# Relative Gain Array for Norm-Bounded Uncertain Systems

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This paper considers the extension of relative gain array (RGA) to norm-bounded uncertain systems. We present a method for calculating a tight bound on the worst-case relative gain and derive necessary and sufficient conditions for the sign change of the relative gain over the uncertainty set. The proposed results improve on recently published results [Chen and Seborg, *AIChE J.* **2002**, *48*, 302]. More importantly, it is shown that the role of RGA is limited for ascertaining the integrity of uncertain systems. This conclusion is in direct contrast with the corresponding result for adjudging integrity of nominal systems, where the usefulness of RGA is well-known. As an offshoot, we present a signal-based representation of the relative gain for uncertain systems.

## 1. Introduction

The Relative Gain Array (RGA)[1] is a well-established tool for the analysis and design of control systems. RGA has many useful algebraic properties, which also have strong control implications (see Skogestad and Postlethwaite[2] for an overview of properties and usefulness of RGA). For example, one property of RGA is that, under minor assumptions, the system has integrity if and only if the pairings are selected such that relative gains of all the principal submatrices of the permuted steady-state gain matrix are positive.[3] The system is said to possess integrity[4] if there exists a diagonal controller with integral action in every output channel, such that closed-loop stability is maintained in the presence of possible controller failures. This paper considers extension of RGA to norm-bounded uncertain systems. This problem is important in practice, because models of real systems always have some uncertainty associated with them. Many results based on RGA hold well for the nominal model of the system but can be difficult to apply to uncertain systems (e.g., verifying the condition for integrity[3] for every member of the set of models describing the uncertain system is computationally intractable in general).

RGA for uncertain systems previously has been considered under the restrictive assumption of element-wise uncertainty or changes in only one element of the gain matrix.[5,6] The more general case, where all the elements of gain matrix are allowed to change simultaneously, is considered by Grosdidier et al.,[7] Skogestad and Morari,[5] and also recently in greater detail by Chen and Seborg.[8] For systems with simultaneous additive perturbations in all of its elements, Chen and Seborg[8] have presented lower and upper bounds on the relative gain. Large relative gains calculated for the nominal model imply strong directionality in the system and potential control problems.[5] Then, the results of Chen and Seborg[8] are useful for analyzing the directionality of the uncertain system. These bounds can also be used to determine an upper bound on the allowable perturbations before the relative gains become negative or the system loses integrity; however, this bound can be loose in general. This may happen because, even when the lower bound on the relative gain is negative, the actual value may remain positive over the given uncertainty set.

The purpose of this paper is to use a more rigorous approach to improve on the results of Chen and Seborg.[8] We present a method for obtaining a bound on the magnitude of the worst-case relative gain, calculated at steady state and also at higher frequencies. Compared to the results of Chen and Seborg,[8] the bound is tight in the sense that there exists an uncertain plant that achieves this bound. We derive the necessary and sufficient conditions for the sign change of the relative gain of norm-bounded uncertain systems. More importantly, we show that the relative gain changes sign, only if the gain of the uncertain system or one of its principal submatrices corresponding to the relative gain becomes singular over the uncertainty set. This result implies that the role of RGA is limited for ascertaining the integrity of uncertain systems, as the nonsingularity of these matrices is trivially necessary for integrity of the uncertain system. This conclusion is in direct contrast with the corresponding result for adjudging the integrity of nominal systems, where the usefulness of RGA is well-known. The discussion is limited to systems with additive norm-bounded perturbations. The results can be easily generalized, however, to systems described by other norm-bounded uncertainty descriptions (e.g., multiplicative uncertainty) and also to systems with multiple sources of perturbations. As an offshoot, we present a signal-based representation of the relative gain for uncertain systems.

## 2. Preliminaries

In this section, we standardize the notation, collect some useful matrix identities, and present a signal-based interpretation of relative gain. The latter result is vital for derivation of the more-important results later in the paper and may be of independent interest to the reader.

**2.1. Notation.** For a given matrix $\mathbf{A} \in \mathcal{R}^{m\times n}$, $\mathbf{A}_{ij}$ and $\mathbf{A}^{ij}$ denote the *ij*th element (or block) and the submatrix of $\mathbf{A}$ with the *i*th row and *j*th column removed, respectively. $\mathbf{A}_{i*}$ and $\mathbf{A}_{*i}$ denote the *i*th row and *i*th column of $\mathbf{A}$, respectively. A matrix made of elements $a_{11} \ldots a_{1n} \ldots a_{mn}$ is represented as $[a_{ij}]$. Let $\mathbf{\Delta} = \{\mathrm{diag}(\mathbf{\Delta}_i)\}$ denote a set of complex matrices with a given block-diagonal structure, where some of the blocks may

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be repeated and may be restricted to be a real matrix. The structured singular value of $\mathbf{A}$ is given as[9]

$$\mu_{\mathbf{\Delta}}(\mathbf{A}) = (\min \bar{\sigma}(\mathbf{\Delta}):\ \det(\mathbf{I} - \mathbf{A}\mathbf{\Delta}) = 0)^{-1} \tag{1}$$

unless no allowable $\mathbf{\Delta}$ makes $(\mathbf{I} - \mathbf{A}\mathbf{\Delta})$ singular, in which case $\mu_{\mathbf{\Delta}}(\mathbf{A}) = 0$. For a partitioned matrix, $\mathcal{F}_u(\mathbf{A},\mathbf{\Delta})$ denotes the upper linear fractional transform (upper-LFT), which is defined as

$$\mathcal{F}_u(\mathbf{A},\mathbf{\Delta}) = \mathbf{A}_{22} + \mathbf{A}_{21}\mathbf{\Delta}(\mathbf{I} - \mathbf{A}_{11}\mathbf{\Delta})^{-1}\mathbf{A}_{12} \tag{2}$$

In this paper, we use the following identities, which are related to the determinant of matrices, frequently:[10]

(1) For $\mathbf{A} \in \mathcal{R}^{m\times n}$ and $\mathbf{B} \in \mathcal{R}^{n\times m}$, $\det(\mathbf{I} + \mathbf{AB}) = \det(\mathbf{I} + \mathbf{BA})$.

(2) For the partitioned matrix $\mathbf{A}$, with $\mathbf{A}_{22}$ square and nonsingular,

$$\det\left(\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}\right) = \det(\mathbf{A}_{22})\cdot\det(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})$$

The latter identity is also called the Schur complement Lemma.

We denote the nominal model of the rational, stable, linear time-invariant and square system as $\mathbf{G}(s)$ and its steady-state gain matrix as $\mathbf{G} \in \mathcal{R}^{n\times n}$. The set of steady-state gain matrices of the perturbed plants with additive norm-bounded uncertainty is represented as

$$\Pi_{\mathrm{A}}: \qquad \mathbf{G}_p = \mathbf{G} + \mathbf{W}\mathbf{\Delta}\mathbf{V} \qquad (\text{for } \bar{\sigma}(\mathbf{\Delta}) \le 1) \tag{3}$$

The uncertainty set at higher frequencies is defined similarly. For nonsingular $\mathbf{G}$, the steady-state relative gain between the $i$th output and $j$th input is defined as[1]

$$\lambda_{ij}(\mathbf{G}) = \mathbf{G}_{ij}[\mathbf{G}^{-1}]_{ji} \tag{4}$$

which represents the ratio of the open-loop gain and the apparent closed-loop gain, when all other loops are closed using controllers with integral action. The relative gain for uncertain systems can be defined similar to eq 4, where

$$\lambda_{ij}(\mathbf{G}_p) = [\mathbf{G}_p]_{ij}[\mathbf{G}_p^{-1}]_{ji} \tag{5}$$

for every $\mathbf{G}_p \in \Pi_A$. Here, $\mathbf{G}_p^{-1}$ is assumed to exist for all $\mathbf{G}_p \in \Pi_{\mathrm{A}}$ (see Remark 1). We use the notation $\lambda_{ij}(\mathbf{G}_p)$ to explicitly show the element of $\Pi_{\mathrm{A}}$, for which the relative gain is calculated. Steady-state RGA, which contains relative gains for all input−output pairs, is denoted as $\Lambda(\mathbf{G}_p) = [\lambda_{ij}(\mathbf{G}_p)]$.

*Remark 1.* For $\lambda_{ij}(\mathbf{G}_p)$ in eq 5 to be well-defined, it is necessary that $\mathbf{G}_p^{-1}$ exists or $\mathbf{G}_p$ is nonsingular over the set $\Pi_{\mathrm{A}}$. The assumption of existence of $\mathbf{G}_p^{-1}$ is not restrictive, because, if some $\mathbf{G}_p \in \Pi_{\mathrm{A}}$ is singular, integral control is not possible, because of the presence of a hidden mode. Numerically, the nonsingularity of $\mathbf{G}_p \in \Pi_{\mathrm{A}}$ can be verified by evaluating $\mu_{\mathbf{\Delta}}(\mathbf{V}\mathbf{G}^{-1}\mathbf{W})$. When $\mu_{\mathbf{\Delta}}(\mathbf{V}\mathbf{G}^{-1}\mathbf{W}) < 1$, it follows from the definition of structured singular value in eq 1 that $\det(\mathbf{I} + \mathbf{V}\mathbf{G}^{-1}\mathbf{W}\mathbf{\Delta}) = \det(\mathbf{I} + \mathbf{G}^{-1}\mathbf{W}\mathbf{\Delta}\mathbf{V}) = \det(\mathbf{G}^{-1})\det(\mathbf{G}_p)$ is nonsingular over the uncertainty set.

**2.2. Signal-Based Interpretation of Relative Gain.** We next present a signal-based interpretation of $\lambda_{ij}(\mathbf{G}_p)$, or, more specifically, $\lambda_{ii}(\mathbf{G}_p)$. Based on eq 5, $\lambda_{ii}(\mathbf{G}_p)$ can be alternately denoted as

$$\lambda_{ii}(\mathbf{G}_p) = (\mathbf{e}_i^{\mathrm{T}}\,\mathbf{G}_p\,\mathbf{e}_i)\,(\mathbf{e}_i^{\mathrm{T}}\,\mathbf{G}_p^{-1}\,\mathbf{e}_i) \qquad \forall\ \mathbf{G}_p \in \Pi_{\mathrm{A}} \tag{6}$$

where $\mathbf{e}_i$ is the unit column vector with its $i$th entry being 1 and the remaining entries being 0 (not to be confused with the $i$th element of vector $\mathbf{e}$). On the basis of eq 3,

$$\mathbf{G}_p^{-1} = [\mathbf{G}(\mathbf{I} + \mathbf{G}^{-1}\mathbf{W}\mathbf{\Delta}\mathbf{V})]^{-1} = (\mathbf{I} + \mathbf{G}^{-1}\mathbf{W}\mathbf{\Delta}\mathbf{V})^{-1}\mathbf{G}^{-1} \tag{7}$$

Now let $y = \lambda_{ii}(\mathbf{G}_p)u$. Based on eqs 6 and 7, $\lambda_{ii}(\mathbf{G}_p)$ can be represented by the signal flow diagram shown in Figure 1. Here, we have used the fact that eq 7 represents an inverse additive uncertainty representation for $\mathbf{G}^{-1}$ (see, for example, Skogestad and Postlethwaite[2]).

*Remark 2.* In Figure 1, the term $\mathbf{G}^{-1}$ can be further expanded as $(\mathbf{I} + (\mathbf{G} - \mathbf{I}))^{-1}$, which can be represented as a negative feedback loop with $\mathbf{I}$ in the forward path and $(\mathbf{G} - \mathbf{I})$ in the feedback path. Figure 1 then does not contain any nonlinear functions of $\mathbf{G}$; however, this additional manipulation is not deemed necessary here.

## 3. Worst-Case Relative Gain

Skogestad and Morari[5] have shown that large relative gains calculated based on the nominal model of the system demonstrate fundamental control problems. Large (positive or negative) elements of RGA imply ill-conditioning (large condition number) and, thus, the presence of strong directionality in the system. The arguments of Skogestad and Morari[5] also carry over to uncertain systems, where large worst-case relative gains imply poor controllability. In the following discussion, we present a method for calculating the magnitude of worst-case relative gain. The derived result is for diagonal elements of RGA, $\lambda_{ii}(\mathbf{G}_p)$. This result, however, can be used for calculating the worst-case magnitudes of $\lambda_{ij}(\mathbf{G}_p)$ for any $i, j \le n$ by permuting $\mathbf{G}_p$ such that $[\mathbf{G}_p]_{ij}$ appears as one of the diagonal elements of the permuted matrix. This happens because permuting the gain matrix results in similar permutations in the RGA matrix.[1]

As a shorthand notation, we denote $\mathbf{E} = \mathbf{e}_i\mathbf{e}_i^{\mathrm{T}}$. By analyzing the signal flow diagram in Figure 1, the following relationships can be established:

$$[\mathbf{y}_{\mathbf{\Delta}}^{\mathrm{T}}\ \ \boldsymbol{\nu}_{\mathbf{\Delta}}^{T}\ \ y\,]^{\mathrm{T}} = \mathbf{M}[\boldsymbol{u}_{\mathbf{\Delta}}^{\mathbf{T}}\ \ \boldsymbol{\psi}_{\mathbf{\Delta}}^{\mathrm{T}}\ \ u\,]^{\mathrm{T}} \tag{8}$$

$$[\mathbf{u}_{\mathbf{\Delta}}^{\mathbf{T}}\ \ \boldsymbol{\psi}_{\mathbf{\Delta}}^{\mathbf{T}}]^{\mathbf{T}} = \bar{\mathbf{\Delta}}[\mathbf{y}_{\mathbf{\Delta}}^{\mathbf{T}}\ \ \boldsymbol{\nu}_{\mathbf{\Delta}}^{\mathbf{T}}]^{\mathrm{T}} \tag{9}$$

where

$$\mathbf{M} = \begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12}\\ \mathbf{M}_{21} & \mathbf{M}_{22}\end{bmatrix} = \left[\begin{array}{cc|c} 0 & -\mathbf{V}\mathbf{E}\mathbf{G}^{-1}\mathbf{W} & \mathbf{V}\mathbf{E}\mathbf{G}^{-1}\mathbf{e}_i\\ 0 & -\mathbf{V}\mathbf{G}^{-1}\mathbf{W} & \mathbf{V}\mathbf{G}^{-1}\mathbf{e}_i\\ \hline \mathbf{e}_i^{\mathrm{T}}\mathbf{W} & -\mathbf{e}_i^{\mathrm{T}}\,\mathbf{G}\mathbf{E}\mathbf{G}^{-1}\mathbf{W} & \mathbf{e}_i^{\mathrm{T}}\,\mathbf{G}\mathbf{E}\mathbf{G}^{-1}\,\mathbf{e}_i\end{array}\right] \tag{10}$$

and

$$\bar{\mathbf{\Delta}} = \begin{bmatrix}\mathbf{\Delta} & 0\\ 0 & \mathbf{\Delta}\end{bmatrix} \tag{11}$$

At this stage, we can combine eqs 8 and 9 to express the relationship between $u$ and $y$ in the form of an upper-LFT, as shown in Figure 2. This representation is commonly used for analysis of norm-bounded uncertain systems (see, e.g., Skogestad and Postlethwaite[2]). Based on Figure 2, we have

$$\lambda_{ii}(\mathbf{G}_p) = \mathcal{F}_u(\mathbf{M},\bar{\mathbf{\Delta}}) \tag{12}$$

which serves as an alternate representation of $\lambda_{ii}(\mathbf{G}_p)$. Note that, for $\mathcal{F}_u(\mathbf{M},\bar{\mathbf{\Delta}})$ to be well-defined, it is assumed that $(\mathbf{I} - \mathbf{M}_{11}\bar{\mathbf{\Delta}})$

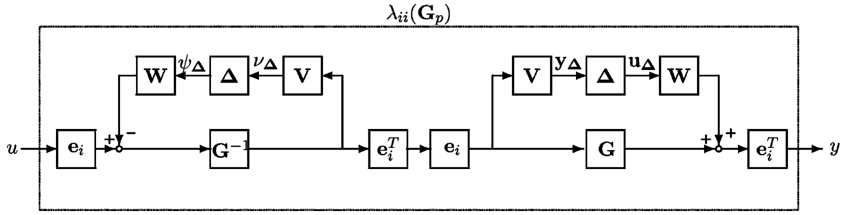


**Figure 1.** Signal-based representation of relative gain for the uncertain system described in eq 6.

is invertible. This assumption is not restrictive, as is shown by Lemma 1 below.

When $\mathbf{\Delta} = 0$ (no uncertainty),

$$\mathscr{F}_u(\mathbf{M},\bar{\mathbf{\Delta}}) = \mathbf{M}_{22} = \mathbf{e}_i^{\mathrm{T}}\mathbf{G}\mathbf{E}\mathbf{G}^{-1}\mathbf{e}_i = \lambda_{ii}(\mathbf{G})$$

and we recover relative gain for the nominal system, as expected.

*Remark 3.* We note that the uncertainty associated with relative gain always has a block-diagonal structure, irrespective of the structure of $\mathbf{\Delta}$ (see eq 11). This follows from the definition of relative gain for uncertain systems in eq 5, where the uncertainty enters the expression once through $[\mathbf{G}_p]_{ij}$ and again through the inverse term $[\mathbf{G}_p^{-1}]_{ji}$.

We next use the equivalence in eq 12 to present the bound on the magnitude of worst-case relative gain.

*Lemma 1.* Let $\mathbf{G}_p$ be nonsingular over the set $\Pi_A$ in eq 3. Then, $(\mathbf{I} - \mathbf{M}_{11}\bar{\mathbf{\Delta}})$ is invertible for allowable $\bar{\mathbf{\Delta}}$, where $\mathbf{M}_{11}$ and $\bar{\mathbf{\Delta}}$ are given by eqs 10 and 11, respectively.

Lemma 1 implies that the relative gain represented as an upper-LFT in eq 12 is well-defined, whenever $\mathbf{G}_p$ is nonsingular. The fact that $\lambda_{ii}(\mathbf{G}_p)$ is well-defined for nonsingular $\mathbf{G}_p$ also follows intuitively, from the definition of relative gain for uncertain systems in eq 5 (also see Remark 1).

*Proposition 1.* Let $\mathbf{G}_p$ be nonsingular over the set $\Pi_A$ in eq 3, such that $\lambda_{ii}(\mathbf{G}_p)$ in eq 5 is well-defined. Assume that the positive real scalar $\gamma$ satisfies

$$\mu_{\tilde{\mathbf{\Delta}}}\begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12}\\ \gamma\mathbf{M}_{21} & \gamma\mathbf{M}_{22}\end{bmatrix} \leq 1 \tag{13}$$

where

$$\tilde{\Delta} = \begin{bmatrix}\bar{\mathbf{\Delta}} & 0\\ 0 & \delta\end{bmatrix}$$

$|\delta| \leq 1$, and $\mathbf{M}$, with the specified partitioning, is given in eq 10. Then,

$$\max_{\mathbf{G}_p \in \Pi_A} |\lambda_{ii}(\mathbf{G}_p)| \leq \gamma^{-1} \tag{14}$$

To find the magnitude of worst-case relative gain, one must find the largest $\gamma$ that satisfies eq 13. Numerically, such a value of $\gamma$ can be found by replacing the inequality in eq 13 by an equality and solving

$$\mu_{\tilde{\mathbf{\Delta}}}\begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12}\\ \gamma\mathbf{M}_{21} & \gamma\mathbf{M}_{22}\end{bmatrix} = 1$$

using a bisection algorithm. The structured singular value calculated in this fashion is called skewed-$\mu$ and is useful for robust performance analysis (see, e.g., Skogestad and Postlethwaite[2]).

Braatz et al.[11] have shown that the exact calculation of $\mu$ is computationally intractable. Thus, computation of the exact value of the worst-case relative gain can be difficult, but tight bounds can be obtained using the upper bound on $\mu$ calculated through the well-known *D*-scaling method.[9] When the (bound on) worst-case relative gain is large, controllability of the uncertain system can be poor, because of strong directionality. Note that, although Proposition 3 is stated only for steady-state relative gain for notational simplicity, it also holds at higher frequencies and can be used to calculate the magnitude of the worst-case frequency-dependent relative gain.

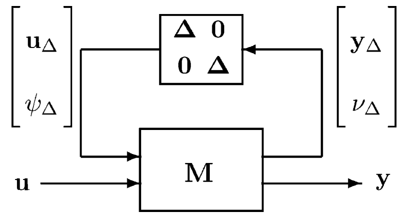


**Figure 2.** Representation of relative gain as an upper-Linear Fractional Transform (upper-LFT).

## 4. Robust Integrity and Relative Gain

In this section, we explore the usefulness of RGA for adjudging the integrity of uncertain systems. Before proceeding with the main developments, we provide a formal definition of integrity for uncertain systems (also referred as robust integrity), which is a minor extension of the definition of integrity for nominal systems presented by Campo and Morari.[4]

*Definition 1.* The system $\mathbf{G}(s)$ is said to have *robust integrity*, if there exists a diagonal controller $\hat{\mathbf{K}}(s) = \mathbf{H}\mathbf{K}(s)$ with integral action, which stabilizes $\mathbf{G}_p(s)$ for all $\mathbf{G}_p(s) \in \Pi_A(s)$ and $\mathbf{H} \in \mathscr{H}$, where

$$\mathscr{H} = \{\mathbf{H} = \mathrm{diag}(h_i) \mid h_i = \{0,1\},\ i = 1, ..., n\} \tag{15}$$

The above definition requires the existence of a diagonal controller, with integral action in every output channel, which provides robust stability of the system, when all of the individual subcontrollers are in operation and also when any combination of the subcontrollers fail. It is inherently assumed that the subcontrollers that fail are immediately taken out of service after instantaneous detection, i.e., the corresponding entries in the diagonal controller matrix are replaced by zero.

Next, we recall the necessary and sufficient conditions for integrity of nominal systems[3] with pairings selected on the diagonal elements of $\mathbf{G}$:

N1. $\mathbf{G}$ is nonsingular, and

N2. The diagonal elements of the RGA calculated for all principal submatrices of $\mathbf{G}$ are positive.

For integrity, it is also necessary that the principal submatrices of $\mathbf{G}$ (including $\mathbf{G}_{ii}$ and $\mathbf{G}^{ii}$) be nonsingular, but nonsingularity of these submatrices is inherent in N2. When one or more of the principal submatrices of $\mathbf{G}$ are singular, the corresponding relative gains are zero and N2 is violated (cf. eq 4).

For integrity of uncertain systems, clearly the minimum requirement is that the conditions N1 and N2 hold for every $\mathbf{G}_p \in \Pi_A$. Specifically, for robust integrity with pairings selected on the diagonal elements of $\mathbf{G}_p$, it is necessary that

R1. $\mathbf{G}_p$ is nonsingular, and

R2. All the principal submatrices of $\mathbf{G}_p$ are nonsingular, and

R3. The diagonal elements of the RGA calculated for all principal submatrices of $\mathbf{G}_p$ are positive over the uncertainty set.

First note that, for robust integrity, we explicitly consider the nonsingularity of the principal submatrices of $\mathbf{G}_p$. This is required, because, for uncertain systems, the relative gains can remain positive (e.g., systems with diagonal multiplicative uncertainty) or stay constant (see, e.g., Example 1) over the uncertainty set, even if one or more of the principal submatrices of $\mathbf{G}_p$ are singular. Second, conditions R1, R2, and R3 have only been claimed to be necessary (and not also sufficient as N1 and N2). This happens because we have inherently allowed the use of different controllers over the subsets of the uncertainty set. Establishing the sufficiency (or insufficiency) of conditions R1, R2, and R3 and the existence of a linear-time invariant controller with integral action that provides robust integrity, upon the satisfaction of these conditions, are open problems. We stress that when condition R1 is violated, integral control is not possible using any controller, including full multivariable controllers. Conditions R2 and R3 specifically involve robust integrity using decentralized controllers.

To evaluate the importance of RGA for the robust integrity problem, we explore the relation between conditions R2 and R3 next. Using the Schur complement Lemma,[10] we have

$$\lambda_{ii}(\mathbf{G}_p) = \frac{[\mathbf{G}_p]_{ii}\,\det(\mathbf{G}_p^{ii})}{\det(\mathbf{G}_p)}\ \forall\ \mathbf{G}_p \in \Pi_A$$

When $\lambda_{ii}(\mathbf{G}) > 0$, the minimal requirement for violating condition R3 is that at least one of $[\mathbf{G}_p]_{ii}$, $\det(\mathbf{G}_p^{ii})$, and $\det(\mathbf{G}_p)$ changes sign over the uncertainty set. Since $\mathbf{\Delta}$ is a closed set, the sign change depends directly on these gains being singular over the uncertainty set. These arguments are easily extended to relative gains calculated for principal submatrices of $\mathbf{G}_p$. Therefore, if $\mathbf{G}_p$ and its principal submatrices are nonsingular and $\lambda_{ii}(\mathbf{G}) > 0$, the relative gains are always positive over the uncertain set in eq 3. Similarly, if one or more of the relative gains change sign, $\mathbf{G}_p$ or one of its principal submatrices become singular over the uncertainty set. Based on these observations, we conclude that once the nonsingularity of $\mathbf{G}_p$ and its principal submatrices is established, checking the signs of relative gains over the uncertainty set is redundant. This somewhat surprising conclusion implies that, in contrast to nominal systems, the role of RGA is limited for ascertaining the integrity of uncertain systems described by eq 3.

For uncertain systems, it is sometimes beneficial to find the size of allowable perturbations before the system loses integrity. Based on the findings of this section, we also conclude that, for determining an upper bound on the allowable additive perturbations before the system in eq 3 loses integrity, RGA is not useful either and it suffices to find the smallest perturbation that makes $[\mathbf{G}_p]_{ii}$, $\mathbf{G}_p^{ii}$, or $\mathbf{G}_p$ singular. Note that these same arguments also hold for systems with uncertainty descriptions other than additive uncertainty, as long as $\mathbf{\Delta}$ is a closed set.

## 5. Sign Change of Relative Gain

In a recent paper, Chen and Seborg[8] presented lower and upper bounds on the relative gain for uncertain systems. The relative gains are considered to be positive over the uncertainty set, if the lower bounds are positive. However, it may be difficult to adjudge whether the relative gains change sign from these bounds. This may happen, because even when the lower bound on the relative gain is negative, the actual value may remain positive over the given uncertainty set.

It was shown in the previous section that, for the relative gains to change sign, it is necessary that the gain of the uncertain system or one of its principal submatrices becomes singular over the uncertainty set. However, the singularity of these gains is not sufficient for sign change of the relative gains, e.g., for systems with diagonal multiplicative uncertainty. In this section, we improve upon the results of Chen and Seborg[8] by deriving the necessary and sufficient conditions for sign change of relative gains, using the structured singular value framework. We point out that the results derived in this section are of algebraic interest only.

*Proposition 2*. Let $\mathbf{G}_p$ be nonsingular over the set $\Pi_A$ in eq 3, such that $\lambda_{ii}(\mathbf{G}_p)$ in eq 5 is well-defined. Then, $\lambda_{ii}(\mathbf{G}_p)$ changes sign over the set $\Pi_A$ if and only if (iff),

$$\mu_{\bar{\mathbf{\Delta}}}(\mathbf{N}) > 1 \qquad (16)$$

where

$$\mathbf{N} = \begin{bmatrix} -\lambda_{ii}^{-1}(\mathbf{G})\mathbf{VEG}^{-1}\mathbf{EW} & -\mathbf{VEG}^{-1}(\mathbf{I} - \lambda_{ii}^{-1}(\mathbf{G})\mathbf{EGEG}^{-1})\,\mathbf{W} \\ -\lambda_{ii}^{-1}(\mathbf{G})\,\mathbf{VG}^{-1}\mathbf{EW} & -\mathbf{VG}^{-1}\,(\mathbf{I} - \lambda_{ii}^{-1}(\mathbf{G})\mathbf{EGEG}^{-1})\,\mathbf{W} \end{bmatrix} \qquad (17)$$

where $\bar{\Delta} = \mathrm{diag}(\mathbf{\Delta}, \mathbf{\Delta})$ and $\mathbf{E} = \mathbf{e}_i\mathbf{e}_i^T$.

In this paper, we have considered systems with additive perturbations only. By finding equivalent signal- and LFT-based representations of relative gain, it is possible to derive conditions similar to eq 16 for systems with other forms of norm-bounded uncertainties (e.g., multiplicative uncertainty) and also for systems with multiple sources of perturbations. In the case of additive perturbations, it is possible to simplify the condition described by eq 16 further, as demonstrated next.

*Corollary 1*. Under the same conditions of Proposition 2, $\lambda_{ii}(\mathbf{G}_p)$ in eq 5 changes sign iff there exists an allowable $\mathbf{\Delta}$ such that one of $[\mathbf{G}_p]_{ii}$ and $\mathbf{G}_p^{ii}$ become singular over the set $\Pi_A$ in eq 3.

Based on Corollary 1, it follows that, for systems with norm-bounded additive uncertainty, checking whether relative gains change sign is equivalent to checking whether $[\mathbf{G}_p]_{ii}$ and $\mathbf{G}_p^{ii}$ become singular over the uncertainty set in eq 3. In comparison to assessing the sign change of relative gains based on the lower and upper bound,[8] this result is not conservative. We note that the nonsingularity of $[\mathbf{G}_p]_{ii}$ and $\mathbf{G}_p^{ii}$ over the uncertainty set can be verified similar to that of $\mathbf{G}_p$ (see Remark 1).

For some uncertainty descriptions, the relative gain may not change sign even when $[\mathbf{G}_p]_{ii}$ and $\mathbf{G}_p^{ii}$ are singular over the uncertainty set. For example, for the diagonal input or output multiplicative uncertainty description, the uncertainty can be effectively treated as a diagonal scaling matrix. Because RGA is scaling invariant,[1] the relative gains remain constant for arbitrarily large diagonal multiplicative uncertainty.

*Remark 4.* Systems with diagonal multiplicative uncertainty can also be represented in a form with an additive uncertainty description. For example, the set of plants $\mathbf{G}\,(\mathbf{I} + \mathrm{diag}(\delta_i)\mathbf{R})$ can be represented in the form of eq 3 with $\mathbf{W} = \mathbf{G}$, $\mathbf{V} = \mathbf{R}$, where $\mathbf{R} = \mathrm{diag}(r_i)$. Clearly, for $r_i > 1$ for some $i$, there exists $\delta_i < 1$ that makes $\mathbf{G}_{ii}$ singular. As the relative gains are scaling independent, this may seem to contrast the findings of Corollary 5. This is not the case, because $\det(\mathbf{G}_p) = \det(\mathbf{G})\cdot\det(\mathbf{I} + \mathbf{G}^{-1}\mathbf{G}\,\mathrm{diag}(\delta_i)\mathbf{R})$ and there exists $\mathrm{diag}(\delta_i)$ with $\delta_i < 1$ for all $i$ that also makes $\mathbf{G}_p$ singular. This violates the assumptions of Proposition 2 and, thus, the results of Corollary 1 hold.

## 6. Examples

In this section, we illustrate the usefulness of the results presented in this paper through two case studies on distillation columns. Example 1 primarily involves the computation of the magnitude of the worst-case relative gain, whereas Example 2 focuses on the results presented in Section 5. (The Matlab files for these examples are available in the Supporting Information.)

**Example 1.** In this example, we consider the Wood−Berry distillation column.[12] For this process, the nominal steady-state gain matrix and RGA are given as follows:

$$\mathbf{G} = \begin{bmatrix} 12.8 & -18.9 \\ 6.6 & -19.4 \end{bmatrix}$$

$$\mathbf{\Lambda}(\mathbf{G}) = \begin{bmatrix} 2.001 & -1.001 \\ -1.001 & 2.001 \end{bmatrix}$$

The uncertainty description is assumed to be of the form

$$[\mathbf{G}_p]_{ij} = \mathbf{G}_{ij} + \alpha\cdot\delta_{ij}\cdot|\mathbf{G}_{ij}|, \qquad |\delta_{ij}| \le 1 \tag{18}$$

which corresponds to independent variations of the individual elements. To represent the set of uncertain plants in the standard form given in eq 3, we use $\mathbf{\Delta} = \mathrm{diag}(\delta_{ij})$, where

$$\mathbf{W} = \begin{bmatrix} |g_{11}| & |g_{12}| & 0 & 0 \\ 0 & 0 & |g_{21}| & |g_{22}| \end{bmatrix}$$

$$\mathbf{V} = \alpha\cdot\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$$

Using Proposition 1, we then have $\max_{\mathbf{G}_p\in\Pi_A}|\lambda_{11}(\mathbf{G}_p)| \le 2.095$ and 4.0057, for $\alpha = 0.01$ and 0.1, respectively. For $\alpha > 0.17$, eq 13 has no solution for arbitrarily small values of $\gamma$ (including $\gamma = 0$) and, thus, $\max_{\mathbf{G}_p\in\Pi_A}|\lambda_{11}(\mathbf{G}_p)| = \infty$. Indeed, it can be confirmed that $\mathbf{G}_p$ becomes singular for $\alpha > 0.17$ and, thus, $\lambda_{11}(\mathbf{G}_p)$ is not well-defined.

For $\alpha = 0.17$, $\max_{\mathbf{G}_p\in\Pi_A}|\lambda_{11}(\mathbf{G}_p)| = 549.95$. The worst-case relative gain occurs when $\delta_{11},\delta_{12} = -1$, $\delta_{21},\delta_{22} = 1$, and

$$\mathbf{G}_p = [\mathbf{G}_{ij} + 0.17\cdot\delta_{ij}\cdot|\mathbf{G}_{ij}|] = \begin{bmatrix} 10.624 & -22.113 \\ 7.722 & -16.102 \end{bmatrix} \tag{19}$$

For $\mathbf{G}_p$ in eq 19, the maximum and minimum singular values are 30.344 and 0.01, respectively, which shows that the uncertain system has strong directionality and the controllability is poor.[5] The magnitudes of the worst-case relative gain, as presented here, match the corresponding results of Chen and Seborg;[8] however, such a conclusion generally does not hold.

Next, we consider the alternate uncertainty description,

$$[\mathbf{G}_p]_{ij} = G_{ij} + \alpha\cdot\delta\cdot|\mathbf{G}_{ij}|, \qquad \text{for } |\delta| \le 1 \tag{20}$$

For this uncertainty description, the elements of $\mathbf{G}_p$ are highly correlated and $\lambda_{11}(\mathbf{G}_p)$ is independent of the perturbation $\delta$, i.e., $\max_{\mathbf{G}_p\in\Pi_A}|\lambda_{11}(\mathbf{G}_p)| = |\lambda_{11}(\mathbf{G})| = 2.009$. This result can also be established analytically by noting

$$\begin{aligned}\lambda_{11}(\mathbf{G}_p) &= 1 - \cfrac{1}{\cfrac{[\mathbf{G}_p]_{12}[\mathbf{G}_p]_{21}}{[\mathbf{G}_p]_{11}[\mathbf{G}_p]_{22}}} \\ &= 1 - \cfrac{1}{\cfrac{\mathbf{G}_{12}(1-\alpha\cdot\delta)\mathbf{G}_{21}(1+\alpha\cdot\delta)}{\mathbf{G}_{11}(1+\alpha\cdot\delta)\mathbf{G}_{22}(1-\alpha\cdot\delta)}} \\ &= 1 - \cfrac{1}{\cfrac{\mathbf{G}_{12}\mathbf{G}_{21}}{\mathbf{G}_{11}\mathbf{G}_{22}}} = \lambda_{11}(\mathbf{G})\end{aligned}$$

which is independent of $\delta$. Furthermore, for the alternate uncertainty description with $\alpha = 1$, clearly there exists allowable perturbations that yield $\mathbf{G}_{11}$ or $\mathbf{G}_{22}$ singular, but relative gains do not change signs. This may seem to violate the findings of Proposition 2, but note that there also exists an allowable perturbation that makes $\mathbf{G}_\mathrm{p}$ singular (also see Remark 4).

**Example 2.** To illustrate the results of Section 5, we consider the example of binary distillation column,[13] whose nominal steady-state model and RGA are given as

$$\mathbf{G} = \begin{bmatrix} 0.66 & -0.61 & -0.0049 \\ 1.11 & -2.36 & -0.012 \\ -33.68 & 46.2 & 0.87 \end{bmatrix}$$

$$\mathbf{\Lambda}(\mathbf{G}) = \begin{bmatrix} 1.95 & -0.67 & -0.27 \\ -0.66 & 1.90 & -0.23 \\ -0.28 & -0.23 & 1.51 \end{bmatrix} \tag{21}$$

The $\mathbf{\Lambda}(\mathbf{G})$ shows that the system can have integrity only if the pairings are selected on the diagonal elements of $\mathbf{G}$. It can be further verified that the relative gains of all the $2 \times 2$ principal submatrices are positive. Thus, the existence of a controller is guaranteed such that the nominal system has integrity.[3]

Next, let the model of the uncertain system be of the form in eq 18, which was earlier considered by Chen and Seborg.[8] The objective is to determine the largest value of $\alpha$ such that $\lambda_{ii}(\mathbf{G}_p)$ remain positive for $i = 1, ..., 3$, where Chen and Seborg[8] suggested that values of $\alpha = 0.5$ can be easily tolerated. We next show that this conclusion is incorrect and only smaller perturbations can actually be accommodated. For this purpose, we assume that the uncertain system has the form of eq 20, which can only allow for smaller uncertainties than eq 18. Thus, if $\alpha = 0.5$ cannot be tolerated by the uncertain system described in eq 20, it cannot also be tolerated by the uncertain system described in eq 18.

The uncertain system described in eq 20 can be represented in the standard form (see eq 3) with $\mathbf{W} = \alpha\cdot\mathbf{I}$, $\mathbf{V} = |\mathbf{G}|$, and $\Delta = \delta\cdot\mathbf{I}$. It follows from Proposition 2 and the definition of $\mu$ that the largest value of $\alpha$, such that $\lambda_{ii}(\mathbf{G}_p) > 0$ for all $i$, is given as $\min_i \mu_{\mathbf{\Delta}}{}^{-1}(\mathbf{N})$, where $\mathbf{V} = |\mathbf{G}|$, $\mathbf{W} = \mathbf{I}$, and $\Delta = \delta\cdot\mathbf{I}$. Because $\mathbf{\Delta}$ is a repeated scalar, $\mu_{\mathbf{\Delta}}(\mathbf{N}) = \rho(\mathbf{N})$,[9] where $\rho$ denotes the spectral radius. Now, for $i = 1$, 2, and 3, we have $\rho(\mathbf{N}) =$ 1, 3.3114, and 1, respectively, and, thus, the largest value of allowable $\alpha$ is given as $1/3.3114 = 0.302$. We also note that, for $\alpha = 0.302$, $\mathbf{G}_p^{22} = \mathbf{G}^{22} - \alpha|\mathbf{G}^{22}|$ is singular. Furthermore, $[\mathbf{G}_p]_{ii}$ and $\mathbf{G}_p^{ii}$ remain nonsingular for $0 \le \alpha < 0.302$, which confirms the findings of Corollary 1.

## 7. Conclusions

Using the structured singular value ($\mu$) framework, we have presented a method for calculation of the magnitude of the worst-case relative gain. This result is useful for analyzing the presence of strong directionality in the uncertain system. We have derived the necessary and sufficient conditions for sign change of relative gain for norm-bounded uncertain systems. More importantly, it is shown that the relative gain changes sign, only if the gain of the uncertain system or one of its principal submatrices corresponding to the relative gain becomes singular over the uncertainty set. Note that the nonsingularity of these matrices is trivially necessary for integrity of the uncertain system. This result implies that, in contrast to nominal systems, the role of RGA is limited for ascertaining the integrity of uncertain systems. To derive the necessary and sufficient conditions for integrity of uncertain systems, one must consider alternate methods, e.g., the approach based on parametrization of all stabilizing controller with integral action, as used by Gündes and Kabuli.[14]

## Appendix

**Proof of Lemma 1.** Based on eq 10, we have

$$\det(\mathbf{I} - \mathbf{M}_{11}\bar{\mathbf{\Delta}}) = \det\left(\begin{bmatrix} \mathbf{I} & \mathbf{VEG}^{-1}\mathbf{W\Delta} \\ 0 & \mathbf{I} + \mathbf{VG}^{-1}\mathbf{W\Delta} \end{bmatrix}\right)$$
$$= \det(\mathbf{I} + \mathbf{VG}^{-1}\mathbf{W\Delta}) = \det(\mathbf{I} + \mathbf{G}^{-1}\mathbf{W\Delta V})$$
$$= \det(\mathbf{G}^{-1})\cdot\det(\mathbf{G} + \mathbf{W\Delta V}) = \det(\mathbf{G}^{-1})\cdot\det(\mathbf{G}_{\mathrm{p}}{}^{-1})$$

Thus, the nonsingularity of $\mathbf{G}_p$ over the set $\Pi_{\mathrm{A}}$ implies that $(\mathbf{I} - \mathbf{M}_{11}\bar{\mathbf{\Delta}})$ is invertible.

**Proof of Proposition 1.** Based on the definition of structured singular value in eq 1, any value of $\gamma$ satisfying eq 13 implies that

$$\det\left(\mathbf{I} - \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \gamma\mathbf{M}_{21} & \gamma\mathbf{M}_{22} \end{bmatrix}\begin{bmatrix} \bar{\mathbf{\Delta}} & 0 \\ 0 & \delta \end{bmatrix}\right)$$
$$= \det\left(\begin{bmatrix} \mathbf{I} - \mathbf{M}_{11}\bar{\mathbf{\Delta}} & -\mathbf{M}_{12}\delta \\ -\gamma\mathbf{M}_{21}\bar{\mathbf{\Delta}} & 1 - \gamma\mathbf{M}_{22}\delta \end{bmatrix}\right)$$
$$= \det(\mathbf{I} - \mathbf{M}_{11}\bar{\mathbf{\Delta}})\cdot\det((1 - \gamma\mathbf{M}_{22}\delta) - (-\gamma\mathbf{M}_{21}\bar{\mathbf{\Delta}})(\mathbf{I} - \mathbf{M}_{11}\bar{\mathbf{\Delta}})^{-1}(-\mathbf{M}_{12}\delta))$$

is nonzero. The last equality follows using the Schur complement Lemma,[10] because $(\mathbf{I} - \mathbf{M}_{11}\bar{\mathbf{\Delta}})$ is nonsingular for allowable $\bar{\mathbf{\Delta}}$ (see Lemma 1). The last expression, in turn, implies that

$$(1 - \gamma\mathbf{M}_{22}\delta) - (-\gamma\mathbf{M}_{21}\bar{\mathbf{\Delta}})(\mathbf{I} - \mathbf{M}_{11}\bar{\mathbf{\Delta}})^{-1}(-\mathbf{M}_{12}\delta)$$
$$= 1 - \gamma(\mathbf{M}_{22} + \mathbf{M}_{21}\bar{\mathbf{\Delta}}(\mathbf{I} - \mathbf{M}_{11}\bar{\mathbf{\Delta}})^{-1}\mathbf{M}_{12})\delta$$
$$= 1 - \gamma\mathcal{F}_u(\mathbf{M},\bar{\mathbf{\Delta}})\delta$$

is nonzero for allowable $\bar{\mathbf{\Delta}}$. From the definition of structured singular value in eq 1, we have

$$\max_{\mathbf{G}_p\in\Pi_{\mathrm{A}}} \mu_\delta(\gamma\mathcal{F}_u(\mathbf{M},\bar{\mathbf{\Delta}})) = \max_{\mathbf{G}_p\in\Pi_{\mathrm{A}}} |\gamma\mathcal{F}_u(\mathbf{M},\bar{\mathbf{\Delta}})| \le 1 \tag{22}$$

which establishes eq 14, because $\lambda_{ii}(\mathbf{G}_p) = \mathcal{F}_u(\mathbf{M},\bar{\mathbf{\Delta}})$. The equality in eq 22 follows, because, for full-block perturbations, $\mu$ is equal to the maximum singular value of the matrix, which is the magnitude of the scalar in the present case.

**Proof of Proposition 2.** Since $\mathbf{\Delta}$ is a closed set, $\lambda_{ii}(\mathbf{G}_p)$ changes sign over the set $\Pi_{\mathrm{A}}$ in eq 3, iff

$$\lambda_{ii}(\mathbf{G}_p) = \mathcal{F}_u(\mathbf{M},\bar{\mathbf{\Delta}}) = 0 \tag{23}$$

for some $\mathbf{G}_p\in\Pi_{\mathrm{A}}$, where $\mathbf{M}$ is given by eq 10. The requirement in eq 23 can be written, somewhat crudely, as $(\mathcal{F}_u(\mathbf{M},\bar{\mathbf{\Delta}}))^{-1} = \infty$ for some $\mathbf{G}_p\in\Pi_{\mathrm{A}}$. Now, $(\mathcal{F}_u(\mathbf{M},\bar{\mathbf{\Delta}}))^{-1}$ is given as $\mathcal{F}_u(\tilde{\mathbf{M}},\bar{\mathbf{\Delta}})$, where[2]

$$\tilde{\mathbf{M}} = \begin{bmatrix} \mathbf{M}_{11} - \mathbf{M}_{12}\mathbf{M}_{22}^{-1}\mathbf{M}_{21} & \mathbf{M}_{12}\mathbf{M}_{22}^{-1} \\ \mathbf{M}_{22}^{-1}\mathbf{M}_{21} & \mathbf{M}_{22}^{-1} \end{bmatrix}$$

We notice, from the expression for upper-LFT in eq 2, that $(\mathcal{F}_u(\mathbf{M},\bar{\mathbf{\Delta}}))^{-1} = \infty$, iff $(\mathbf{I} - \mathbf{N}\,\bar{\mathbf{\Delta}})$ is singular for some allowable $\bar{\mathbf{\Delta}}$, where $\mathbf{N} = \tilde{\mathbf{M}}_{11} = \mathbf{M}_{11} - \mathbf{M}_{12}\mathbf{M}_{22}^{-1}\mathbf{M}_{21}$. The expression for $\mathbf{N}$ in eq 17 is obtained by substituting for $\mathbf{M}_{ij}$ using eq 10 and simplifying the resulting expression. Now, it follows from the definition of structured singular value in eq 1 that $(\mathbf{I} - \mathbf{N}\bar{\mathbf{\Delta}})$ is singular for some allowable $\bar{\mathbf{\Delta}}$ iff eq 16 holds.

**Proof of Corollary 1.** Without loss of generality, we assume that $i = 1$. After some lengthy, but straightforward algebraic manipulations, it can be shown that the (1,2) element of $\mathbf{N}$ in eq 17 is zero. Then, $(\mathbf{I} - \mathbf{N}\bar{\mathbf{\Delta}})$ is a block-triangular matrix and

$$\mu_{\bar{\mathbf{\Delta}}}(\mathbf{N}) = \max(\mu_{\mathbf{\Delta}}(\mathbf{N}_{11}), \mu_{\mathbf{\Delta}}(\mathbf{N}_{22})) \tag{24}$$

Using $\mathbf{E} = \mathbf{e}_1\mathbf{e}_1^{\mathrm{T}}$, we have

$$\mathbf{N}_{11} = \lambda_{11}{}^{-1}(\mathbf{G})(\mathbf{Ve}_1)(\mathbf{e}_1^{\mathrm{T}}\mathbf{G}^{-1}\mathbf{e}_1)(\mathbf{e}_1^{\mathrm{T}}\mathbf{W})$$
$$= \lambda_{11}{}^{-1}(\mathbf{G})\mathbf{V}_{*1}[\mathbf{G}^{-1}]_{11}\mathbf{W}_{1*}$$
$$= (\mathbf{G}_{11}[\mathbf{G}^{-1}]_{11})^{-1}\mathbf{V}_{*1}[\mathbf{G}^{-1}]_{11}\mathbf{W}_{1*} = \mathbf{V}_{*1}\mathbf{G}_{11}{}^{-1}\mathbf{W}_{1*}$$

and, thus,

$$\det(\mathbf{I} - \mathbf{N}_{11}\mathbf{\Delta}) = \det(\mathbf{I} - \mathbf{V}_{*1}\mathbf{G}_{11}{}^{-1}\mathbf{W}_{1*}\mathbf{\Delta})$$
$$= \det(\mathbf{I} - \mathbf{G}_{11}{}^{-1}\mathbf{W}_{1*}\mathbf{\Delta V}_{*1})$$
$$= \mathbf{G}_{11}{}^{-1}\cdot\det(\mathbf{G}_{11} - \mathbf{W}_{1*}\mathbf{\Delta V}_{*1})$$
$$= \mathbf{G}_{11}{}^{-1}\cdot\det([\mathbf{G}_p]_{11})$$

Thus, $\mu_{\mathbf{\Delta}}{}^{-1}(\mathbf{N}_{11})$ denotes the smallest perturbation that makes $[\mathbf{G}_p]_{11}$ singular.

Now, let $\mathbf{G}^{-1}$ be partitioned as

$$\mathbf{G}^{-1} = \begin{bmatrix} [\mathbf{G}^{-1}]_{11} & [\mathbf{G}^{-1}]_{12} \\ [\mathbf{G}^{-1}]_{21} & [\mathbf{G}^{-1}]_{22} \end{bmatrix} \tag{25}$$

where $[\mathbf{G}^{-1}]_{11}$ is a scalar. Then, using $\mathbf{E} = \mathbf{e}_1\mathbf{e}_1^{\mathrm{T}}$,

$$\mathbf{G}^{-1}(\mathbf{I} - \lambda_{11}{}^{-1}(\mathbf{G})\mathbf{EGEG}^{-1})$$
$$= \mathbf{G}^{-1} - \lambda_{11}{}^{-1}(\mathbf{G})(\mathbf{G}^{-1}\mathbf{e}_1)(\mathbf{e}_1^{\mathrm{T}}\mathbf{Ge}_1)(\mathbf{e}_1^{\mathrm{T}}\mathbf{G}^{-1})$$
$$= \mathbf{G}^{-1} - \begin{bmatrix} [\mathbf{G}^{-1}]_{11} \\ [\mathbf{G}^{-1}]_{21} \end{bmatrix}\lambda_{11}{}^{-1}(\mathbf{G})\mathbf{G}_{11}\begin{bmatrix} [\mathbf{G}^{-1}]_{11} & [\mathbf{G}^{-1}]_{12} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & [\mathbf{G}^{-1}]_{22} - [\mathbf{G}^{-1}]_{21}[\mathbf{G}^{-1}]_{11}{}^{-1}[\mathbf{G}^{-1}]_{12} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{G}_{22}{}^{-1} \end{bmatrix}$$

The last equality follows because $[\mathbf{G}^{-1}]_{22} - [\mathbf{G}^{-1}]_{21}[\mathbf{G}^{-1}]_{11}{}^{-1}\cdot[\mathbf{G}^{-1}]_{12}$ is the Schur complement of $[\mathbf{G}^{-1}]_{11}$ in $\mathbf{G}^{-1}$, and, thus, its inverse is given by $\mathbf{G}_{22}$.[10] Let the matrices $\mathbf{W}$ and $\mathbf{V}$ be partitioned to have compatible dimensions with the partitioned

$\mathbf{G}^{-1}$ in eq 25. Using the same arguments as those used for singularity of $[\mathbf{G}_p]_{11}$, it can be shown that $\mu_{\mathbf{\Delta}}{}^{-1}(\mathbf{N}_{22})$ denotes the smallest perturbation that makes $[\mathbf{G}_p]_{22}$ singular. Then, $\mu_{\bar{\mathbf{\Delta}}}(\mathbf{N}) > 1$ iff any allowable $\mathbf{\Delta}$ makes one of $[\mathbf{G}_p]_{ii}$ and $\mathbf{G}_p^{ii}$ singular and the result follows.

**Supporting Information Available:** Matlab files for Examples 1 and 2 (TXT). This material is available free of charge via the Internet at http://pubs.acs.org.

## Literature Cited

(1) Bristol, E. H. On a new measure of interaction for multivariable process control. *IEEE Trans. Autom. Control* **1966**, *11*, 133.

(2) Skogestad, S.; Postlethwaite, I. *Multivariable Feedback Control: Analysis and Design*, 2nd Edition; Wiley: Chichester, U.K., 2005.

(3) Kariwala, V.; Forbes, J. F.; Meadows, E. S. Integrity of systems under decentralized integral control. *Automatica* **2005**, *41*, 1575.

(4) Campo, P. J.; Morari, M. Achievable closed loop properties of systems under decentralized control: Conditions involving the steady-state gain. *IEEE Trans. Autom. Control* **1994**, *39*, 932.

(5) Skogestad, S.; Morari, M. Implications of large RGA-elements on control performance. *Ind. Eng. Chem. Res.* **1987**, *26*, 2323.

(6) Yu, C. C.; Luyben, W. L. Robustness with respect to integral controllability. *Ind. Eng. Chem. Res.* **1987**, *26*, 1043.

(7) Grosdidier, P.; Morari, M.; Holt, B. R. Closed-loop properties from steady-state gain information. *Ind. Eng. Chem. Fundam.* **1985**, *24*, 221.

(8) Chen, D.; Seborg, D. E. Relative gain array analysis for uncertain process models. *AIChE J.* **2002**, *48*, 302.

(9) Doyle, J. C. Analysis of feedback systems with structured uncertainties. *IEE Proc.—D. Control Theory Appl.* **1982**, *129*, 242.

(10) Horn, R. A.; Johnson, C. R. *Matrix Analysis*; Cambridge University Press: Cambridge, U.K., 1990.

(11) Braatz, R. D.; Young, P. M.; Doyle, J. C.; Morari, M. Computational complexity of $\mu$ calculation. *IEEE Trans. Autom. Control* **1994**, *39*, 1000.

(12) Wood, R. K.; Berry, M. W. Terminal composition control of a binary distillation column. *Chem. Eng. Sci.* **1973**, *28*, 1707.

(13) Ogunnaike, B. A.; Lemaire, J.; Morari, M.; Ray, W. H. Advanced multivariable control of a pilot plant distillation column. *AIChE J.* **1983**, *29*, 632.

(14) Gündes, A. N.; Kabuli, M. G. Reliable decentralized integral-action controller design. *IEEE Trans. Autom. Control* **2001**, *46*, 296.

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