Explicit MPC with output feedback using self-optimizing control

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Abstract: Model predictive control (MPC) is a favored method for handling constrained linear control problems. Normally, the MPC optimization problem is solved on-line, but in 'explicit MPC' an explicit precomputed feedback law is used for each region of active constraints (Bemporad et al., 2002). In this paper we make a link between this and the 'self-optimizing control' idea of finding simple policies for implementing optimal operation. The 'nullspace' method (Alstad and Skogestad, 2007) generates optimal variable combinations, $c = u - Kx$, which for the case with perfect state measurements are equivalent to the explicit MPC feedback laws, where K is the optimal state feedback matrix in a given region. More importantly, this link makes it possible to derive explicit feedback laws for cases with (1) state measurement error included and (2) measurement (rather than state) feedback. We further show how to generate optimal low-order controllers for unconstrained optimal control, also in the presence of noise.

Keywords: Explicit model predictive control; Self-optimizing control; Invariants

1. INTRODUCTION

Consider the general static optimization problem (Alstad and Skogestad, 2007):

$$\begin{aligned} \min_{u_0,x} \quad & J_0(x, u_0, d) \\ \text{s.t.} \quad & f_i(x, u_0, d) = 0, \quad i \in \mathcal{E} \\ & h_i(x, u_0, d) \geq 0, \quad i \in \mathcal{I}, \end{aligned} \tag{P1}$$

where $x \in \mathbb{R}^{n_x}$ are the states, $u_0 \in \mathbb{R}^{n_{u_0}}$ are the inputs, and $d \in \mathcal{D} \subset \mathbb{R}^{n_d}$ are disturbances. By discretization and reformulation this may also represent some dynamic optimization problems. Usually f is a model of the physical system, whilst h is a set of inequality constraints that limits the operation (e.g., physical limits on temperature measurements or flow constraints). In addition to (P1) we have measurements on the form

$$y_0 = f^y(x, u_0, d). \tag{1}$$

In this work the emphasis is on *implementation of the solution to* (P1). This means that the optimization problem (P1) is solved off-line to generate a 'control policy' which is suitable for on-line implementation, with particular emphasis on remaining close to optimal solution when there are unknown disturbances.

In our previous work on 'self-optimizing control' we have looked for simple control policies to implement optimal operation, and in particular 'what should we control' (choice of controlled variables (CV's)). Using off-line optimization we may determine regions where different sets of active constraints are active, and implementation of optimal operation is then in each region to:

(1) Control the active constraints.
(2) For the remaining unconstrained degrees of freedom: Control 'self-optimizing' variables $c = Hy$ which have the property that keeping them constant ($c = c_s$) indirectly achieves close-to optimal operation (with a small loss), in spite of disturbances d.

A key result, which is the basis for this paper, is

For a quadratic optimization problem there exists (infinitely many) linear measurement combinations $c = Hy$ that are optimally invariant to disturbances d.

One sees immediately that there may be some link to explicit MPC, because the discrete form MPC problem can be written as a static quadratic problem. The link is: If we let y contain the inputs u and the states x, then the 'self-optimizing' variable combination $c = Hy$ is the same as the explicit MPC feedback law, i.e. $c = u - Kx$. (This is shown in section 3.)

Based on this, we provide in this contribution some *new* ideas on explicit MPC:

(1) We propose that tracking the variables c (deviation from optimal feedback law) for all regions, may be used as a local method to detect when to switch between regions. (This is discussed in Manum et al. (2008b).)
(2) We extend the results to output feedback ($c = u - Ky$) by including in y present and past outputs.
(3) For unconstrained optimal control, we show how the links can be used to give low-order controllers that give a small loss from optimality also for noisy measurements.
(4) We also extend the results to the case where only a subset of the states are measured (but in this case

there will be a loss, which we can quantify). This may be of interest even in the unconstrained LQ case.

2. RESULTS FROM SELF-OPTIMIZING CONTROL

In this section we will present results from previous work on self-optimizing control and relate them to quadratic optimization problems.

2.1 Steady state conditions

Once the set of active constraints is known, we can form the reduced problem and the unconstrained degrees of freedom u can be determined. The unconstrained measurements are

$$y = G^y u + G^y_d d, \tag{2}$$

and y contain information about the present state and disturbances (y may include u_0 and d, but not the active constraints.) The (measured) value of y_m available for implementation is

$$y_m = y + n^y, \tag{3}$$

where n^y represents uncertainty in the measurement of y including uncertainty of implementation in u.

The following theorem describes a method to find linear invariants that yields zero loss from optimality when the invariants are controlled at constant setpoint. The theorem is based on the 'nullspace method' presented in Alstad and Skogestad (2007).

Theorem 1. (Linear invariants for quadratic optimization problem (Alstad et al., 2008)) Consider an unconstrained quadratic optimization problem in the variables u (input vector of length n_u) and d (disturbance vector of length n_d)

$$\min_u J(u,d) = [u \;\; d] \begin{bmatrix} J_{uu} & J_{ud} \\ J_{ud}{}^{\mathrm{T}} & J_{dd} \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix} \tag{4}$$

In addition, there are 'measurement variables' $y = G^y u + G^y_d d$.

If there exists $n_y \geq n_u + n_d$ independent measurements (where 'independent' means that the matrix $\tilde{G}^y = \left[G^y \; G^y_d\right]$ has full rank), then the optimal solution to (4) has the property that there exists $n_c = n_u$ linear variable combinations (constraints) $c = Hy$ that are invariant to the disturbances d. The optimal measurement combination matrix H is found by either: (1): Let $F = \frac{\partial y^{\mathrm{opt}}}{\partial d^{\mathrm{T}}}$ be the optimal sensitivity matrix evaluated with constant active constraints. Under the assumptions stated above possible to select the matrix H in the left nullspace of F, $H \in \mathcal{N}(F^{\mathrm{T}})$, such that

$$HF = 0 \tag{5}$$

(2): If $n_y = n_u + n_d$:

$$H = M_n^{-1} \tilde{J} (\tilde{G}^y)^{-1}, \tag{6}$$

where $\tilde{J} = \left[J_{uu}^{1/2} \;\; J_{uu}^{-1/2} J_{ud}\right]$ and $\tilde{G}^y = \left[G^y \; G^y_d\right]$ is the augmented plant. M_n^{-1} may be seen as a free parameter. (Note that $M_n = J_{cc}$ is the Hessian of the cost with respect to the c-variables; in most cases we select $M_n = I$ for convenience.)

Remark 2. The sensitivity F matrix can be obtained from

$$F = -\left(G^y J_{uu}^{-1} J_{ud} - G^y_d\right). \tag{7}$$

Theorem 1 may be extended:

Lemma 3. (Linear invariants for constrained quadratic optimization methods (Manum et al., 2008b)) Consider an optimization problem of the form

$$\begin{aligned} \min_{u_0, x} J_0 = [x \;\; u_0 \;\; d] \, S \begin{bmatrix} x \\ u_0 \\ d \end{bmatrix} \\ \text{s.t. } Ax + Bu + Cd = 0 \\ \tilde{A}x + \tilde{B}u + \tilde{C}d \leq 0, \end{aligned} \tag{8}$$

with $\det(A) \neq 0$ and $[\tilde{A} \; \tilde{B}]$ full row rank.

Assume that the disturbance space has been partitioned into n_a critical regions. In each region there are $n_u^i = n_{u_0} - n_A^i \geq 0$ unconstrained degrees of freedom, where $n_A^i \leq n_m$ is the number of optimally active constraints in region i.

If there exists a set of independent unconstrained measurements $y^i = (G^y)^i u^i + (G^y_d)^i d$ in each region i, such that $n_{y^i} \geq n_{u^i} + n_d$, the optimal solution to (8) has the property that there exists variable combinations $c^i = H^i y^i$ that for critical region i are invariant to the disturbances d. The corresponding optimal H^i may be obtained from Theorem 1. Within each region, optimality requires that $c^i - c^i_s = 0$ (where c^i_s is a constant). From continuity of the solution we have that c^i is continuous across the boundary of region i. This implies that the elements in the variable vector $c^i - c^i_s$ will change sign or remain zero when crossing into or from a neighboring region.

2.2 Including noise

For the noise-free problem, adding the constraints $c = Hy = c_s$ does not change the optimal solution (Theorem 1). However with measurement noise there will be some loss, which can be minimized if H is selected as given in Theorem 4.

Theorem 4. (Loss by introducing linear constraint for noisy quadratic optimization problem (Alstad et al., 2008)) Consider the unconstrained quadratic optimization problem in Theorem 1:

$$\min_u J(u,d) = [u \;\; d] \begin{bmatrix} J_{uu} & J_{ud} \\ J_{ud}{}^{\mathrm{T}} & J_{dd} \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix}$$

and a set of noisy measurements $y_m = y + n^y$. Assume that $n_c = n_u$ constraints $c = Hy_m = c_s$ are added to the problem, which will result in a non-optimal solution with loss $L = J(u,d) - J_{\mathrm{opt}}(d)$. Consider the disturbances d and the noise n^y with magnitudes:

$$d = W_d d'; \quad n^y = W_{n^y} n^{y'}; \quad \| \begin{bmatrix} d' \\ n^{y'} \end{bmatrix} \| \leq 1. \tag{9}$$

Then, for a given H, the worst-case loss is $L_{\mathrm{wc}} = \bar{\sigma}(M)^2/2$, where $M = [M_d \;\; M_{n^y}]$ is given by

$$M_d = -J_{uu}^{1/2} (HG^y)^{-1} HFW_d, \tag{10}$$

$$M_{n^y} = -J_{uu}^{1/2} (HG^y) HW_{n^y}., \tag{11}$$

and the optimal H that minimizes $\bar{\sigma}(M)$ is given by

$$H^{\mathrm{T}} = \left(\tilde{F}\tilde{F}^{\mathrm{T}}\right)^{-1} G^y (G^{y\mathrm{T}} (\tilde{F}\tilde{F}^{\mathrm{T}})^{-1} G^y)^{-1} J_{uu}^{1/2}, \tag{12}$$

where $\tilde{F} = [FW_d \;\; W_{n^y}]$. This solution also minimizes the average loss $\|M\|_F$.

Remark 5. The optimal H can also be found by solving the constrained optimization problem

$$H = \arg\min_{H} \bar{\sigma}(H\tilde{F}) \text{ subject to } HG^y = J_{uu}^{1/2} \tag{13}$$

3. APPLICATION TO EXPLICIT MPC

Pistikopoulos et al. (2002) show that by substitution of the model equations, the linear MPC problem can be rewritten to the form

$$\begin{aligned} &\min_U \frac{1}{2}U^{\mathrm{T}}HU + x(t)^{\mathrm{T}}FU + \frac{1}{2}x(t)^{\mathrm{T}}Yx(t) \\ &\text{s.t. } GU \leq W + Ex(t) \end{aligned} \tag{14}$$

The MPC control law is based on the following idea: At time t, compute the optimal solution $U^*(t) = \{u_t^*, \ldots, u_{t+N_u-1}^*\}$ and apply $u(t) = u_t^*$ (Bemporad et al., 2002).

If we let the initial state $x(t)$ be treated as a disturbance, (14) can be written as:

$$\begin{aligned} &\min_U \frac{1}{2}\begin{bmatrix}U^{\mathrm{T}} & d^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}H & F\\ F & Y\end{bmatrix}\begin{bmatrix}U\\ d\end{bmatrix} \\ &\text{s.t. } GU \leq W + Ed, \end{aligned} \tag{15}$$

and we observe that (15) is on the same form as (8), where the model equations $f(x, u_0, d) = 0$ have already been substituted into the objective function.

A property of the solution to (15) is that the disturbance space (initial state space) will be divided into critical regions. In the i'th critical region there will be $n_u^i = n_U - n_A^i$ unconstrained degrees of freedom, where n_A^i is the number of active constraints in region i.

As we will discuss in section 3.1, a possible set of measurements y is the current state and the inputs, $y^{\mathrm{T}} = \begin{bmatrix}x^{\mathrm{T}} & u^{\mathrm{T}}\end{bmatrix}$. We further note that causality is not an issue here, as we have the information at the current time.

3.1 Exact measurements of all states (state feedback)

The following theorem is well known, but we shown in (Manum et al., 2008b) that it can be derived using the nullspace method. The proof is left out here due to space limitations.

Theorem 6. (Optimal state feedback (Bemporad et al., 2002)) The control law $u(t) = f(x(t))$, $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, defined by the MPC problem, is continuous and piecewise affine

$$f(x) = K^i x + g^i \quad \text{if } H^i x \leq k^i, \quad i = 1, \ldots, N_{\mathrm{mpc}} \tag{16}$$

where the polyhedral sets $\{H^i x \leq k^i\}$, $i = 1, \ldots, N_{\mathrm{mpc}} \leq N_r$ are a partition of the given set of states X.

Remark 7. (Comparison with previous results on unconstrained MPC) In the proof shown in Manum et al. (2008b) the state feedback gain matrix is given as $J_{uu}^{-1}J_{ud}$. This is gives the same result as conventional MPC, see equation (3) in Rawlings and Muske (1993).

Remark 8. Our alternative proof of Theorem 6 leads to some new insights. The most important is probably that the 'self-optimizing' variables $c^i = u - (K^i x + g^i)$ which are optimally zero in region i, may be used for identifying when to switch between regions (Theorem 9) rather than using a 'centralized' approach, for example based on a state tree structure search. This seems to be new. Another insight is to understand why a simple feedback solution must exist in the first place. A third is to allow for new extensions.

Theorem 9. (Optimal region for explicit MPC detection using feedback law (Manum et al., 2008b)) The variables $c = u_k - (Kx_k + g)$ can be used to identify region changes.

An algorithm for implementing the region detection scheme is presented in Manum et al. (2008b).

We present a simple example from Bemporad et al. (2002) that confirms that our switching policy based on tracking the sign of the c-variables works in practice.

Example 3.1. (Optimal switching). This example is taken from Bemporad et al. (2002) (with correction), and is included here to demonstrate optimal switching using $c = u - Kx$ as criterion. For more details on this example see (Manum et al., 2008b). The system is:

$$y(t) = \frac{2}{s^2 + 3s + 2}u(t).$$

With a sampling time $T = 0.1$ seconds the following state-space representation is obtained:

$$\begin{aligned} x(t+1) &= \begin{bmatrix}0.7326 & -0.0861\\ 0.1722 & 0.9909\end{bmatrix}x(t) + \begin{bmatrix}0.0609\\ 0.0064\end{bmatrix}u(t) \\ y(t) &= \begin{bmatrix}0 & 1.4142\end{bmatrix}x(t) \end{aligned}$$

One observes that only the last state is measured, but it will be assumed that both states are known (measured) in the remainder of this example.

The task is to regulate the system to the origin while fulfilling the input constraint

$$-2 \leq u(t) \leq 2. \tag{17}$$

The objective function to be minimized is

$$\min x_{t+2|t}{}^{\mathrm{T}}Px_{t+2|t} + \sum_{k=0}^{1}\left[x_{t+k|t}{}^{\mathrm{T}}x_{t+k|t} + 0.01u_{t+k}^2\right] \tag{18}$$

subject to the constraints and $x_{t|t} = x(t)$.

P solves the Lyapunov equation $P = A^{\mathrm{T}}PA + Q$, where $Q = I$ in this case. The P-matrix is numerically $P = \left[\begin{smallmatrix}5.5461 & 4.9873\\ 4.9873 & 10.4940\end{smallmatrix}\right]$. The optimal control problem can be solved for example using the MPT toolbox (Kvasnica et al., 2004).

To illustrate the ideas, we show a simulation where the control objective is to bring the process from $x_0 = (1, 1)$ and back to $x = (0, 0)$. State space trajectories and inputs are shown in figures 1 and 2 (dotted line). As long as the state is in the input-constrained region where $u^{\mathrm{opt}} = -2$, the linear combination $c = u_k - Kx_k$ remains positive. One chooses to leave the input-constrained region when c becomes zero. The state trajectory is the same as in Bemporad et al. (2002).

3.2 Output feedback with no noise

Consider now the case where all the states x are not measured. The objective is to find linear combinations

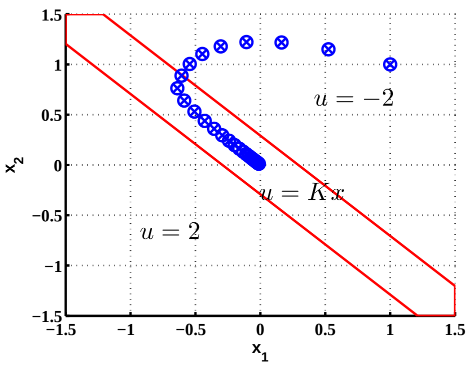

Fig. 1. Partition of state space for first input. (Example 3.1.)

$c = Hy$ that are optimally constant in each optimal region. From the nullspace method, this requires that we have as many independent measurements y as there are inputs and disturbances.

With no measurement error, the optimal combination $c = Hy$ can be obtained from the nullspace method. This requires that $\tilde{G}_y$ has full rank, which again implies that all d's can be observed from the outputs y. Because of causality, $\tilde{G}_y$ will not be full rank initially (just after the disturbance occurs), but the rank condition will be satisfied if we consider a disturbance entering sufficiently long ($n_x - 1$ steps) back in time. From this time and on the solution is the same as the state feedback solution.

In terms of detecting region changes, we suggested for the state feedback case to use the deviation c from the optimal feedback laws $c = u - Kx$ as tracking variables. This simple strategy may not work as well with output feedback, partly because output feedback is not truly optimal, and partly because the outputs do not contain accurate information about the present state. (It can however be applied in the following example.)

Example 3.2. (Output feedback). Consider the same model and optimal control problem as in example 3.1, but assume that only the output $y(t)$ is available (and not both states). Recall from figure 1 that the state space is optimally partitioned into 3 regions with 3 different state feedback laws. As before, let $d = x_k$.

One approach is to find the optimal sensitivity F from $F = -(G^y J_{uu}^{-1} J_{ud} - G_d^y)$, where $y = (y_k, y_{k+1}, U)$, and

$$y = \begin{bmatrix} y_k \\ y_{k+1} \\ u_k \\ u_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ CB & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{G^y} \begin{bmatrix} u_k \\ u_{k+1} \end{bmatrix} + \underbrace{\begin{bmatrix} C \\ CA \end{bmatrix}}_{G_d^y} x_k \tag{19}$$

By finding an H such that $HF = 0$, this method yields feedback gains from the outputs to the inputs. Note that we can always 'decouple' the invariants in the inputs u when all inputs are included in the candidate vector y. This is because $n_c = n_u$ and we have a degree of freedom in H such that multiplying by a non-singular $n_c \times n_c$ matrix on the left yields the same loss as before. Write $H = [H^y \; H^u]$, then a combination matrix that is decoupled in u is $\hat{H} = (H^u)^{-1} H$.

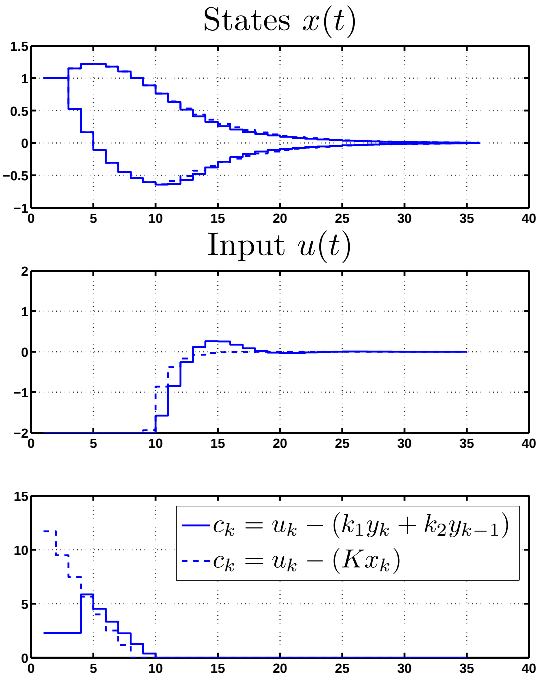

Fig. 2. Simulation of output feedback configuration, where the output feedback law is used both for switching and control. Dotted line is optimal switching and control when both states assumed measured. (Examples 3.1 and 3.2.)

We here get two invariants, one between (u_{k+1}, y_{k+1}, y_k), and one between u_k, y_{k+1}, y_k, where only the first one is implementable because of causality.

The controller gains for the central region are $(k_1, k_2) = (-16.7, 13.7)$, with control equation $u_k = -(k_1 y_k + k_2 y_{k-1})$.

Another approach for finding F and H is to use the optimal solution $u_k = -Kx_k$ *a priori*, which we did not do above. If we use the knowledge of the optimal feedback law, we can for example find that

$$F'^{\mathrm{T}} = \left(\frac{\partial [u_k \; y_k \; y_{k+1}]^{\mathrm{T}}}{\partial x_k} \right)^{\mathrm{opt}} = \left[K^{\mathrm{T}} \; C^{\mathrm{T}} \; (C(A + BK))^{\mathrm{T}} \right], \tag{20}$$

and by solving $H'F' = 0$ we get an invariant between (u_k, y_k, y_{k+1}). This invariant is not implementable, but by using the same idea we can find another invariant between (y_k, y_{k+1}, y_{k+2}), shift this invariant one time-step back and then combine with the first one. The resulting output feedback law becomes the same as for the method above, where we did not use the optimal state feedback law in the derivations.

Figure 2 shows the result of a simulation of the output feedback control from $x_0 = (1, 1)$. Note that we use the output feedback control law for the unconstrained region to decide when to leave the constrained region. The previous optimal control with both states assumed measured is shown as the dotted line. One observers that the optimal control scheme leaves the constrained region one time instant before the output feedback scheme. This is expected from the discussion above. The 'discontinuity' in c_k at sample 1 is due to initialisation issues.

4. LOW-ORDER CONTROLLERS FOR UNCONSTRAINED OPTIMAL CONTROL IN THE PRESENCE OF NOISE

In example 3.2 we used Theorem 3 to derive an output feedback law, and moreover this feedback law could be used for region detection. We will now focus on unconstrained optimization problems and show, by using Theorem 4, how we can find optimal invariants between *noisy* measurements y. To achieve this we use the weights W_d and W_{n^y}, see (9). The approach we use is summarized in algorithm 1. The algorithm can easily be extended to cover a non-stable system matrix A by not setting $u_k = 0$ for $k \geq N$.

Algorithm 1 Finding low-order controllers. (For stable system matrix A)

1: Define cost function $\hat{J}(u,x) = \sum_{k=0}^{\infty} x_k{}^{\mathrm{T}} Q x_k + u_k{}^{\mathrm{T}} R u_k$.
2: Choose N_y, where $u_k = 0,\ k \geq N_y$.
3: Rewrite cost function to $J(u,x) = \sum_{k=0}^{N_y-1}\left(x_k{}^{\mathrm{T}} Q x_k + u_k{}^{\mathrm{T}} R u_k\right) + x_{N_y} P x_{N_y}$, where $P = A^{\mathrm{T}} P A + Q$.
4: Treat x_0 as a disturbance d and find J_{uu} and J_{ud}.
5: Decide candidate variables y, for example $y = (y_k, u_{k+1}, u_k, \ldots, u_{k+N-1})$ and form the "open-loop" model $y = G^y U + G_d^y d$, $U = (u_k, \ldots, u_{k+N-1})$.
6: Decide disturbance weight W_d and noise weight W_{n^y}.
7: Find $\tilde{H}$ by solving either the optimization problem (13) or use (12).
8: Decouple the inputs in $\tilde{H} = [\tilde{H}_x\ \tilde{H}_u]$ by setting $H = \tilde{H}_u^{-1}[\tilde{H}_x\ \tilde{H}_u] = [\tilde{H}_u^{-1}\tilde{H}_x\ I]$.

In step 2 one has to choose the input horizon N_y. In practical applications we found that this should be set rather high to give good performance in the low-order controllers. In the following example, where we focus on the central region (unconstrained) for example 3.2, we had to increase N_y from 1 (as it was in example 3.2) to 10 to get acceptable performance. To reduce complexity in constrained explicit MPC one can decrease the input horizon to get less number of regions, but the resulting controller will have a poorer performance, so obviously there is a trade-off. In future work we will investigate the possibility of using different input horizons for solving the parametric program and for deriving the controllers in each region.

The disturbance and noise weights W_d, W_{n^y} in step 6 should contain information about the expected variation in disturbances versus measurement noise.

In the following example we compare different low-order controllers found by using algorithm 1 with optimal LQG controllers for the central region of example 3.2 using output feedback.

Example 4.1. (Low-order controllers and comparison with LQG: output feedback) In this example we investigate the same process as before, but with noisy measurements, i.e.

$$x_{k+1} = \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix} x_k + \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix} u_k + w_k$$
$$y_k = [0\ \ 1.4142]\, x_k + v_k, \tag{21}$$

where the process noise w_k are two uniformly distributed random numbers drawn from a uniform distribution on a $[-\beta, \beta]$ interval, and the measurement noise v_k is a uniformly distributed random number drawn from a uniform distribution on a $[-\alpha, \alpha]$ interval. There is no correlation between the noises. This implies variances $\mathrm{var}(w_k) = \frac{\beta^2}{3} I$, and $\mathrm{var}(v_k) = \frac{\alpha^2}{3}$.

The objective is to find low-order controllers that can give comparable performance with the well-known LQG controller.

In this example we investigate the following controllers for controlling the noisy process:

(1) LQG from y_k to u_k.
(2) Invariant (u_k, y_k).
(3) Invariant (u_k, y_k, y_{k-1}).
(4) Invariant $(u_k, y_k, \ldots, y_{k-2})$.
(5) Invariant $(u_k, y_k, \ldots, y_{k-3})$.

Algorithm 1 can directly be applied to find invariants between inputs u_k and output $y_k, y_{k-1}, \ldots,$ also when there is noise on the measurements. Here we choose $N_y = 10$, which will give a good performance in the resulting controllers. Apart from this, the cost function is the same as in (18).

In Manum et al. (2007) analytical expressions for the derivatives J_{uu} and J_{ud} are given. These can be derived by substituting the state space model into the objective function to get an unconstrained optimization problem as a function of (U, x_k), where again we treat x_k as a disturbance.

The open-loop model follows from the model equations. For example, for $y = (y_k, y_{k+1,U})$, where $U = (u_0, \ldots, u_9)$, we establish the model:

$$y = \begin{bmatrix} y_k \\ y_{k+1} \\ U \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ CB & 0 \\ I & 0 \\ 0 & I \end{bmatrix}}_{G_y} U + \underbrace{\begin{bmatrix} C \\ CA \\ 0 \end{bmatrix}}_{G_d^y} x_k. \tag{22}$$

The disturbance weight W_d should reflect the variation in disturbances, whilst the noise weight W_{n^y} the noises on measurements and inputs. In (Manum et al., 2008a) it is shown that the resulting controllers are not affected by the noise on the inputs using the current formulation. We therefore choose:

$$W_d = \frac{\beta^2}{3} I_{2\times 2} \qquad W_{n^y} = \begin{bmatrix} \frac{\alpha^2}{3} I_{\tilde{n}_y \times \tilde{n}_y} & 0 \\ 0 & I_{N_y \times N_y} \end{bmatrix}, \tag{23}$$

where $\tilde{n}_y \leq N_y$ is the number of measurements we want to include in the implementation (i.e. the order of the resulting controller).

This framework was used to generate the controllers shown in Table 1. The LQR and LQG controllers were designed using standard software, and the tuning was based on the known distributions of the process and measurement noises.

Table 1. Simulated costs for example 4.1. Noise levels for w_k, v_k: $(\alpha, \beta) = (0.8, 1)$.

Number	Control equation	J_1	J_2
0	$u_k = -[6.08\ 6.07]x_k$ (noise free, perfect measurement)	2.86	0.284
1	$u_k = -[6.08\ 6.07]\hat{x}_k$	**3.40**	0.400
2	$u_k = -(3.25y_k)$	5.27	0.569
3	$u_k = -(1.54y_k + 0.5y_{k-1})$	3.88	0.401
4	$u_k = -(0.78y_k + 0.44y_{k-1} - 0.03y_{k-2})$	3.88	**0.394**
5	$u_k = -(0.39y_k + 0.28y_{k-1} + 0.12y_{k-2} - 0.09y_{k-3})$	4.11	0.416

The reference controller is an LQR using full state information (available in Matlab).

The LQG controller (from y_k to u_k) is implemented as:

$$\begin{aligned} \hat{x}_{n+1|n} &= A\hat{x}_{n|n-1} + Bu_n + L(y_n - C\hat{x}_{n|n-1}) \\ \hat{x}_{n|n} &= \hat{x}_{n|n-1} + M(y_n - C\hat{x}_{n|n-1}) \\ u_k &= -K\hat{x}_{n|n} \end{aligned} \tag{24}$$

with $L^{\mathrm{T}} = [0.04\ 0.59]$ and $M^{\mathrm{T}} = [0.12\ 0.57]$.

The simulated costs for the different controllers are shown in Table 1. We investigate two cases, one where the process noise (i.e. disturbances) occurs at all time instants (J_1) and one where the process noise occurs only every tenth instant (J_2). The simulated costs are the values of the objective function divided by the simulation length.

When the process noise is occurring at all time instants (see J_1), the LQG controller is optimal. The best variable combination between the present input and the outputs back in time, controller no. 4, has a simulated cost 13% higher than the LQG controller. However, if the process noise occurs only every tenth time instant (see J_2), a simple combination between y_k, y_{k-1}, y_{k-2} actually yields slightly better performance than the LQG controller.

As we increase the order of the controller we will reduce the noise sensitivity but we will be more sensitive to startup problems. The control law using $y_k, \ldots, y_{k-3}$ is only optimal 3 time-steps after the disturbance occurs, this is the reason why it has a higher simulated cost than controller number 4.

This example shows that our approach for deriving low-order controllers has some inherent problems regarding causality; to achieve optimal operation in the noise-free case we need at least $n_y = n_u + n_d$ measurements, and in the presence of noise we should include even more to reduce the sensitivity of noise. However, increasing the number of y's in the control law makes the causality problem more significant as we need to 'wait' until the rank conditions from the disturbance to the measurements becomes fulfilled.

The example further shows that the method works, and we get controllers comparable with the LQG controller. For disturbances occurring at every time instant the LQG controller will be optimal at all times. However, in most practical cases we do not expect that the disturbances will change in a random manner from one time step to the next, so the assumption of d_k changing for example only every tenth time step may not be too wrong. Further, if we are allowed to change the sample time we can always increase it to be faster than the dynamics of the disturbances and our method can be applied.

5. DISCUSSION AND EXTENSIONS

In this paper we have discussed that feedback laws may be viewed as additional constraints (invariants) to the original optimization problem, and based on this, we have shown that optimal linear feedback laws can be derived for quadratic optimization problems.

Further, we have presented a mathematical framework, Theorem 4 that gives optimal invariants of *noisy* measurements. This theorem can also be used in the case of too few measurements, which can be of interest even for the unconstrained LQ case.

Currently we are working on how to determine changes in the active set for noisy measurements and how to optimally include integral action in the low-order controllers.

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