular value of M [6], this H is also a solution to $\min_H \bar{\sigma}(M)$.

Remark 3: From [4] we have that any optimal H premultiplied by a non-singular matrix $n_c \times n_c$ D, i.e. $H_1 = DH$ is still optimal. One implication of this is that for a square plant, $n_c = n_u$, we can write $c = H_1 y = H_1^{y_m} y_m + Iu$. To see this, assume $y = (y_m, u)$, so $H = [H^{y_m} \ H^u]$, where H^u is a non-singular $n_u \times n_u$ matrix. Now, $H_1 = (H^u)^{-1}[H^{y_m} \ H^u] = [(H^u)^{-1} H^{y_m} \ I]$.

Remark 4: More generally, for the case when $\tilde{F}\tilde{F}^{\mathrm{T}}$ is singular, we can solve the convex problem (7) using for example CVX, a package for specifying and solving convex programs [7], with the following code:

```
cvx_begin
   variable H(N*nu,ny+nu*N);
   minimize norm(H*Ftilde,'fro')
   subject to
      H*Gy == sqrtm(Juu);
cvx_end
```

An important comment regrading Theorem 2 for LQ

It is assumed in this work that the problem can be formulated as a static problem at time $t = k$ (with all the measurements available at time k) This assumption is satisfied for the PID controller with direct measurements of the present output y_k, the derivatives y_k' and the sum $y_k^I = \sum y_i$ (for integration).

However, if we only have available present output measurements (y_k), then the derivatives must be obtained by using previous measurements, e.g. $y_k' = y_k - y_{k-1}$. In this case, there will then be an additional "start-up" loss, in addition to that given in Theorem 2, and it is not guaranteed that the solution obtained from Theorem 2 is optimal (although it is likely to be reasonably close to the optimal case)

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III. Full state information

A. No noise

Assume that noise-free measurements of all the states are available. It is well known that the LQ problem (1) can be rewritten on the form in (3) (see for example [8]) by treating x_0 as the disturbance d, and letting $u = (u_0, u_1, \dots, u_{N-1})$. Thus, from Theorem 1 we know that for this problem there exists *infinitely* many invariants, but only one of these involves only present states.

Without loss of generality consider a stable process that can be described by the following linear model:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \quad k = 0, 1, 2, \dots \\ x_0 &: \text{ known} \end{aligned} \tag{9}$$

Let $y = (x_k, u_k, u_{k+1}, \dots, u_{k+N-1}) = (x_k, u)$. Note that this includes also future inputs, but we will use the normal "trick" in MPC of implementing only the present (first) input change u_k. Since we have $n_y = n_d + n_u$ and no noise, we can use Theorem 1. The open loop model becomes:

$$\begin{aligned} y &= G^y u + G^y_d d \\ G^y &= \begin{bmatrix} 0_{n_x \times (n_u N)} \\ I_{n_u N} \end{bmatrix} \in \mathbb{R}^{(n_x + n_u N) \times (n_u N)} \\ G^y_d &= \begin{bmatrix} I_{n_u N} \\ 0_{(n_u N) \times n_x} \end{bmatrix} \in \mathbb{R}^{(n_x + n_u N) \times n_x} \end{aligned} \tag{10}$$

Here I_m is an $m \times m$ identity matrix and $0_{m \times n}$ is a $m \times n$ matrix of zeros.

The matrices J_{uu} and J_{ud} are the derivatives of the linear quadratic objective function. Here we will consider the following infinite horizon objective function:

$$J = \sum_{k=0}^{\infty} \left(x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k \right). \tag{11}$$

If we assume that for $k \geq 0$ the solution to the optimization problem of minimizing (11), it can be shown [8] that this particular objective function can be rewritten as

$$J = \sum_{k=0}^{N-1} \left(x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k \right) + x_N^{\mathrm{T}} P x_N, \tag{12}$$

where P is a solution to the discrete Lyapunov equation $P = A^{\mathrm{T}} P A + Q$. (For an unstable process we can set $u_k = -Kx_k$ for $k \geq N$, where K is a state feedback gain matrix such that $(A - BK)$ has no eigenvalues outside the unit circle. For the objective function in (11) we can convert the problem to finite horizon by using a final state weight matrix for example from [9].)

For the objective in (12) with the process model in (9) we show in [10] that

$$\frac{J_{uu}}{2} = \begin{bmatrix} B^{\mathrm{T}} P B + R & B^{\mathrm{T}} A^{\mathrm{T}} K B & \dots & B^{\mathrm{T}} (A^{N-1})^{\mathrm{T}} P B \\ B^{\mathrm{T}} P A B & B^{\mathrm{T}} P B + R & \dots & B^{\mathrm{T}} (A^{N-2})^{\mathrm{T}} P B \\ \vdots & \vdots & \ddots & \vdots \\ B^{\mathrm{T}} P A^{N-1} B & B^{\mathrm{T}} P A^{N-2} B & \dots & B^{\mathrm{T}} P B + R \end{bmatrix} \tag{13}$$

and

$$\frac{J_{ud}}{2} = \begin{bmatrix} B^{\mathrm{T}} & & & \\ & B^{\mathrm{T}} & & \\ & & \ddots & \\ & & & B^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} P \\ PA \\ \vdots \\ PA^{N-1} \end{bmatrix} A \tag{14}$$

The sensitivity matrix (optimal change in y when d is perturbed) becomes:

$$F = \frac{\partial y^{\mathrm{opt}}}{\partial d^{\mathrm{T}}} = -(G^y J_{uu}^{-1} J_{ud} - G^y_d) = \begin{bmatrix} I_{n_x} \\ -J_{uu}^{-1} J_{ud} \end{bmatrix} \tag{15}$$

Since there is no noise we can use Theorem 1 to get the combination matrix H, i.e. find an H such that $HF = 0$:

$$\begin{bmatrix} H_1 & H_2 \end{bmatrix} \begin{bmatrix} I_{n_x} \\ J_{uu}^{-1} J_{ud} \end{bmatrix} = H_1 - H_2 (J_{uu}^{-1} J_{ud}) = 0 \tag{16}$$

To ensure a non-trivial solution we can choose $H_2 = I_{n_u N}$ and get the following optimal combination of x_k and u:

$$c = Hy = J_{uu}^{-1} J_{ud} x_k + u, \tag{17}$$

which reads out as $(u_k = K^k x_k), (u_{k+1} = K^{k+1} x_k), \dots, (u_{k+N-1} = K^{k+N-1}) x_k$, of which the first invariant $u_k = K^k x_k$ is the one to be implemented.

In [10] we prove that this gives the same result as conventional linear quadratic control, by conventional meaning for example equation (3) in Rawlings and Muske 1993 [8].

B. Noisy measurement of state vector

Assume now that *noisy* measurements of the state vector are available, and that the noise-level on all states is the same, i.e. $x_{m,k} = x_k + \alpha$. As before, we treat the initial state as a disturbance, $d = x_0$, and assume the following bounds on the disturbance and noise:

$$\begin{gathered} d = W_d d', \ n^y = W_{n^y} n^{y'}, \ W_d = I, \ W_{n^y} = \alpha I, \\ \text{and } \left\| \begin{bmatrix} d' \\ n^{y'} \end{bmatrix} \right\|_2 \leq 1. \end{gathered} \tag{18}$$

Here α is the *noise-to-disturbance* ratio and we have assumed that the combined two-norm describes the disturbance and noise variations. Further assume that an optimal state feedback K for the case of no noise ($\alpha = 0$) has already been found. By using Theorem 2 and the analytical expression for H (8), we prove in appendix A that

$$u_k = \frac{1}{1 + \alpha^2} K x_k \tag{19}$$

Thus, $(1 + \alpha^2)$ is the optimal reduction in state feedback gain when $\alpha > 0$.

IV. Output feedback without noise

In this section we will consider a second order SISO process with noise-free output measurements. For clarity of presentation, we present the theory by way of an example.

We will consider two cases. First, the full-information case where we measure the derivatives and where $(y_k, \frac{\partial y_k}{\partial t})$ and the inputs are combined using Theorem 1. The controller is equivalent to a Luenberger observer with poles at $-\infty$

in continuous time domain and 0 in discrete time, but is derived using Theorem 1, and *not* using observer theory. In the second case a low-order controller using the explicit expression (8) from Theorem 2 with $W_{n^y} = 0$ is derived.

Example 1: (SISO process) Consider the process $y(t) = \frac{2}{s^2+3s+2}u(t)$ taken from example 7.1 in [11]. This can be written on "observer canonical form" (see for example [12]):

$$G(s) = \left[\begin{array}{cc|c} -3 & 1 & 0 \\ -2 & 0 & 2 \\ \hline 1 & 0 & 0 \\ -3 & 1 & 0 \end{array}\right], \tag{20}$$

where have used the notation $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ represents the transfer function $G(s) = C(sI - A)^{-1}B + D$. The derivative $y'(t)$ is included as the second output from the process. We sample the process with $T = 0.1$ and get the following *discrete* realization:

$$G(z) = \left[\begin{array}{cc|c} 0.7326 & -0.0861 & 0.009 \\ -0.1722 & 0.9909 & 0.200 \\ \hline 1 & 0 & 0 \\ -3 & 1 & 0 \end{array}\right]. \tag{21}$$

The task is to regulate the system to the origin, and to achieve this we minimize the following objective:

$$J(u, x_0) = x_N^{\mathrm{T}} P x_N + \sum_{i=0}^{N-1} x_i^{\mathrm{T}} Q x_i + u_i^{\mathrm{T}} R u_i, \tag{22}$$

subject to model equations.

We want to bring both (y, y') to the origin and set $Q = C^{\mathrm{T}} I_2 C$, while $R = 0.01$. P solves the Lyapunov equation $P = A^{\mathrm{T}} P A + Q$, i.e. we assume that for $k \geq N$, $u_k = 0$. (See [8]).

Full information case: We here use Theorem 1 as in section III-A, but with

$$G^y = \begin{bmatrix} 0_{2,n_u N} \\ I_{n_u} \end{bmatrix}, \quad G_d^y = \begin{bmatrix} C \\ 0_{n_u N, n_x} \end{bmatrix} \tag{23}$$

After doing the calculations we find the feedback law

$$u_k = -\begin{bmatrix} 3.59 & 3.58 \end{bmatrix} \begin{bmatrix} y_k \\ y_k' \end{bmatrix}. \tag{24}$$

It is not surprising that the controller puts almost equal gains on the process output and the derivative, since we use $Q = C^{\mathrm{T}} I C$ as a weight on the states in the objective function.

Reduced-order controller case. In the full information case we got a controller of order 1, i.e. we had one differentiation of the output. This is the same order as a reduced-order Luenberger observer, which is of order $n_x - n_y$ [13]. A reduced-order controller for this process is to use only the process output y_k. This was done in [2], but is here repeated with focus on the relation to the full information controller. We here use (8), with

$$G^y = \begin{bmatrix} 0_{1,n_u N} \\ I_{n_u} \end{bmatrix}, \quad G_d^y = \begin{bmatrix} c_1 \\ 0_{n_u N, n_x} \end{bmatrix}. \tag{25}$$

Here $c_1 = [1 \quad 0]$ is the first row of the C-matrix. In this case we need to solve an optimization problem to get the optimal combination, since $n_d \leq n_y = N n_u + (n_d - 1) \leq N n_u + n_d$. The optimal H can now be found solving the convex optimization problem shown in Theorem 1.

We end up with the feedback law

$$u_k = -7.14 y_k. \tag{26}$$

Note that this gain is the double of the gain for the full information case.

Numerical comparison: A simulation was run with disturbances drawn from a uniform distribution with $\|d\|_1 \leq 1$, and by computing the average stage costs under closed loop, $J_{\text{avg}} = \frac{1}{N}\sum_{i=0}^{N} x_i^{\mathrm{T}} Q x_i + u_i^{\mathrm{T}} R u_i$ we found that $J_{\text{avg, full information}} = 6.4$, while $J_{\text{avg, reduced information}} = 22.7$. As expected, there is a loss with only output feedback.

V. OUTPUT FEEDBACK WITH NOISE

In this section, we will use Theorem 2 to find low-order controllers when noisy measurements are available. We will show the methodology on a small-scale laboratory plant, which is shown in figure 2. The low-order controller that we want to use is a PID controller, and therefore we first show how to derive a LQ-optimal PID controller and then apply the controller to the laboratory plant. The laboratory-scale plant is rather small and likely to be affected by disturbances such as opening of lab doors, air conditioning, other lamps switched on/off etc., hence integral action seems necessary for controlling the plant

The PID controller is synthesized using Theorem 2, but before finding the controller some more preliminaries are needed. We need to

1) Augment the model with a disturbance model.
2) Modify the objective function to penalize input change rather than absolute value of the inputs. This is necessary in order let the outputs reach their setpoints when integrating disturbances occur. (We want to use the inputs to counteract disturbances at steady state, hence we should not require the inputs to return to the nominal point of operation.)

We start by augmenting the model with integrating disturbances. The formulation, see (27), includes both input and output disturbances. In addition we add integrators for summing up the outputs. These correspond to the integrators in the controller. (For the example, the number of integrators in the controller n_s equals number of integrating disturbances $n_s = n_d = n_y = 1$).We also add as an output the output change $y_{k+1} - y_k = (CAx_k + CBu_k) - Cx_k$, where d_{k+1} was assumed to be $d_{k+1} = d_k$. (The derivatives may also be added by starting from a continuous model on observer canonical form and then discretizing, as in example 1.) We

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then get the following model of the plant and controller:

$$\begin{bmatrix} x_{k+1} \\ d_{k+1} \\ \sigma_{k+1} \end{bmatrix} = \begin{bmatrix} A_{\text{plant}} & B_d & 0 \\ 0 & I & 0 \\ C & C_d & I \end{bmatrix} \begin{bmatrix} x_k \\ d_k \\ \sigma_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} u_k$$
$$\begin{bmatrix} y_k^P \\ y_k^I \\ y_k^D \end{bmatrix} = \begin{bmatrix} C & C_d & 0 \\ 0 & 0 & I \\ C(A_{\text{plant}} - I) & 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ d_k \\ \sigma_k \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ CB \end{bmatrix} u_k \tag{27}$$

We now modify the objective function to penalize $\Delta u_k \triangleq u_{k+1} - u_k$. Assume the original objective function was on the form $J(x,u) = x_N^{\mathrm{T}} P x_N + \sum_{i=0}^{N-1} x_i^{\mathrm{T}} Q x_i + u_i^{\mathrm{T}} R u_i + 2 x_i^{\mathrm{T}} N u_i$, hence no term $\Delta u^{\mathrm{T}} R_\Delta \Delta u_k$. First note that in continuous time, $u = Kx \Rightarrow \dot{u} = K\dot{x}$ in closed loop. In discrete time $\dot{x} \approx x_{k+1} - x_k$ and we get that

$$\dot{u} = K\dot{x} \approx K(x_{k+1} - x_k) = K\left((A - I)x_k + Bu_k\right) \tag{28}$$

The term $\Delta u_k^{\mathrm{T}} R_\Delta \Delta u_k$ becomes

$$\begin{aligned} \Delta u_k^{\mathrm{T}} R_\Delta \Delta u_k &= x_k^{\mathrm{T}} (A - I)^{\mathrm{T}} K^{\mathrm{T}} R_\Delta K (A - I) x_k + \\ &+ u_k B^{\mathrm{T}} K^{\mathrm{T}} R_\Delta K B u_k + 2 x_k^{\mathrm{T}} (A - I)^{\mathrm{T}} K^{\mathrm{T}} R_\Delta K B u_k. \end{aligned} \tag{29}$$

This formulation is useful because we can use, for example, the function `'lqr'` in Matlab directly to get the K_{lqr} feedback matrix. This matrix is needed for the calculation of the final weight matrix P. In earlier examples we calculated P from $P = A^{\mathrm{T}} P A + Q$, by assuming that $u_k = 0$ for $k \geq N$. With integral action this is wrong, since at steady state we use the inputs to counteract the integrating disturbances. The following final weight can be used to change the problem from infinite to finite horizon: (See Appendix B for a derivation.)

$$\begin{aligned} P = &(A - BK_{\text{lqr}})^{\mathrm{T}} P (A - BK_{\text{lqr}}) + \\ &+ K_{\text{lqr}}^{\mathrm{T}} R K_{\text{lqr}} + Q - N K_{\text{lqr}}. \end{aligned} \tag{30}$$

Let us summarize the method for finding a (MIMO) PID controller with quadratic objective function and noisy measurements:

1) Choose weights (Q, R_Δ) for the LQ problem.
2) Determine weights W_d, W_{n^y} from operating data and/or process knowledge.
3) Augment the process model as shown in (27).
4) Solve LQ problem, for example with `'lqr'` in Matlab, *iteratively* on K, with the following objective:

$$\begin{aligned} J = \sum_{i=0}^{\infty} &x_i^{\mathrm{T}} (Q + (A - I)^{\mathrm{T}} K^{\mathrm{T}} R_\Delta K (A - I)) x_i + \\ &+ u_i^{\mathrm{T}} B^{\mathrm{T}} K^{\mathrm{T}} R_\Delta K B u_i + \\ &+ 2 x_k^{\mathrm{T}} (A - I)^{\mathrm{T}} K^{\mathrm{T}} R_\Delta K B u_k. \end{aligned} \tag{31}$$

The following iteration scheme was used:

while $\|\Delta K\| > \beta$ **do**
$Q_{\text{it}} \leftarrow Q + (A - I)^{\mathrm{T}} K^{\mathrm{T}} R_\Delta K (A - I)$
$R_{\text{it}} \leftarrow B^{\mathrm{T}} K^{\mathrm{T}} R_\Delta K B$
$N_{\text{it}} \leftarrow (A - I)^{\mathrm{T}} K^{\mathrm{T}} R_\Delta K B$
$K_{\text{new}} \leftarrow \text{lqr}(G, Q_{,\text{it}}\, R_{\text{it}}, N_{\text{it}})$
$\Delta K \leftarrow K_{\text{new}} - K$
$K \leftarrow \alpha K + (1 - \alpha) K_{\text{new}}$
end while

Here G is the state-space representation of the process, $0 \leq \alpha < 1$ is a numerical damping factor and β is the convergence criterion.

5) Find P from (30), for example by `'dlyap'` in Matlab.
6) Use theorem 2 to find the optimal combination between $(y_k^P, y_k^I, y_k^D, u_k)$.

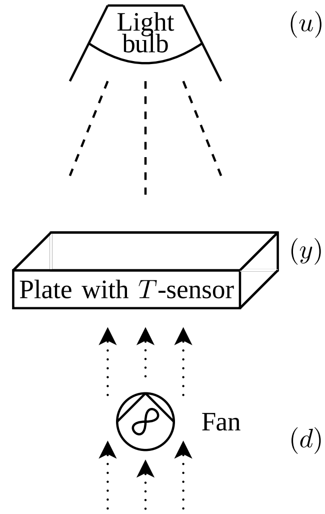

Fig. 2. Laboratory thermal plant.

Example 2: (Laboratory experiment: Thermal Plant [14]) In this example we want to control the temperature $y_k = T$ by changing the power inlet to a light bulb (u_k). A sketch of the plant is shown in figure 2. We observe that a fan is blowing air onto the plate with the temperature sensor. We will use this fan to generate disturbances for the plant. A model of the plant has been found experimentally, is

$$G(z) = \left[\begin{array}{cc|c} 0.9771 & -0.0210 & 10^{-3} \cdot 0.1978 \\ -0.0319 & 0.9430 & 10^{-3} \cdot 0.1955 \\ \hline 525.1 & -1.982 & \end{array}\right]. \tag{32}$$

The sample time in the above model is 1 second. For this process we choose

$$Q = C^{\mathrm{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} C \tag{33}$$

and

$$R_\Delta = 1. \tag{34}$$

We further set $W_d = I_4$. For the noise weight W_{n^y}, we choose

$$W_{n^y} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 100 & \\ & & & I_{n_u N} \end{bmatrix} \tag{35}$$

This matrix should be related to the noise-to-disturbance ratio. Here the disturbances are the disturbances to the initial

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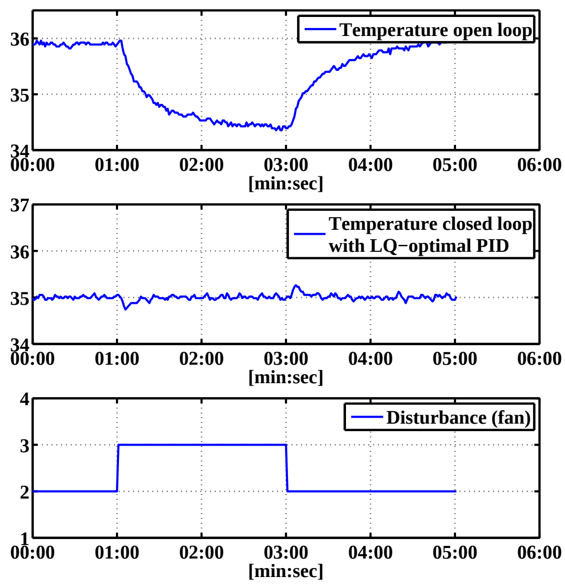

Fig. 3. Experimental data.

states x_0. In this example however, we use this matrix as a tuning matrix in which we set a high noise-term on the differential (the third output) and let the other terms have same weight as the disturbance weight matrix. This is because we do not want too much derivative action , but at the same time we want to demonstrate the mathematical framework for deriving a PID controller. (Of course, if we do not want a D-term in the controller, we should have excluded the differential as a possible "measurement" before using Theorem 2 to find the controller.)

For the disturbance model we choose

$$B_d^{\mathrm{T}} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad C_d = 0. \tag{36}$$

Notice that

$$\operatorname{rank} \begin{bmatrix} I - A_{\text{plant}} & -B_d \\ C_{\text{plant}} & C_d \end{bmatrix} = 3 = n_x + n_y, \tag{37}$$

which indicates that offset-free control at steady state should be possible [15], [16].

As input horizion we set $N = 20$ in this example.

Using the above method, we first find that $K_{\text{lqr}} = 10^3 \begin{bmatrix} 3.1590 & -0.1174 & 0.0010 & 0.0013 \end{bmatrix}$, and that

$$P = 10^6 \begin{bmatrix} 6.8224 & -0.2768 & 0.0029 & 0.0035 \\ -0.0993 & 0.0547 & -0.0011 & 0.0000 \\ 0.0010 & -0.0010 & 0.1811 & -0.0000 \\ 0.0045 & -0.0000 & 0.0000 & 0.0000 \end{bmatrix}. \tag{38}$$

Since we now have penalty on the input change, $\Delta u_k^{\mathrm{T}} R_\Delta u_k$, the J_{uu} matrix in (13) needs to be changed slightly. This can be done by letting $U = (u_0, u_1, \ldots, u_{N-1})$ and $\Delta U = MU$ where

$$M = \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix} \in \mathbb{R}^{n_u(N-1) \times n_u N}. \tag{39}$$

The matrix J_{uu} is now

$$\begin{aligned} \frac{J_{uu}}{2} = & \begin{bmatrix} B^{\mathrm{T}}PB{+}R & B^{\mathrm{T}}A^{\mathrm{T}}KB & \ldots & B^{\mathrm{T}}(A^{N-1})^{\mathrm{T}}PB \\ B^{\mathrm{T}}PAB & B^{\mathrm{T}}PB{+}R & \ldots & B^{\mathrm{T}}(A^{N-2})^{\mathrm{T}}PB \\ \vdots & \vdots & \ddots & \vdots \\ B^{\mathrm{T}}PA^{N-1}B & B^{\mathrm{T}}PA^{N-2}B & \ldots & B^{\mathrm{T}}PB{+}R \end{bmatrix} + \\ & + M^{\mathrm{T}} \begin{bmatrix} R_\Delta & & \\ & \ddots & \\ & & R_\Delta \end{bmatrix} M \end{aligned} \tag{40}$$

The structure of J_{ud} is the same as in (14).

The open loop model $y = G^y u + G_d^y d$, with $d = x_0$, is for this example

$$\begin{aligned} G^y &= \begin{bmatrix} D & 0_{3n_y \times (N-1)n_u} \\ I & 0 \\ 0 & I \end{bmatrix} \\ G_d^y &= \begin{bmatrix} C \\ 0_{Nn_u \times (n_x + n_d + n_s)} \end{bmatrix}. \end{aligned} \tag{41}$$

Here n_d is the number of integrating disturbances and n_s is the number of integrators in the controller. We have that $n_d = n_s = n_y = 1$.

We can now calculate $\tilde{F}$, and solve the convex optimization problem that finds the minimum of $\|H\tilde{F}\|$ subject to $HG^y = J_{uu}^{1/2}$. As indicated above, H can be written as $H = [H_m^y \;\; H^u]$ and another matrix that minimizes the norm is $H^1 = (H^u)^{-1}H = [(H^u)^{-1}H^{y_m} \;\; I]$. By considering the *first row* of this matrix we find that

$$u_k + 5.04 y_k + 0.53 \sum_{i=0}^{k} y_i + 0.11(y_k - y_{k-1}) = 0. \tag{42}$$

This variable combination that gives the minimum loss when we impose a PID-structure for the controller to the original problem. In feedback form:

$$u_k = -\underbrace{5.04 y_k}_{\text{P}} - \underbrace{0.53 \sum_{i=0}^{k} y_i}_{\text{I}} - \underbrace{0.11(y_k - y_{k-1})}_{\text{D}} \tag{43}$$

Note that in the original problem formulation we obtain $y_{k+1} - y_k$ for the derivative, but since this is non-causal we have shifted the derivative one step back in the implementation.

Figure 3 shows a plot of the temperature loop in open and closed loop, where in closed loop we implemented the LQ-optimal PID controller. No filter on the derivative part was used. One observes that under closed loop the temperature is kept at its set-point at 35°C, even with the integrating disturbances from the fan, whilst in open loop the temperature drifts away when the plant is subjected to the same disturbances. In closed loop is seems like the noise is slightly amplified, this is probably due to the derivative term in the controller. This can be fixed by placing a filter in front of the derivative term. Here we want to demonstrate the design of the controller rather than tuning, so we will not pursue this issue further.

Discussion

Above we used pure integrators in the derivation, and we used a penalty on $\Delta u_k^{\mathrm{T}} R \Delta u_k$ rather than $u_k^{\mathrm{T}} R u_k$ in the objective function. The reasoning was that since we want integral action we want to use the input to counteract the integrating disturbances, therefore it is not reasonable to require that the inputs return to the nominal point. We saw that by using $\dot{u} = K\dot{x}$ we could fit the penalty of the input-change into the normal objective function $J = \sum_{k=0}^{\infty} x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k + 2x_k^{\mathrm{T}} N u_k$, but in order to get the optimal controller we had to iterate, since the weights (Q, R, N) are functions of the controller itself.

Another obvious approach is to not use pure integrators, but rather add disturbances with very large time constants. This way we do not have to iterate on the controller. In this setting we can also add a weight on $u_k^{\mathrm{T}} R u_k$, since the states eventually will be driven back to the origin. The main gain from the method above seems to be that we can reduce the input horizon N, and hence the number of the degrees of freedom, compared to the approach of adding disturbances with large time constants, as in order to capture the behaviour of the process a larger input horizon N is needed.

VI. Conclusion

In this work we have presented a convex approach to the design of fixed order linear quadratic controllers. In particular, we have shown how to derive PD and PID controllers for a linear plant with a quadratic control objective. From Theorem 2 we have derived expressions for fixed-order controller both for the case of noisy and noise-free measurements.

In example we 2 gave all steps necessary to derive a PID controller for a given linear plant, and we tested the controller on a laboratory temperature loop. The framework is general in the sense that it can be applied directly to MIMO systems to get MIMO PID controller. In a forthcoming contribution we will indeed give guidelines for setting up a MIMO PID controller using the ideas presented here.

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Appendix

A. Proof of gain reduction for LQ control

Assume $W_d = I$ and

$$W_{n^y} = \begin{bmatrix} \alpha I & \\ & \beta I \end{bmatrix}$$

Here α is the measurement noise and β is additive noise to the inputs. (We will show that β does not affect the solution.) Define $J = -J_{uu}^{-1} J_{ud}$. We have that $\tilde{F}\tilde{F}^{\mathrm{T}} = F W_d W_d^{\mathrm{T}} F^{\mathrm{T}} + W_{n^y} W_{n^y}^{\mathrm{T}}$. By the above assumptions we get that

$$F \underbrace{W_d W_d^{\mathrm{T}}}_{I} F^{\mathrm{T}} = F F^{\mathrm{T}} = \begin{bmatrix} I & J^{\mathrm{T}} \\ J & J J^{\mathrm{T}} \end{bmatrix} \quad (44)$$

Due to the assumptions on W_{n^y} we get

$$\tilde{F}\tilde{F}^{\mathrm{T}} = F W_d W_d^{\mathrm{T}} F^{\mathrm{T}} + W_{n^y} W_{n^y}^{\mathrm{T}} = \begin{bmatrix} (1+\alpha^2) I & J^{\mathrm{T}} \\ J & J J^{\mathrm{T}} + \beta^2 I \end{bmatrix} \quad (45)$$

This matrix has to be inverted. This can be done using Lemma A.2 (Inverse of a partitioned matrix) in [12], with $A_{11} = (1+\alpha^2) I$, $A_{12} = J^{\mathrm{T}}$, $A_{21} = J$, $A_{22} = J J^{\mathrm{T}} + \beta^2 I$. Further we have $X = A_{22} - A_{21} A_{11}^{-1} A_{12} = \cdots = \left(\frac{\alpha^2}{1+\alpha^2} J J^{\mathrm{T}} + \beta^2 I \right)$. We observe that the inverse of X exists. Using the Lemma, we get that the inverse of $\tilde{F}\tilde{F}^{\mathrm{T}}$ is:

$$\left(\tilde{F}\tilde{F}^{\mathrm{T}} \right)^{-1} = \begin{bmatrix} \frac{1}{1+\alpha^2} I + \frac{1}{(1+\alpha^2)^2} J^{\mathrm{T}} X^{-1} J & -\frac{1}{1+\alpha^2} J^{\mathrm{T}} X^{-1} \\ -\frac{1}{1+\alpha^2} X^{-1} J & X^{-1} \end{bmatrix} \quad (46)$$

We now need to evaluate $G^{y\mathrm{T}} (\tilde{F}\tilde{F}^{\mathrm{T}})^{-1} G^y$. For the current problem formulation we have that $G^{y\mathrm{T}} = [0_{n_x \times n_U} \; I_{n_U \times n_U}]$,

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and after doing the multiplication we get that

$$G^{y\mathrm{T}}(\tilde{F}\tilde{F}^{\mathrm{T}})^{-1}G^{y} = X^{-1} \;\Rightarrow\; \left(G^{y\mathrm{T}}\left(\tilde{F}\tilde{F}^{\mathrm{T}}\right)^{-1}G^{y}\right)^{-1} = X \tag{47}$$

Further,

$$\left(\tilde{F}\tilde{F}^{\mathrm{T}}\right)^{-1}G^{y} = \begin{bmatrix} -\frac{1}{1+\alpha^2}J^{\mathrm{T}}X^{-1} \\ X^{-1}\end{bmatrix} \tag{48}$$

and finally we get that

$$H^{\mathrm{T}} = \left(\tilde{F}\tilde{F}^{\mathrm{T}}\right)^{-1}G^{y}\left(G^{y\mathrm{T}}\left(\tilde{F}\tilde{F}^{\mathrm{T}}\right)^{-1}G^{y}\right)^{-1}J_{uu}^{1/2} \tag{49}$$

$$= \begin{bmatrix} \frac{1}{1+\alpha^2}(J_{uu}^{-1}J_{ud})^{\mathrm{T}}J_{uu}^{1/2} \\ J_{uu}^{1/2}\end{bmatrix}, \tag{50}$$

or

$$H = \begin{bmatrix} \frac{1}{1+\alpha^2}J_{uu}^{1/2}J_{uu}^{-1}J_{ud} & J_{uu}^{1/2}\end{bmatrix} \tag{51}$$

We now scale H matrix by $J_{uu}^{-1/2}$ to decouple the inputs and to get an expression for the controller gains:

$$(J_{uu}^{1/2})^{-1}H = \begin{bmatrix} \frac{1}{1+\alpha^2}J_{uu}^{-1}J_{ud} & I\end{bmatrix}, \tag{52}$$

and we observe that optimally we should reduce the controller gains by $1/(1+\alpha^2)$ when there is noise on the states on the form αI. To see this, remember that $y=(x,u)$, and hence we get c's on the form

$$c = Hy = \frac{1}{1+\alpha^2}J_{uu}^{-1}J_{ud}x + U, \tag{53}$$

which is on exactly the same form as (17).

Remark 5: From the above derivation one notes that noise entering on the inputs does not affect the optimal solution. This may also be seen from the norm of Hy:

$$\begin{aligned}\|Hy\| &= \|(u_k + Kx_k) + n^u + Kn^x\| \\ &\le \|Kx_k + u_k\| + \|Kn^x\| + \|n^u\|\end{aligned} \tag{54}$$

We observe that there is a trade-off by using u_k to keep $\|Kx_k + u_k\|$ small, but avoiding amplification of $\|Kn^x\|$. However, n^u does not affect this trade-off. Remember that for the noise-free case with full information, the optimal setpoint for $c = Hy = u - Kx = 0$.

B. Change from infinite to finite horizon problem with cross-term

Assume we have the following objective

$$J(u,x) = \sum_{i=0}^{\infty}\left[x_i^{\mathrm{T}}Qx_i + u_i^{\mathrm{T}}Ru_i + 2x_i^{\mathrm{T}}Nu_i\right]. \tag{55}$$

This infinite horizon optimization problem can be changed to finite horizon by assuming $u_k = -K_{\text{lqr}}x_k$ for $k \ge N$. Then, for $k \ge N$ $x_{N+i} = (A - BK_{\text{lqr}})^i x_N$ and $u_{N+i} = -K_{\text{lqr}}(A - BK_{\text{lqr}})^i x_N$. This implies that

$$\begin{aligned}&\sum_{i=N}^{\infty}\left[x_i^{\mathrm{T}}Qx_i + u_i^{\mathrm{T}}Ru_i + 2x_i^{\mathrm{T}}Nu_i\right] = \\ &= x_N^{\mathrm{T}}\{\sum_{i=0}^{\infty}(A - BK_{\text{lqr}})^{i\mathrm{T}}(Q + K_{\text{lqr}}^{\mathrm{T}}RK_{\text{lqr}} \\ &- NK_{\text{lqr}})(A - BK_{\text{lqr}})\}x_N\end{aligned} \tag{56}$$

Now consider $P = \sum_{i=0}^{\infty}X^{i\mathrm{T}}WX^i = W + X^{\mathrm{T}}WX + X^{2\mathrm{T}}WX^2 + \cdots = W + X^{\mathrm{T}}\left(W + X^{\mathrm{T}}WX + X^{2\mathrm{T}}WX^2 + \ldots\right)X = W + X^{\mathrm{T}}PX$. Let $X = (A - BK_{\text{lqr}})$ and $W = Q + K_{\text{lqr}}^{\mathrm{T}}RK_{\text{lqr}} - NK_{\text{lqr}}$ and we get that

$$\begin{aligned}J(u,x) &= \sum_{i=0}^{\infty}\left[x_i^{\mathrm{T}}Qx_i + u_i^{\mathrm{T}}Ru_i + 2x_i^{\mathrm{T}}Nu_i\right] \\ &\le \sum_{i=0}^{N-1}\left[x_i^{\mathrm{T}}Qx_i + u_i^{\mathrm{T}}Ru_i + 2x_i^{\mathrm{T}}Nu_i\right] + \\ &+ x_N^{\mathrm{T}}Px_N,\end{aligned} \tag{57}$$

where P is the solution to the discrete Lyapunov equation

$$P = Q + K_{\text{lqr}}^{\mathrm{T}}RK_{\text{lqr}} - NK_{\text{lqr}}(A - BK_{\text{lqr}})^{\mathrm{T}}P(A - BK_{\text{lqr}}). \tag{58}$$

We have equality in (57) if, for the original problem, the solution is $u_k = -K_{\text{lqr}}x_k$ for $k \ge N$.