ACHIEVABLE H_∞-PERFORMANCE OF MULTIVARIABLE SYSTEMS WITH UNSTABLE ZEROS AND POLES

Kjetil Havre [*,1] **Sigurd Skogestad** [*,2]

[*] *Chemical Engineering, Norwegian University of Science and Technology, N-7034 Trondheim, Norway*

Abstract: This paper examines the limitations imposed by Right Half Plane (RHP) zeros and poles in multivariable feedback systems. The main result is to provide lower bounds on $\|WXV(s)\|_\infty$ where X is S, S_I, T or T_I (sensitivity and complementary sensitivity). Previously derived lower bounds on the $\mathcal{H}_\infty$-norm of S and T are thus generalized to the case with matrix-valued weights, including bounds for reference tracking, disturbance rejection, and input usage.

Keywords: System theory, Achievable $\mathcal{H}_\infty$-performance, Unstable systems, RHP-zeros and poles, Stabilization.

1. INTRODUCTION

It is well known that the presence of RHP zeros and poles pose fundamental limitations on the achievable control performance. This was quantified for SISO systems by Bode (1945) more than 50 years ago, and most control engineers have an intuitive feeling of the limitations for scalar systems. Rosenbrock (1966; 1970) was one of the first to point out that multivariable RHP-zeros pose similar limitations.

The main results in this paper are explicit lower bounds on the $\mathcal{H}_\infty$-norm of closed-loop transfer functions. Of course, it is relatively straightforward to compute the exact minimum value of the $\mathcal{H}_\infty$-norm for a given case using standard software, and a direct computation of the value of the $\mathcal{H}_\infty$-norm is also possible, e.g. using the Hankel-norm as explained in (Francis, 1987). Therefore, we want to stess that the objective is to derive explicit (analytical) bounds that yield direct insight into the limitations imposed by RHP-poles and zeros.

The basis of our results is the work by Zames (1981), who made use of the interpolation constraint $y_z^H S(z) = y_z^H$ and the maximum modulus theorem to derive bounds on $\mathcal{H}_\infty$-norm of S for plants with one RHP-zero. The results by Zames were generalized to plants with RHP-poles by Doyle *et al.* (1992) in the SISO case, and by Skogestad and Postlethwaite (1996), Havre and Skogestad (1996; 1997*a*) in the MIMO case.

In this paper we extend the work of Zames (1981) and Havre and Skogestad (1996; 1997*a*) and quantify the fundamental limitations imposed by RHP zeros and poles in terms of lower bounds on the $\mathcal{H}_\infty$-norm of important closed-loop transfer functions. The main generalization of the previous result is that from the results in this paper we can derive lower bounds on $\mathcal{H}_\infty$-norm of other closed-loop transfer functions than sensitivity and complementary sensitivity. To do this it was necessary to generalize the previous results to include multivariable, unstable and non-minimum phase weights.

One important application of the lower bounds, is that we can *quantify* the minimum usage needed to stabilize an unstable plant in the presence of the worst case disturbance, measurement noise and reference changes. An additional important contribution of this

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[1] Present address: ABB Corporate Research, P.O.Box 90 N-1361 Billingstad, Norway. Fax: (+47) 66 84 35 41, E-mail: Kjetil.Havre@nocrc.abb.no.
[2] Fax: (+47) 73 59 40 80, E-mail: skoge@chembio.ntnu.no.

paper is that we prove that the lower bounds are *tight* in a large number of cases. That is, we give analytical expressions for controllers which *achieve* an $\mathcal{H}_\infty$-norm of the closed-loop transfer function which is equal to the lower bound.

2. ZEROS AND POLES IN MIMO SYSTEMS

Zeros and zero directions. Zeros of a system arise when competing effects, internal to the system, are such that the output is zero even when the inputs and the states are not identically zero. Here we apply the following definition of zeros (MacFarlane and Karcanias, 1976).

DEFINITION 1. (ZEROS). *$z_i \in \mathbb{C}$ is a zero of $G(s)$ if the rank of $G(z_i)$ is less than the normal rank of $G(s)$.*

The normal rank of $G(s)$ is defined as the rank of $G(s)$ at all s except a finite number of singularities (which are the zeros).

DEFINITION 2. (ZERO DIRECTIONS). *If $G(s)$ has a zero for $s = z \in \mathbb{C}$ then there exist non-zero vectors, denoted the input zero direction $u_z \in \mathbb{C}^m$ and the output zero direction $y_z \in \mathbb{C}^l$, such that $u_z^H u_z = 1$, $y_z^H y_z = 1$ and*

$$G(z)u_z = 0; \qquad y_z^H G(z) = 0 \tag{1}$$

For a system $G(s)$ with state-space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, the zeros z of the system, the input zero directions u_z and the state input zero vectors $\boldsymbol{x}_{zi} \in \mathbb{C}^n$ (n is the number of states) can all be computed from the generalized eigenvalue problem

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{zi} \\ u_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2}$$

Similarly one can compute the zeros z and the output zero directions y_z from G^T, see (Havre, 1998, Section 2.3) for further details.

Poles and pole directions. Bode (1945) states that *the poles are the singular points at which the transfer function fails to be analytic*. In this work we replace "fails to be analytic" with "is infinite", which certainly implies that the transfer function is *not analytic*. When we evaluate [3] the transfer function $G(s)$ at $s = p$, $G(p)$ is infinite in some directions at the input and the output. This is the basis for the following definition of input and output pole directions.

DEFINITION 3. (POLE DIRECTIONS). *If $s = p \in \mathbb{C}$ is a distinct pole of $G(s)$ then there exist one input direction $u_p \in \mathbb{C}^m$ and one output direction $y_p \in \mathbb{C}^l$ with infinite gain for $s = p$.*

For a system $G(s)$ with minimal state-space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ the pole directions u_p and y_p for a *distinct* pole p can be computed from (Havre, 1998, Section 2.4)

$$u_p = B^H x_{pi} / \left\| B^H x_{pi} \right\|_2 \tag{3}$$

$$y_p = C x_{po} / \left\| C x_{po} \right\|_2 \tag{4}$$

where $x_{pi} \in \mathbb{C}^n$ and $x_{po} \in \mathbb{C}^n$ are the eigenvectors corresponding to the two eigenvalue problems

$$x_{pi}^H A = p x_{pi}^H; \qquad A x_{po} = p x_{po}$$

Note, that the pole directions are normalized, i.e. $\|u_p\|_2 = 1$ and $\|y_p\|_2 = 1$. For the sake of simplicity we will only consider distinct poles in this paper, for computation and definition of pole directions in the case when the pole p is not distinct refer to (Havre, 1998, Chapter 2).

All-pass factorizations of RHP zeros and poles. A transfer function matrix $B(s)$ is all-pass if $B^T(-s)B(s) = I$, which implies that all singular values of $B(j\omega)$ are equal to one.

A rational transfer function matrix $M(s)$ with RHP-poles $p_i \in \mathbb{C}_+$, can be factorized either at the input (subscript i) or at the output (subscript o) of $M(s)$ as follows [4]

$$M(s) = M_{si}\, \mathcal{B}_{pi}^{-1}(M(s)) \tag{5}$$

$$M(s) = \mathcal{B}_{po}^{-1}(M)\, M_{so}(s) \tag{6}$$

M_{si}, M_{so} – Stable (subscript s) versions of M with the RHP-poles mirrored across the imaginary axis.

$\mathcal{B}_{pi}(M)$, $\mathcal{B}_{po}(M)$ – Stable all-pass rational transfer function matrices containing the RHP-poles (subscript p) of M as RHP-zeros.

The all-pass filters are (Havre, 1998, Appendix A)

$$\mathcal{B}_{pi}(M(s)) = \prod_{i=1}^{N_p} (I - \frac{2\mathrm{Re}(p_i)}{s + \bar{p}_i} \hat{u}_{p_i} \hat{u}_{p_i}^H) \tag{7}$$

$$\mathcal{B}_{po}(M(s)) = \prod_{i=N_p}^{1} (I - \frac{2\mathrm{Re}(p_i)}{s + \bar{p}_i} \hat{y}_{p_i} \hat{y}_{p_i}^H) \tag{8}$$

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[3] Strictly speaking, the transfer function $G(s)$ can *not* be evaluated at $s = p$, since $G(s)$ is not analytic at $s = p$.

[4] Note that the notation on the all-pass factorizations of RHP zeros and poles used in this paper is reversed compared to the notation used in (Green and Limebeer, 1995; Skogestad and Postlethwaite, 1996; Havre and Skogestad, 1996). The reason for this change of notation is to get consistent with what the literature generally defines as an all-pass filter.

$\mathcal{B}_{po}(M)$ is obtained by factorizing at the output one RHP-pole at a time, starting with

$$M = \mathcal{B}_{p_1 o}^{-1}(M) M_{p_1 o}$$

where

$$\mathcal{B}_{p_1 o}^{-1}(M(s)) = I + \frac{2\mathrm{Re}(p_1)}{s - p_1} \hat{y}_{p_1} \hat{y}_{p_1}^H$$

and $\hat{y}_{p_1} = y_{p_1}$ is the output pole direction of M for p_1. This procedure may be continued to factor out p_2 from $M_{p_1 o}$ where $\hat{y}_{p_2}$ is the output pole direction of $M_{p_1 o}$ (which need not coincide with y_{p_2}, the pole direction [5] of M) and so on. A similar procedure may be used to factorize the poles at the input of M. Note that the sequence get reversed in the input factorization compared to the output factorization.

In a similar sequential manner, the RHP-zeros can be factorized either at the input or at the output of M (Havre, 1998, Appendix A)

$$M(s) = M_{mi} \mathcal{B}_{zi}(M(s)) \tag{9}$$

$$M(s) = \mathcal{B}_{zo}(M) M_{mo}(s) \tag{10}$$

M_{mi}, M_{mo} – Minimum phase (subscript m) versions of M with the RHP-zeros mirrored across the imaginary axis.

$\mathcal{B}_{zi}(M), \mathcal{B}_{zo}(M)$ – Stable all-pass rational transfer function matrices containing the RHP-zeros (subscript z) of M.

3. LOWER BOUNDS ON THE $\mathcal{H}_\infty$-NORM OF CLOSED-LOOP TRANSFER FUNCTIONS

We define the following sensitivity and complementary sensitivity functions

$$S \triangleq (I + GK_2)^{-1} \tag{11}$$

$$T \triangleq I - S \;=\; GK_2(I + GK_2)^{-1} \tag{12}$$

$$S_I \triangleq (I + K_2 G)^{-1} \tag{13}$$

$$T_I \triangleq I - S_I = K_2 G (I + K_2 G)^{-1} \tag{14}$$

where K_2 denotes the feedback part of the controller.

In this section we derive general lower bounds on the $\mathcal{H}_\infty$-norm of closed-loop transfer functions when the plant G has one or more RHP zeros and/or poles, by using the interpolation constraints and the maximum modulus principle. The bounds are applicable to closed-loop transfer functions on the form

$$W(s)X(s)V(s) \tag{15}$$

where X may be S, T, S_I or T_I. The idea is to derive lower bounds on $\|WXV(s)\|_\infty$ which are independent of the controller K. In general, we assume that WXV is stable. The "weights" W and V must be independent of K. They may be unstable, but it must be possible to stabilize all transfer functions through the outputs. This implies that the unstable modes of W and V also appear in $L = GK_2$. Otherwise, the system is not stabilizable. The results are stated in terms of four theorems.

Theorems 1 and 2 provide lower bounds on the $\mathcal{H}_\infty$-norm of closed-loop transfer functions on the forms WSV and WS_IV caused by one or more RHP-zeros in G. By maximizing over all RHP-zeros, we find the largest lower bounds on $\|WSV(s)\|_\infty$ and $\|WS_IV(s)\|_\infty$ which takes into account one RHP-zero and all RHP-poles in the plant.

THEOREM 1. (LOWER BOUND ON $\|WSV(s)\|_\infty$). *Consider a plant G with $N_z \geq 1$ RHP-zeros z_j, output directions y_{z_j} and $N_p \geq 0$ RHP-poles $p_i \in \mathbb{C}_+$. Let W and V be rational transfer function matrices, and assume that W has no RHP-poles at locations corresponding to RHP poles or zeros in G. Assume that the closed-loop transfer function WSV is (internally) stable. Then the following lower bound on $\|WSV(s)\|_\infty$ applies:*

$$\|WSV(s)\|_\infty \geq \max_{\text{RHP-zeros } z_j \text{ in } G} \|W_{mo}(z_j)\, y_{z_j}\|_2 \cdot \|y_{z_j}^H V \mathcal{B}_{zi}^{-1}(\mathcal{B}_{po}(G)\, V)\big|_{s=z_j}\|_2 \tag{16}$$

Proof. The main steps in the proof of (16) are given in Section 3.

THEOREM 2. (LOWER BOUND ON $\|WS_IV(s)\|_\infty$). *Consider a plant G with $N_z \geq 1$ RHP-zeros z_j, input directions u_{z_j} and $N_p \geq 0$ RHP-poles $p_i \in \mathbb{C}_+$. Let W and V be rational transfer function matrices, and assume that V has no RHP-poles at locations corresponding to RHP poles or zeros in G. Assume that the closed-loop transfer function WS_IV is (internally) stable. Then the following lower bound on $\|WS_IV(s)\|_\infty$ applies:*

$$\|WS_IV(s)\|_\infty \geq \max_{\text{RHP-zeros, } z_j \text{ in } G} \|\mathcal{B}_{zo}^{-1}(W\mathcal{B}_{pi}(G))\, W\big|_{s=z_j} u_{z_j}\|_2 \cdot \|u_{z_j}^H V_{mi}(z_j)\|_2 \tag{17}$$

Theorems 3 and 4 provide lower bounds on the $\mathcal{H}_\infty$-norm of closed-loop transfer functions on the forms WTV and WT_IV caused by one or more RHP-poles in G. By maximizing over all RHP-poles, we find the largest lower bounds on $\|WTV(s)\|_\infty$ and $\|WT_IV(s)\|_\infty$ which takes into account one RHP-pole and all RHP-zeros in the plant.

THEOREM 3. (LOWER BOUNDS ON $\|WTV(s)\|_\infty$). *Consider a plant G with $N_p \geq 1$ RHP-poles p_i, output directions y_{p_i} and $N_z \geq 0$ RHP-zeros $z_j \in \mathbb{C}_+$. Let W and V be rational transfer function matrices, and assume that V has no RHP-poles at locations*

[5] In fact: $\hat{y}_{p_2} = \mathcal{B}_{p_1 o}^{-H}(M)\big|_{s=p_2} y_{p_2}$. Here $\mathcal{B}\big|_{s=s_0}$ means the rational transfer function matrix $\mathcal{B}(s)$ evaluated at the complex number $s = s_0$. Thus, it provides an alternative to $\mathcal{B}(s_0)$, and it will mainly be used to avoid double parenthesis.

corresponding to RHP *zeros or poles in* G. *Assume that the closed-loop transfer function* WTV *is (internally) stable. Then the following lower bound on* $\|WTV(s)\|_\infty$ *applies:*

$$\|WTV(s)\|_\infty \geq \max_{\text{RHP-poles, } p_i \text{ in } G} \|\mathcal{B}_{zo}^{-1}(W\mathcal{B}_{zo}(G))\, W|_{s=p_i} y_{p_i}\|_2 \cdot \|y_{p_i}^H V_{mi}(p_i)\|_2 \quad (18)$$

THEOREM 4. (LOWER BOUNDS ON $\|WT_IV(s)\|_\infty$). *Consider a plant* G *with* $N_p \geq 1$ RHP-*poles* p_i, *input directions* u_{p_i} *and* $N_z \geq 0$ RHP-*zeros* $z_j \in \mathbb{C}_+$. *Let* W *and* V *be rational transfer function matrices, and assume that* W *has no* RHP-*poles at locations corresponding to* RHP *zeros or poles in* G. *Assume that the closed-loop transfer function* WT_IV *is (internally) stable. Then the following lower bound on* $\|WT_IV(s)\|_\infty$ *applies:*

$$\|WT_IV(s)\|_\infty \geq \max_{\text{RHP-poles, } p_i \text{ in } G} \|W_{mo}(p_i)\, u_{p_i}\|_2 \cdot \|u_{p_i}^H V \mathcal{B}_{zi}^{-1}(\mathcal{B}_{zi}(G)\, V)|_{s=p_i}\|_2 \quad (19)$$

Remarks:

1) The somewhat messy notation can easily be interpreted. As an example take the last factor of (16): Factorize the RHP-poles at the output of G into an all-pass filter $\mathcal{B}_{po}(G)$ (yields RHP-zeros), multiply on the right with V (may add RHP-zeros if V is non-minimum phase), then factorize at the input the RHP-zeros of the product into an all-pass transfer function, take its inverse, multiply on the left with $y_{z_j}^H V$ and finally evaluate the result for $s = z_j$.
2) The lower bounds (16)–(19) are independent of the feedback controller K_2 if the weights W and V are independent of K_2.
3) The internal stability assumption on the closed-loop transfer functions WXV, where $X \in \{S, S_I, T, T_I\}$, means that WXV are stable, and we have no RHP pole/zero cancellations between the plant G and the feedback controller K_2.

Main steps in the proof of Theorem 1. To provide some more insight to the results, we give the main steps in the proof of the lower bound (16) on $\|WSV(s)\|_\infty$. The main steps in the proof of the lower bounds (17)–(19) are similar.

1) **Factor out RHP zeros in WSV to obtain $(WSV)_m$.**
   (a) Factor out RHP-zeros of S due to RHP-poles in G at the input of S.
   (b) Factor out RHP zeros of $\mathcal{B}_{pi}(G)V$ at the input of WSV.
   (c) Factor out RHP zeros of W at the output of WSV.

   Note, make sure that no RHP-zeros in S due to poles in G, which cancel RHP-poles in V and W, are factorized:
   1) We avoid factorizing RHP-zeros in S which cancel poles in V, by factorizing the zeros of $\mathcal{B}_{po}(G)\, V$.
   2) With the assumption on the poles in Theorem 1 we avoid factorizing RHP-zeros in S which cancel poles in W.
2) **Introduce the stable scalar function $f(s)$ on the minimum phase version $(WSV)_m$ of WSV.**
3) **Apply the maximum modulus theorem to $f(s)$ at the RHP-zeros of G.**
4) **Resubstitute the factorization of RHP-zeros in S.**
5) **Use the interpolation constraint for RHP-zeros in G.**
6) **Evaluate the lower bound.**

4. TIGHTNESS OF LOWER BOUNDS

The theorems provide lower bounds on $\|WXV(s)\|_\infty$ where $X \in \{S, S_I, T, T_I\}$. The question is whether these bounds are tight, meaning that there exist controllers which achieve these bounds? The answer is "yes" if there is only one RHP-zero or one RHP-pole. Specifically, we find that the bounds on $\|WSV(s)\|_\infty$ and $\|WS_IV(s)\|_\infty$ are tight if the plant G has one RHP-zero and any number of RHP-poles. Similarly, we find that the bounds on $\|WTV(s)\|_\infty$ and $\|WT_IV(s)\|_\infty$ are tight if the plant G has one RHP-pole and any number of RHP-zeros. We prove tightness of the lower bounds by constructing controllers which achieve the bounds.

THEOREM 5. *Consider a plant* G *with one* RHP-*zero* z, *output direction* y_z, *and* $N_p \geq 0$ RHP-*poles* $p_i \in \mathbb{C}_+$. *Let* W *and* V *be rational transfer function matrices, and assume that* W *has no* RHP-*poles at locations corresponding to* RHP *poles or zeros in* G. *A feedback controller (possible improper) which stabilizes* WSV, *is given by*

$$K_2(s) = G_{smo}^{-1}(s)\, P(s)\, Q^{-1}(s) \quad (20)$$

where

$$Q(s) = W_{mo}^{-1}(s)\, W_{mo}(z)\, V_0\, \mathcal{B}_{po}^{-1}(G)|_{s=z}\, M_{mi}(z)\, M_{mi}^{-1}(s) \quad (21)$$

$$P(s) = \mathcal{B}_{zo}^{-1}(G_{so})\, (I - \mathcal{B}_{po}(G)\, Q) \quad (22)$$

$$V_0 = y_z y_z^H + k_0^2 U_0 U_0^H \quad \textit{and}$$

$$M_{mi}(s) = (\mathcal{B}_{po}(G)\, V(s))_{mi}$$

where the columns of the matrix $U_0 \in \mathbb{R}^{l\times(l-1)}$ *together with* y_z *forms an orthonormal basis for* $\mathbb{R}^l$ *and* k_0 *is any constant.* $P(s)$ *is stable since the* RHP-*zero for* $s = z$ *in* $I - \mathcal{B}_{po}(G)\, Q$ *cancels the* RHP-*pole for* $s = z$ *in* $\mathcal{B}_{zo}^{-1}(G_{so})$, *in a minimal realization of* P. *With this controller we have*

$$\lim_{k_0 \to 0} \|WSV(s)\|_\infty = \|W_{mo}(z)\, y_z\|_2 \cdot \|y_z^H V \mathcal{B}_{zi}^{-1}(\mathcal{B}_{po}(G)\, V)|_{s=z}\|_2 \quad (23)$$

From Theorem 5 it follows that the bound (16) is tight when the plant has one RHP-zero.

We can proove that the three other bounds in Theorems 2, 3 and 4 are tight, under conditions similar to those given in Theorem 5 (see Havre, 1998, for further details).

5. APPLICATIONS OF LOWER BOUNDS

5.1 *Output performance*

The previously derived bounds in terms of the $\mathcal{H}_\infty$-norms of S and T given in (Zames, 1981; Skogestad and Postlethwaite, 1996) and in Havre and Skogestad (1996; 1997a) follow easily, and further generalizations involving output performance can be derived. Here we assume that the performance weights W_P and W_T are stable and minimum phase.

Weighted sensitivity, $W_P S$. Select $W = W_P$, $V = I$, and apply the bound (16) to obtain

$$\|W_P S(s)\|_\infty \geq \max_{\text{RHP-zeros, } z_j} \|W_P(z_j)\, y_{z_j}\|_2 \cdot \|y_{z_j}^H \mathcal{B}_{po}^{-1}(G)|_{s=z_j}\|_2 \quad (24)$$

Disturbance rejection and reference tracking. Select $W = W_P$, $V = G_d$, and apply the bound (16) to obtain

$$\|W_P S G_d(s)\|_\infty \geq \max_{\text{RHP-zeros, } z_j} \|W_P(z_j)\, y_{z_j}\|_2 \cdot \|y_{z_j}^H G_d \mathcal{B}_{zi}^{-1}(\mathcal{B}_{po}(G)\, G_d)|_{s=z_j}\|_2 \quad (25)$$

For reference tracking the same bound applies (but here G_d is usually a diagonal matrix R representing the magnitude of the reference changes).

Note, we can also look at the combined effect of disturbances and references by selecting $V = [G_d \quad R]$.

5.2 *Input usage*

The above provide generalizations of previous results, but we can also derive some new bounds in terms of input usage from Theorems 3 and 4. These new bounds provide very interesting insights, for example, into the possibility of stabilizing an unstable plant with inputs of bounded magnitude.

The basis of these new bounds is to note that the transfer function from the outputs to the inputs, $K_2 S$, can be rewritten as $K_2 S = T_I G^{-1}$ or $K_2 S = G^{-1} T$. When G is unstable, G^{-1} has one or more RHP-zeros, so it is important that the bounds in Theorem 4 can handle the case when $V = G^{-1}$ has RHP-zeros. Otherwise, G^{-1} evaluated at the pole of G, would be zero in a certain direction, and we would not derive any useful bounds. Here we assume that the weight W_u on the input u is stable and minimum phase.

Disturbance rejection. Apply the equality $K_2 S = T_I G^{-1}$, select $W = W_u$, $V = G^{-1} G_d$, and use the bound (19) to obtain

$$\|W_u K_2 S G_d(s)\|_\infty \geq \max_{\text{RHP-poles, } p_i} \|W_u(p_i)\, u_{p_i}\|_2 \cdot \|u_{p_i}^H G^{-1} G_d \mathcal{B}_{zi}^{-1}(G_{mi}^{-1} G_d)|_{s=p_i}\|_2 \quad (26)$$

where we have used the identity $\mathcal{B}_{zi}(G)\, G^{-1} = G_{mi}^{-1}$. Again, reference tracking si included by replacing G_d by R.

5.3 *Two degrees-of-freedom control*

For a 2-DOF controller the closed-loop transfer function from references $\tilde{r}$ to outputs $z_1 = W_p(y - r)$ becomes

$$W_P(SGK_1 - I)R \quad (27)$$

We then have the following "special" lower bound on this transfer function.

THEOREM 6. *Consider a plant G with $N_z \geq 1$ RHP-zeros z_j and $N_p \geq 0$ RHP-poles $p_i \in \mathbb{C}_+$. Let the performance weight W_P be minimum phase and let (for simplicity) R be stable. Assume that the closed-loop transfer function $W_P(SGK_1 - I)R$ is stable. Then the following lower bound on $\|W_P(SGK_1 - I)R(s)\|_\infty$ applies:*

$$\|W_P(SGK_1 - I)R(s)\|_\infty \geq \max_{\text{RHP-zeros } z_j \text{ in } G} \|W_P(z_j) y_{z_j}\|_2 \cdot \|y_{z_j}^H R_{mi}(z_j)\|_2 \quad (28)$$

The bound (28) is tight if the plant has one RHP-zero z.

Note that this bound does not follow directly from Theorems 1–4. The bound in (28) should be compared to the following bound for a 1-DOF controller (which follows from Theorem 1, assuming that W_P is minimum phase).

$$\|W_P S R(s)\|_\infty \geq \max_{\text{RHP-zeros } z_j \text{ in } G} \|W_P(z_j) y_{z_j}\|_2 \cdot \|y_{z_j}^H R \mathcal{B}_{zi}^{-1}(\mathcal{B}_{po}(G)\, R)|_{s=z_j}\|_2 \quad (29)$$

We see that for the 2-DOF controller only the RHP-zeros pose limitations.

6. EXAMPLE

Consider the following multivariable plant G

$$G(s) = \begin{bmatrix} \frac{s-z}{s-p} & -\frac{0.1s+1}{s-p} \\ \frac{s-z}{0.1s+1} & 1 \end{bmatrix}, \quad z = 2.5 \text{ and } p = 2$$

The plant G has one multivariable RHP-zero $z = 2.5$ and one RHP-pole $p = 2$. The corresponding input and output zero and pole directions are

$$u_z = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; y_z = \begin{bmatrix} 0.371 \\ 0.928 \end{bmatrix}, \; u_p = \begin{bmatrix} 0.385 \\ 0.923 \end{bmatrix}, \; y_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The RHP-pole p can be factorized into $G(s) = \mathcal{B}_{po}^{-1}(G) G_{so}(s)$ where

$$\mathcal{B}_{po}(G) = \begin{bmatrix} \frac{s-p}{s+p} & 0 \\ 0 & 1 \end{bmatrix}, \; G_{so}(s) = \begin{bmatrix} \frac{s-z}{s+p} & -\frac{0.1s+1}{s+p} \\ \frac{s-z}{0.1s+1} & 1 \end{bmatrix}$$

From the lower bound (16), with $W = I$ and $V = I$, we find

$$\begin{aligned} \|S(s)\|_\infty &\geq \|y_z^H \mathcal{B}_{po}^{-1}(G)|_{s=z}\|_2 \\ &= \left\| \begin{bmatrix} 0.371 & 0.928 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \right\|_2 = 3.4691 \end{aligned}$$

Next, consider to minimize the input usage, i.e. to minimize the $\mathcal{H}_\infty$-norm of $K_2 S$. We have two lower bounds on $\|K_2 S(s)\|_\infty$, but they are identical since the bounds are tight. We use the equality $K_2 S = T_I G^{-1}$ and the lower bound (19) with $W = I$ and $V = G^{-1}$, to obtain [6]

$$\begin{aligned} \|K_2 S(s)\|_\infty &\geq \|u_p^H G^{-1} \mathcal{B}_{zi}^{-1}(G_{mi}^{-1})|_{s=p}\|_2 \\ &= \|u_p^H G_{so}^{-1}(p)\|_2 = 3.077 \end{aligned}$$

In (Havre, 1998, Section 5.7) reference tracking is also considered, and the benefit of applying 2-DOF controller when the plant is unstable is illustated.

7. REFERENCES

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For further details regarding the usefulness of the lower bounds in engineering applications, the reader should consult (Havre, 1998, Chapters 4, 5 and 6)

PhD thesis and MATLAB-software to compute the bounds in Theorems 1 to 4 are available on the internet: `http://www.chembio.ntnu.no/users/skoge`.

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[6] We use the following relations: $\mathcal{B}_{zi}(G_{mi}^{-1}) = \mathcal{B}_{po}(G)$ and $G^{-1}\mathcal{B}_{po}^{-1}(G) = G_{so}^{-1}$. The first, follows since the input factorization of RHP-zeros in G does *not* change the output pole directions.