# Achievable $\mathcal{H}_\infty$-performance of multivariable systems with unstable zeros and poles

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**Abstract**

This paper examines the limitations imposed by Right Half Plane (RHP) zeros and poles in multivariable feedback systems. The main result is to provide lower bounds on $\|WXV(s)\|_\infty$ where $X$ is the input or output sensitivity or complementary sensitivity. $W$ and $V$ are matrix valued weights who might depend on the plant and who also might be unstable. Previously derived lower bounds on the $\mathcal{H}_\infty$-norm of the sensitivity and the complementary sensitivity are thus generalized to include bounds for reference tracking and disturbance rejection. Furthermore, new bounds which quantify the minimum *input* usage for stabilization in the presence of measurement noise and disturbances, are derived. From the bounds we find that *output* performance is *only* limited if the plant has RHP-zeros. For a one degree-of-freedom (1-DOF) controller the presence of RHP-poles further deteriorate the response, whereas there is no additional penalty for having RHP-poles if we use a two degrees-of-freedom (2-DOF) controller (where the disturbance and/or reference signal is measured). For large classes of plants we *prove* that the lower bounds given are *tight* in the sense that there exist stable controllers (possible improper) that achieve the bounds.

**Keywords:** System theory; Achievable $\mathcal{H}_\infty$-performance; Unstable systems; RHP-zeros and poles; Stabilization.

## 1 Introduction

It is well known that the presence of RHP zeros and poles pose fundamental limitations on the achievable control performance. This was quantified for SISO systems by Bode (1945) more than $50$ years ago, and most control engineers have an intuitive feeling of the limitations for scalar systems. Rosenbrock (1966; 1970) was one of the first to point out that multivariable RHP-zeros pose similar limitations.

The main results in this paper are explicit lower bounds on the $\mathcal{H}_\infty$-norm of closed-loop transfer functions. Of course, it is relatively straightforward to compute the exact minimum value of the $\mathcal{H}_\infty$-norm for a given case using standard software, and a direct computation of the value of the $\mathcal{H}_\infty$-norm is also possible, e.g. using the Hankel-norm as explained in (Francis, 1987). Therefore, we want

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to stress that the objective is to derive explicit (analytical) bounds that yield direct insight into the limitations imposed by RHP-poles and zeros.

The basis of our results is the *important* work by Zames (1981), who made use of the interpolation constraint $y_z^H S(z) = y_z^H$ and the maximum modulus theorem to derive bounds on $\mathcal{H}_\infty$-norm of $S$ for plants with one RHP-zero. The results by Zames were generalized to plants with RHP-poles by Doyle, Francis and Tannenbaum (1992) in the SISO case, and by Skogestad and Postlethwaite (1996), Havre and Skogestad (1998) in the MIMO case.

In this paper we extend the work of Zames (1981) and Havre and Skogestad (1998) and quantify the fundamental limitations imposed by RHP zeros and poles in terms of lower bounds on the $\mathcal{H}_\infty$-norm of important closed-loop transfer functions. The main generalization of the previous result is that from the results in this paper we can derive lower bounds on $\mathcal{H}_\infty$-norm of closed-loop transfer functions other than sensitivity and complementary sensitivity. Further[1] generalizations include multivariable weights and unstable and non-minimum phase weights.

One important application of the lower bounds, is that we can *quantify* the minimum usage needed to stabilize an unstable plant in the presence of the "worst case" disturbance, measurement noise and reference changes for the "best"[2] possible controller. An additional important contribution of this paper is that we prove that the lower bounds are *tight* in a large number of cases. That is, we give analytical expressions for controllers which *achieve* an $\mathcal{H}_\infty$-norm of the closed-loop transfer function which is equal to the lower bound.

# 2 Elements from linear system theory

## 2.1 Zeros and zero directions.

Zeros of a system arise when competing effects, internal to the system, are such that the output is zero even when the inputs and the states are not identically zero. Here we apply the following definition of zeros (MacFarlane and Karcanias, 1976).

DEFINITION 1 (ZEROS). $z_i \in \mathbb{C}$ *is a zero of* $G(s)$ *if the rank of* $G(z_i)$ *is less than the normal rank of* $G(s)$.

The normal rank of $G(s)$ is defined as the rank of $G(s)$ at all $s$ except a finite number of singularities (which are the zeros).

DEFINITION 2 (ZERO DIRECTIONS). *If* $G(s)$ *has a zero for* $s = z \in \mathbb{C}$ *then there exist non-zero vectors, denoted the input zero direction* $u_z \in \mathbb{C}^m$ *and the output zero direction* $y_z \in \mathbb{C}^l$*, such that* $u_z^H u_z = 1$, $y_z^H y_z = 1$ *and*

$$G(z)u_z = 0; \qquad y_z^H G(z) = 0 \tag{1}$$

For a system $G(s)$ with state-space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, the zeros $z$ of the system, the input zero directions $u_z$ and the state input zero vectors $\boldsymbol{x}_{zi} \in \mathbb{C}^n$ ($n$ is the number of states) can all be computed from the generalized eigenvalue problem

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{zi} \\ u_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2}$$

[1]In order to accomplish lower bounds on $\mathcal{H}_\infty$-norm of general closed-loop transfer functions, it was necessary to generalize the previous results to include multivariable, unstable and non-minimum phase weights.

[2]The best possible controller in the sense that the controller which minimizes the $\mathcal{H}_\infty$-norm of the closed-loop transfer function from the disturbances, measurement noise and references to the outputs.

Similarly one can compute the zeros $z$ and the output zero directions $y_z$ from $G^T$.

## 2.2 Poles and pole directions.

Bode (1945) states that *the poles are the singular points at which the transfer function fails to be analytic*. In this work we replace "fails to be analytic" with "is infinite", which certainly implies that the transfer function is *not analytic*. When we evaluate[3] the transfer function $G(s)$ at $s = p$, $G(p)$ is infinite in some directions at the input and the output. This is the basis for the following definition of input and output pole directions.

DEFINITION 3 (POLE DIRECTIONS). *If $s = p \in \mathbb{C}$ is a distinct pole of $G(s)$ then there exist an input direction $u_p \in \mathbb{C}^m$ and an output direction $y_p \in \mathbb{C}^l$ with infinite gain for $s = p$.*

For a system $G(s)$ with minimal state-space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ the pole directions $u_p$ and $y_p$ for a *distinct* pole $p$ can be computed from (Havre, 1998, Section 2.4)

$$u_p = B^H x_{pi} / \left\| B^H x_{pi} \right\|_2 ; \qquad y_p = C x_{po} / \left\| C x_{po} \right\|_2 \tag{3}$$

where $x_{pi} \in \mathbb{C}^n$ and $x_{po} \in \mathbb{C}^n$ are the eigenvectors corresponding to the two eigenvalue problems

$$x_{pi}^H A = p x_{pi}^H; \qquad A x_{po} = p x_{po}$$

Note, that the pole directions are normalized, i.e. $\left\| u_p \right\|_2 = 1$ and $\left\| y_p \right\|_2 = 1$. For the sake of simplicity we will only consider distinct poles in this paper.

## 2.3 All-pass factorizations of RHP zeros and poles

A transfer function matrix $B(s)$ is all-pass if $B^T(-s)B(s) = I$, which implies that all singular values of $B(j\omega)$ are equal to one.

A rational transfer function matrix $M(s)$ with RHP-poles $p_i \in \mathbb{C}_+$, can be factorized either at the input (subscript $i$) or at the output (subscript $o$) as follows[4]

$$M(s) = M_{si}\, \mathcal{B}_{pi}^{-1}(M); \qquad M(s) = \mathcal{B}_{po}^{-1}(M)\, M_{so}(s) \tag{4}$$

$M_{si}$, $M_{so}$ – Stable (subscript $s$) versions of $M$ with the RHP-poles mirrored across the imaginary axis.

$\mathcal{B}_{pi}(M)$, $\mathcal{B}_{po}(M)$ – Stable all-pass rational transfer function matrices containing the RHP-poles (subscript $p$) of $M$ as RHP-zeros.

The all-pass filters are

$$\mathcal{B}_{pi}(M(s)) = \prod_{i=1}^{N_p} (I - \tfrac{2\mathrm{Re}(p_i)}{s + \bar{p}_i} \hat{u}_{p_i} \hat{u}_{p_i}^H); \quad \mathcal{B}_{pi}^{-1}(M(s)) = \prod_{i=N_p}^{1} (I + \tfrac{2\mathrm{Re}(p_i)}{s - p_i} \hat{u}_{p_i} \hat{u}_{p_i}^H) \tag{5}$$

$$\mathcal{B}_{po}(M(s)) = \prod_{i=N_p}^{1} (I - \tfrac{2\mathrm{Re}(p_i)}{s + \bar{p}_i} \hat{y}_{p_i} \hat{y}_{p_i}^H); \quad \mathcal{B}_{po}^{-1}(M(s)) = \prod_{i=1}^{N_p} (I + \tfrac{2\mathrm{Re}(p_i)}{s - p_i} \hat{y}_{p_i} \hat{y}_{p_i}^H) \tag{6}$$

[3]Strictly speaking, the transfer function $G(s)$ can *not* be evaluated at $s = p$, since $G(s)$ is not analytic at $s = p$.

[4]Note that the notation on the all-pass factorizations of RHP zeros and poles used in this paper is reversed compared to the notation used in (Green and Limebeer, 1995; Skogestad and Postlethwaite, 1996; Havre and Skogestad, 1996). The reason for this change of notation is to be consistent with what the literature generally defines as an all-pass filter.

$\mathcal{B}_{po}(M)$ is obtained by factorizing at the output one RHP-pole at a time, starting with

$$M = \mathcal{B}^{-1}_{p_1 o}(M) M_{p_1 o}$$

where

$$\mathcal{B}^{-1}_{p_1 o}(M(s)) = I + \frac{2\mathrm{Re}(p_1)}{s - p_1}\hat{y}_{p_1}\hat{y}^H_{p_1}$$

and $\hat{y}_{p_1} = y_{p_1}$ is the output pole direction of $M$ for $p_1$. This procedure may be continued to factor out $p_2$ from $M_{p_1 o}$ where $\hat{y}_{p_2}$ is the output pole direction of $M_{p_1 o}$ (which need not coincide with $y_{p_2}$, the pole direction[5] of $M$) and so on. A similar procedure may be used to factorize the poles at the input of $M$. Note that the sequence get reversed in the input factorization compared to the output factorization.

In a similar sequential manner, the RHP-zeros can be factorized either at the input or at the output of $M$

$$M(s) = M_{mi}\mathcal{B}_{zi}(M(s)); \qquad M(s) = \mathcal{B}_{zo}(M)M_{mo}(s) \tag{7}$$

$M_{mi}$, $M_{mo}$ – Minimum phase (subscript $m$) versions of $M$ with the RHP-zeros mirrored across the imaginary axis.

$\mathcal{B}_{zi}(M)$, $\mathcal{B}_{zo}(M)$ – Stable all-pass rational transfer function matrices containing the RHP-zeros (subscript $z$) of $M$.

We get

$$\mathcal{B}_{zi}(M(s)) = \prod_{j=N_z}^{1}(I - \frac{2\mathrm{Re}(z_j)}{s + \bar{z}_j}\hat{u}_{z_j}\hat{u}^H_{z_j}); \quad \mathcal{B}^{-1}_{zi}(M(s)) = \prod_{j=1}^{N_z}(I + \frac{2\mathrm{Re}(z_j)}{s - z_j}\hat{u}_{z_j}\hat{u}^H_{z_j}) \tag{8}$$

$$\mathcal{B}_{zo}(M(s)) = \prod_{j=1}^{N_z}(I - \frac{2\mathrm{Re}(z_j)}{s + \bar{z}_j}\hat{y}_{z_j}\hat{y}^H_{z_j}); \quad \mathcal{B}^{-1}_{zo}(M(s)) = \prod_{j=N_z}^{1}(I + \frac{2\mathrm{Re}(z_j)}{s - z_j}\hat{y}_{z_j}\hat{y}^H_{z_j}) \tag{9}$$

Alternative all-pass factorizations are in use, e.g. the inner-outer factorizations used in (Morari and Zafiriou, 1989) which are the same as (8) and (9) except for the multiplication of a constant unitary matrix. Reasons for using the factorizations given here are:

1) The factorizations of RHP-zeros given here are analytic and in terms of the zeros and the zero directions, whereas the inner-outer factorizations in (Morari and Zafiriou, 1989) are given in terms of the solution to an algebraic Riccati equation.
2) To factorize RHP-poles using the inner-outer factorization one needs to assume that $G^{-1}$ exist.

## 2.4 Closing the loop

In this paper we consider the general two degrees-of-freedom (2-DOF) control configuration shown in Figure 1. In the figure the performance weights are given in dashed lines. We have included both references $r$ and measurement noise $n$ in addition to disturbances $d$ as external inputs. The three matrices $G_d$, $R$ and $N$ can be viewed as weights on the inputs, and the inputs $\tilde{d}$, $\tilde{r}$ and $\tilde{n}$ are normalized in magnitude. Normally, $N$ is diagonal and $[N]_{ii}$ is the inverse of signal to noise ratio. For most practical purposes, we can assume that $R$ and $N$ are stable. However, from a technical point of view it suffices that the unstable modes in $N$ and $R$ can be stabilized through the inputs $u$. For

[5] In fact: $\hat{y}_{p_2} = \mathcal{B}^{-H}_{p_1 o}(M)|_{s=p_2} y_{p_2}$. Here $\mathcal{B}|_{s=s_0}$ means the rational transfer function matrix $\mathcal{B}(s)$ evaluated at the complex number $s = s_0$. Thus, it provides an alternative to $\mathcal{B}(s_0)$, and it will mainly be used to avoid double parenthesis.

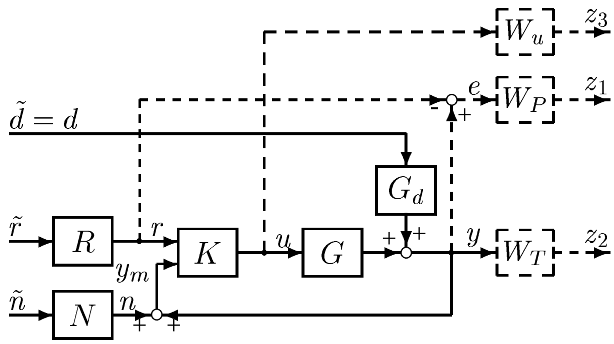


Figure 1: Two degrees-of-freedom control configuration with $K = [K_1\ K_2]$

the disturbance plant $G_d$ we assume that all the unstable modes of $G_d$ also appears in $G$ (which is required if the unstable modes of $G_d$ are state controllable in $u$).

The controller can be divided into a negative feedback part from $y$ ($K_2$) and a feed forward part from $r$ ($K_1$)

$$u = K_1 r - K_2 y_m = K_1 r - K_2(y + n) \tag{10}$$

The closed-loop transfer function $F$ from

$$v = \begin{bmatrix} \tilde{r} \\ \tilde{d} \\ \tilde{n} \end{bmatrix} \quad \text{to} \quad z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}^T = \begin{bmatrix} W_P(y - r) \\ W_T y \\ W_u u \end{bmatrix}$$

is

$$F(s) = \begin{bmatrix} W_P(SGK_1 - I)R & W_P S G_d & -W_P T N \\ W_T S G K_1 R & W_T S G_d & -W_T T N \\ W_u S_I K_1 R & -W_u K_2 S G_d & -W_u K_2 S N \end{bmatrix} \tag{11}$$

where the sensitivity $S$, the complementary sensitivity $T$ and the input sensitivity $S_I$ are defined by

$$S \triangleq (I + GK_2)^{-1} \tag{12}$$

$$T \triangleq I - S = GK_2(I + GK_2)^{-1} \tag{13}$$

$$S_I \triangleq (I + K_2 G)^{-1} \tag{14}$$

We also define the input complementary sensitivity

$$T_I \triangleq I - S_I = K_2 G(I + K_2 G)^{-1} \tag{15}$$

By setting $K_1 = K_2$ in the above equations, the one degree-of-freedom (1-DOF) control configuration can be analyzed.

# 3 Lower bounds on the $\mathcal{H}_\infty$-norm of closed-loop transfer functions

In this section we derive general lower bounds on the $\mathcal{H}_\infty$-norm of closed-loop transfer functions when the plant $G$ has one or more RHP zeros and/or poles, by using the interpolation constraints and the maximum modulus principle. The bounds are applicable to closed-loop transfer functions on the form

$$W(s)X(s)V(s) \tag{16}$$

where $X$ may be $S$, $T$, $S_I$ or $T_I$. The idea is to derive lower bounds on $\|WXV(s)\|_\infty$ which are independent of the controller $K$. In general, we assume that $WXV$ is stable. The "weights" $W$ and $V$ must be independent of $K$, they may be unstable provided that the unstable modes can be stabilized by feedback conntrol of the plant $G$ (e.g.unstable disturbance model $G_d$ or non-minimum phase plant $G$ with an unstable $G^{-1}$). This implies that the unstable modes of $W$ and $V$ also appear in $L = GK_2$. Otherwise, the system is not stabilizable. The results are stated in terms of four theorems.

Theorems 1 and 2 provide lower bounds on the $\mathcal{H}_\infty$-norm of closed-loop transfer functions on the forms $WSV$ and $WS_IV$ caused by one or more RHP-zeros in $G$. By maximizing over all RHP-zeros, we find the largest lower bounds on $\|WSV(s)\|_\infty$ and $\|WS_IV(s)\|_\infty$ which takes into account one RHP-zero and all RHP-poles in the plant.

THEOREM 1 (LOWER BOUND ON $\|WSV(s)\|_\infty$). *Consider a plant $G$ with $N_z \geq 1$ RHP-zeros $z_j$, output directions $y_{z_j}$ and $N_p \geq 0$ RHP-poles $p_i \in \mathbb{C}_+$. Let $W$ and $V$ be rational transfer function matrices, where $W$ is stable. Assume that the closed-loop transfer function $WSV$ is (internally) stable. Then the following lower bound on $\|WSV(s)\|_\infty$ applies:*

$$\|WSV(s)\|_\infty \geq \max_{\text{RHP-zeros } z_j \text{ in } G} \|W_{mo}(z_j)\, y_{z_j}\|_2 \cdot \|y_{z_j}^H V \mathcal{B}_{zi}^{-1}(\mathcal{B}_{po}(G)\, V)|_{s=z_j}\|_2 \tag{17}$$

*Proof.* see Section A.

THEOREM 2 (LOWER BOUND ON $\|WS_IV(s)\|_\infty$). *Consider a plant $G$ with $N_z \geq 1$ RHP-zeros $z_j$, input directions $u_{z_j}$ and $N_p \geq 0$ RHP-poles $p_i \in \mathbb{C}_+$. Let $W$ and $V$ be rational transfer function matrices, where $V$ is stable. Assume that the closed-loop transfer function $WS_IV$ is (internally) stable. Then the following lower bound on $\|WS_IV(s)\|_\infty$ applies:*

$$\|WS_IV(s)\|_\infty \geq \max_{\text{RHP-zeros, } z_j \text{ in } G} \|\mathcal{B}_{zo}^{-1}(W\mathcal{B}_{pi}(G))\, W|_{s=z_j} u_{z_j}\|_2 \cdot \|u_{z_j}^H V_{mi}(z_j)\|_2 \tag{18}$$

Theorems 3 and 4 provide lower bounds on the $\mathcal{H}_\infty$-norm of closed-loop transfer functions on the forms $WTV$ and $WT_IV$ caused by one or more RHP-poles in $G$. By maximizing over all RHP-poles, we find the largest lower bounds on $\|WTV(s)\|_\infty$ and $\|WT_IV(s)\|_\infty$ which takes into account one RHP-pole and all RHP-zeros in the plant.

THEOREM 3 (LOWER BOUNDS ON $\|WTV(s)\|_\infty$). *Consider a plant $G$ with $N_p \geq 1$ RHP-poles $p_i$, output directions $y_{p_i}$ and $N_z \geq 0$ RHP-zeros $z_j \in \mathbb{C}_+$. Let $W$ and $V$ be rational transfer function matrices, where $V$ is stable. Assume that the closed-loop transfer function $WTV$ is (internally) stable. Then the following lower bound on $\|WTV(s)\|_\infty$ applies:*

$$\|WTV(s)\|_\infty \geq \max_{\text{RHP-poles, } p_i \text{ in } G} \|\mathcal{B}_{zo}^{-1}(W\mathcal{B}_{zo}(G))\, W|_{s=p_i} y_{p_i}\|_2 \cdot \|y_{p_i}^H V_{mi}(p_i)\|_2 \tag{19}$$

THEOREM 4 (LOWER BOUNDS ON $\|WT_IV(s)\|_\infty$). *Consider a plant $G$ with $N_p \geq 1$ RHP-poles $p_i$, input directions $u_{p_i}$ and $N_z \geq 0$ RHP-zeros $z_j \in \mathbb{C}_+$. Let $W$ and $V$ be rational transfer function matrices, where $W$ is stable. Assume that the closed-loop transfer function $WT_IV$ is (internally) stable. Then the following lower bound on $\|WT_IV(s)\|_\infty$ applies:*

$$\|WT_IV(s)\|_\infty \geq \max_{\text{RHP-poles, } p_i \text{ in } G} \|W_{mo}(p_i)\, u_{p_i}\|_2 \cdot \|u_{p_i}^H V \mathcal{B}_{zi}^{-1}(\mathcal{B}_{zi}(G)\, V)|_{s=p_i}\|_2 \tag{20}$$

Remarks on Theorems 1–4:

1) The somewhat messy notation can easily be interpreted. As an example take the last factor of (17): Factorize the RHP-poles at the output of $G$ into an all-pass filter $\mathcal{B}_{po}(G)$ (yields RHP-zeros), multiply on the right with $V$ (may add RHP-zeros if $V$ is non-minimum phase), then factorize at the input the RHP-zeros of the product into an all-pass transfer function, take its inverse, multiply on the left with $y_{z_j}^H V$ and finally evaluate the result for $s = z_j$.
2) The lower bounds (17)–(20) are independent of the feedback controller $K_2$ if the weights $W$ and $V$ are independent of $K_2$.
3) The internal stability assumption on the closed-loop transfer functions $WXV$, where $X \in \{S, S_I, T, T_I\}$, means that $WXV$ are stable, and we have no RHP pole/zero cancellations between the plant $G$ and the feedback controller $K_2$.
4) The assumption on stability of $W$ and $V$ in Theorems 1–4 is in practice *not* restrictive, since when the assumption is *not* fulfilled we can generally rewrite the transfer function and apply another theorem instead.

EXAMPLE 1. Consider deriving a bound on $\mathcal{H}_\infty$-norm of the closed-loop transfer function $K_2SG_d$ (input usage due to disturbances). We can use the relation $K_2SG_d = G^{-1}TG_d$ and apply Theorem 3 with $W = G^{-1}$ and $V = G_d$, but we must assume that $G_d$ is *stable*. However, we can use the relation $K_2SG_d = T_I G^{-1} G_d$ and apply Theorem 4 with $W = I$ and $V = G^{-1}G_d$, and in this case we can also allow $G_d$ to be *unstable*.

# 4 Tightness of lower bounds

Theorems 1 to 4 provide lower bounds on $\|WXV(s)\|_\infty$ where $X \in \{S, S_I, T, T_I\}$. The question is whether these bounds are tight, meaning that there exist controllers which achieve these bounds? The answer is "yes" if there is only one RHP-zero or one RHP-pole. Specifically, we find that the bounds on $\|WSV(s)\|_\infty$ and $\|WS_IV(s)\|_\infty$ are tight if the plant $G$ has one RHP-zero and any number of RHP-poles. Similarly, we find that the bounds on $\|WTV(s)\|_\infty$ and $\|WT_IV(s)\|_\infty$ are tight if the plant $G$ has one RHP-pole and any number of RHP-zeros. We prove tightness of the lower bounds by constructing controllers which achieve the bounds.

THEOREM 5 (CONTROLLER WHICH MINIMIZES $\|WSV(s)\|_\infty$). *Consider a plant $G$ with one* RHP-*zero $z$, output direction $y_z$, and $N_p \geq 0$* RHP-*poles $p_i \in \mathbb{C}_+$. Let $W$ and $V$ be rational transfer function matrices, where $W$ is stable. A feedback controller (possible improper) which stabilizes $WSV$, is given by*

$$K_2(s) = G_{smo}^{-1}(s)\, P(s)\, Q^{-1}(s) \tag{21}$$

*where*

$$Q(s) = W_{mo}^{-1}(s)\, W_{mo}(z)\, V_0\, \mathcal{B}_{po}^{-1}(G)|_{s=z}\, M_{mi}(z)\, M_{mi}^{-1}(s) \tag{22}$$

$$P(s) = \mathcal{B}_{zo}^{-1}(G_{so})\; (I - \mathcal{B}_{po}(G)\, Q) \tag{23}$$

$$V_0 = y_z y_z^H + k_0^2 U_0 U_0^H \quad \textit{and} \quad M_{mi}(s) = (\mathcal{B}_{po}(G)\, V(s))_{mi}$$

*where the columns of the matrix $U_0 \in \mathbb{R}^{l\times(l-1)}$ together with $y_z$ forms an orthonormal basis for $\mathbb{R}^l$ and $k_0$ is any constant. $P(s)$ is stable since the* RHP-*zero for $s = z$ in $I - \mathcal{B}_{po}(G)\, Q$ cancels the* RHP-*pole for $s = z$ in $\mathcal{B}_{zo}^{-1}(G_{so})$, in a minimal realization of $P$. With this controller we have*

$$\lim_{k_0 \to 0} \|WSV(s)\|_\infty = \|W_{mo}(z)\, y_z\|_2 \cdot \|y_z^H V \mathcal{B}_{zi}^{-1}(\mathcal{B}_{po}(G)\, V)|_{s=z}\|_2 \tag{24}$$

From Theorem 5 it follows that the bound (17) is tight when the plant has one RHP-zero.

We can proove that the three other bounds in Theorems 2, 3 and 4 are tight, under conditions similar to those given in Theorem 5.

# 5 Applications of lower bounds

The lower bounds on $\|WXV(s)\|_\infty$ in Theorems 1 and 4 can be used to derive a large number of interesting and useful bounds.

## 5.1 Output performance

The previously derived bounds in terms of the $\mathcal{H}_\infty$-norms of $S$ and $T$ given in (Zames, 1981; Skogestad and Postlethwaite, 1996) and in Havre and Skogestad (1998) follow easily, and further generalizations involving output performance can be derived. Here we assume that the performance weights $W_P$ and $W_T$ are stable and minimum phase.

**Weighted sensitivity, $W_P S$.** Select $W = W_P$, $V = I$, and apply the bound (17) to obtain

$$\|W_P S(s)\|_\infty \geq \max_{\text{RHP-zeros, } z_j} \|W_P(z_j)\, y_{z_j}\|_2 \cdot \|y_{z_j}^H \mathcal{B}_{po}^{-1}(G)|_{s=z_j}\|_2 \tag{25}$$

Note, this generalizes the previously found bounds to the case with a matrix valued weight.

**Disturbance rejection.** Select $W = W_P$, $V = G_d$, and apply the bound (17) to obtain

$$\|W_P S G_d(s)\|_\infty \geq \max_{\text{RHP-zeros, } z_j} \|W_P(z_j)\, y_{z_j}\|_2 \cdot \|y_{z_j}^H G_d\, \mathcal{B}_{zi}^{-1}(\mathcal{B}_{po}(G)\, G_d)|_{s=z_j}\|_2 \tag{26}$$

**Reference tracking.** Select $W = W_P$, $V = R$, and apply the bound (17) to obtain

$$\|W_P S R(s)\|_\infty \geq \max_{\text{RHP-zeros, } z_j} \|W_P(z_j)\, y_{z_j}\|_2 \cdot \|y_{z_j}^H R\, \mathcal{B}_{zi}^{-1}(\mathcal{B}_{po}(G)\, R)|_{s=z_j}\|_2 \tag{27}$$

Note that we can also look at the combined effect of disturbances and references by selecting $V = [G_d \quad R]$.

**Measurement noise rejection.** Select $W = W_P$, $V = N$, and apply the bound (19) to obtain

$$\|W_P T N(s)\|_\infty \geq \max_{\text{RHP-poles, } p_i} \|\mathcal{B}_{zo}^{-1}(W_P \mathcal{B}_{zo}(G))\, W_P|_{s=p_i} y_{p_i}\|_2 \cdot \|y_{p_i}^H N_{mi}(p_i)\|_2 \tag{28}$$

where we must assume that $N$ has no RHP-poles corresponding to RHP zeros or poles in $G$. Normally $N$ is stable.

## 5.2 Input usage

The above provide generalizations of previous results, but we can also derive some new bounds in terms of input usage from Theorems 3 and 4. These new bounds provide very interesting insights, for example, into the possibility of stabilizing an unstable plant with inputs of bounded magnitude.

The basis of these new bounds is to note that the transfer function from the outputs to the inputs, $K_2S$, can be rewritten as $K_2S = T_I G^{-1}$ or $K_2S = G^{-1}T$. When $G$ is unstable, $G^{-1}$ has one or more RHP-zeros, so it is important that the bounds in Theorem 4 can handle the case when $V = G^{-1}$ has RHP-zeros. Otherwise, $G^{-1}$ evaluated at the pole of $G$, would be zero in a certain direction, and we would not derive any useful bounds. Here we assume that the weight $W_u$ on the input $u$ is stable and minimum phase.

**Disturbance rejection.** Apply the equality $K_2S = T_I G^{-1}$, select $W = W_u$, $V = G^{-1}G_d$, and use the bound (20) to obtain

$$\|W_u K_2 S G_d(s)\|_\infty \geq \max_{\text{RHP-poles, } p_i} \|W_u(p_i)\, u_{p_i}\|_2 \cdot \|u_{p_i}^H G^{-1} G_d \mathcal{B}_{zi}^{-1}(G_{mi}^{-1} G_d)|_{s=p_i}\|_2 \tag{29}$$

where we have used the identity $\mathcal{B}_{zi}(G)\, G^{-1} = G_{mi}^{-1}$. Again, reference tracking is included by replacing $G_d$ by $R$.

**Measurement noise rejection.** Apply the equality $K_2S = T_I G^{-1}$, select $W = W_u$, $V = G^{-1}N$, and use the bound (20) to obtain

$$\|W_u K_2 S N(s)\|_\infty \geq \max_{\text{RHP-poles, } p_i} \|W_u(p_i)\, u_{p_i}\|_2 \cdot \|u_{p_i}^H G^{-1} N \mathcal{B}_{zi}^{-1}(G_{mi}^{-1} N)|_{s=p_i}\|_2 \tag{30}$$

We may look at the combined effect of reference tracking, disturbance rejection and measurement noise by using (20) with $W = W_u$ and $V = G^{-1}\,[\,G_d \quad R \quad N\,]$.

**Simplified lower bound on $\|K_2S(s)\|_\infty$.** Two useful simplified lower bounds on $\|K_2S(s)\|_\infty$ can easily be derived. First, apply the equality $K_2S = T_I G^{-1}$, select $W = I$, $V = G^{-1}$, and use the bound (20) to obtain

$$\|K_2 S(s)\|_\infty \geq \max_{\text{RHP-poles, } p_i} \|u_{p_i}^H G^{-1} \mathcal{B}_{zi}^{-1}(G_{mi}^{-1})|_{s=p_i}\|_2 = \|u_{p_i}^H G_{so}^{-1}|_{s=p_i}\|_2 \tag{31}$$

where the last identity follows from $\mathcal{B}_{zi}(G_{mi}^{-1}) = \mathcal{B}_{zi}(G^{-1}) = \mathcal{B}_{po}(G)$.

Similarly, we obtain from (19), with $W = G^{-1}$ and $V = I$

$$\|K_2 S(s)\|_\infty \geq \max_{\text{RHP-poles, } p_i} \|\mathcal{B}_{zo}^{-1}(G_{mo}^{-1})\, G^{-1}|_{s=p_i}\, y_{p_i}\|_2 = \|G_{si}^{-1}|_{s=p_i}\, y_{p_i}\|_2 \tag{32}$$

where the last identity follows from $\mathcal{B}_{zo}(G_{mo}^{-1}) = \mathcal{B}_{zo}(G^{-1}) = \mathcal{B}_{pi}(G)$.

# 6 Two degrees-of-freedom control

For a 2-DOF controller the closed-loop transfer function from references $\tilde{r}$ to outputs $z_1 = W_p(y - r)$ becomes

$$W_P(SGK_1 - I)R \tag{33}$$

We then have the following "special" lower bound on this transfer function.

THEOREM 6. *Consider a plant $G$ with $N_z \geq 1$ RHP-zeros $z_j$ and $N_p \geq 0$ RHP-poles $p_i \in \mathbb{C}_+$. Let the performance weight $W_P$ be minimum phase and let (for simplicity) $R$ be stable. Assume that the closed-loop transfer function $W_P(SGK_1 - I)R$ is stable. Then the following lower bound on $\|W_P(SGK_1 - I)R(s)\|_\infty$ applies:*

$$\|W_P(SGK_1 - I)R(s)\|_\infty \geq \max_{\text{RHP-zeros } z_j \text{ in } G} \|W_P(z_j) y_{z_j}\|_2 \cdot \|y_{z_j}^H R_{mi}(z_j)\|_2 \tag{34}$$

*The bound (34) is tight if the plant has one* RHP*-zero $z$.*

Note that this bound does not follow directly from Theorems 1–4. The bound in (34) should be compared to the following bound for a 1-DOF controller (which follows from Theorem 1, assuming that $W_P$ is minimum phase).

$$\|W_P S R(s)\|_\infty \geq \max_{\text{RHP-zeros } z_j \text{ in } G} \|W_P(z_j) y_{z_j}\|_2 \cdot \|y_{z_j}^H R \mathcal{B}_{zi}^{-1} (\mathcal{B}_{po}(G)\, R)|_{s=z_j}\|_2 \tag{35}$$

We see that for the 2-DOF controller only the RHP-zeros pose limitations.

# 7 Example

In this section we consider the following multivariable plant $G$

$$G(s) = \begin{bmatrix} \frac{s-z}{s-p} & -\frac{0.1s+1}{s-p} \\ \frac{s-z}{0.1s+1} & 1 \end{bmatrix}, \quad \text{with} \quad z = 2.5 \quad \text{and} \quad p = 2$$

The plant $G$ has one multivariable RHP-zero $z = 2.5$ and one RHP-pole $p = 2$. The corresponding input and output zero and pole directions are

$$u_z = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad y_z = \begin{bmatrix} 0.371 \\ 0.928 \end{bmatrix}, \qquad u_p = \begin{bmatrix} 0.385 \\ 0.923 \end{bmatrix}, \qquad y_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The RHP-pole $p$ can be factorized into $G(s) = \mathcal{B}_{po}^{-1}(G) G_{so}(s)$ where

$$\mathcal{B}_{po}(G) = \begin{bmatrix} \frac{s-p}{s+p} & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad G_{so}(s) = \begin{bmatrix} \frac{s-z}{s+p} & -\frac{0.1s+1}{s+p} \\ \frac{s-z}{0.1s+1} & 1 \end{bmatrix}$$

From the lower bound (17), with $W = I$ and $V = I$, we find

$$\|S(s)\|_\infty \geq \|y_z^H \mathcal{B}_{po}^{-1}(G)|_{s=z}\|_2 = \left\| [\, 0.371 \quad 0.928 \,] \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \right\|_2 = 3.4691$$

Next, we use Theorem 5 (with $W = I$ and $V = I$) to find the feedback controller which minimizes $\|S(s)\|_\infty$. With $k_0 = 10^{-2}$ we get the following balanced minimal state-space realization of the feedback controller $K_2$

$$K_2(s) = G_{smo}^{-1} P Q^{-1}(s) \stackrel{s}{=} \left[ \begin{array}{c|cc} -10 & 188.4 & -75.49 \\ \hline 0 & 306 & -122.6 \\ 203 & -6508 & 2605 \end{array} \right] \quad \text{which achieves} \quad \|S(s)\|_\infty = 3.4691$$

Note the large gain in the controller (large elements in the $D$ matrix). The reason is the small value of $k_0 = 10^{-2}$, $k_0$ must be small to get the $\mathcal{H}_\infty$-norm of $S$ close to the lower bound. Note, it is not surprising that we get large gains in the controller (and large input usage) since no weight has been put on the transfer function $K_2 S$.

Next, consider minimizing the input usage, i.e. to minimize the $\mathcal{H}_\infty$-norm of $K_2 S$. We have two lower bounds on $\|K_2 S(s)\|_\infty$, but they are identical since the bounds are tight. We use the equality $K_2 S = T_I G^{-1}$ and the lower bound (20) with $W = I$ and $V = G^{-1}$, to obtain[6]

$$\|K_2 S(s)\|_\infty \geq \|u_p^H G^{-1} \mathcal{B}_{zi}^{-1}(G_{mi}^{-1})|_{s=p}\|_2 = \|u_p^H G_{so}^{-1}(p)\|_2 = 3.077$$

In (Havre, 1998, Section 5.7) reference tracking is also considered, and the benefit of applying 2-DOF controller when the plant is unstable is illustated.

# 8 Conclusion

- We have derived tight lower bounds on closed-loop transfer functions valid for multivariable plants. The bounds are independent of the controller and therefore reflects the controllability of the plant.
- The bounds extend and generalize the results by Zames (1981), Doyle et al. (1992), Skogestad and Postlethwaite (1996) and the results given in Havre and Skogestad (1998), to also handle non-minimum phase and unstable weights. This allows us to derive *new* lower bounds on input usage due to disturbances, measurement noise and reference changes.
- The new lower bounds on input usage make it possible to *quantify* the minimum input usage for stabilization of unstable plants in the presence of worst case disturbances, measurement noise and reference changes.
- It is proved that the lower bounds are *tight*, by deriving analytical expressions for stable controllers which achieves an $\mathcal{H}_\infty$-norm of the closed-loop transfer functions equal to the lower bound for large classes of systems.
- Theorem 6 expresses the benefit of applying a 2-DOF controller compared to a 1-DOF controller when the plant is unstable and has a RHP-zero.

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[6]We use the following relations: $\mathcal{B}_{zi}(G_{mi}^{-1}) = \mathcal{B}_{po}(G)$ and $G^{-1}\mathcal{B}_{po}^{-1}(G) = G_{so}^{-1}$. The first, follows since the input factorization of RHP-zeros in $G$ does *not* change the output pole directions.

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PhD thesis and MATLAB-software to compute the bounds in Theorems 1 to 4 are available on the internet: `http://www.chembio.ntnu.no/users/skoge`.

# A Proofs of the results

*Proof of Theorem 1.* We prove (17) by applying the following six steps:

1) Factor out RHP-zeros in $WSV$: RHP-poles in $G$ appears as RHP-zeros in $S$. Factor out $S = \widetilde{S}\,\mathcal{B}_{po}(G)$ to obtain

$$\begin{aligned} WSV(s) &= \mathcal{B}_{zo}(W)\, W_{mo}\,\widetilde{S}\,\mathcal{B}_{po}(G)\,V \\ &= \mathcal{B}_{zo}(W)\, \underbrace{W_{mo}\,\widetilde{S}\,(\mathcal{B}_{po}(G)\,V)_{mi}}_{(WSV)_m}\, \mathcal{B}_{zi}(\mathcal{B}_{po}(G)\,V) \end{aligned}$$

$WSV$ is stable by assumption. From the assumption on internal stability it follows that $S$ is stable (if one closed-loop transfer function is stable then internal stability implies that all the other closed-loop transfer functions are stable). Then it is only the RHP-zeros in $S$ which can cancel RHP-poles in $V$ and $W$. So, factorizing the zeros in $\mathbb{C}_+$ of $W$ does not introduce instability in $(WSV)_m$, since none of these cancel unstable modes in $S$ or $V$ . Similarly, we can factorize the zeros in $\mathbb{C}_+$ of $V$. However, when factorizing the zeros in $S$ we must avoid factorizing the zeros which cancel poles in $\mathbb{C}_+$ of $V$. Otherwise, $(WSV)_m$ becomes unstable. By factorizing only the zeros in a minimal realization of $\mathcal{B}_{po}(G)\,V$ we accomplish this. Since $W$ is stable there are no canncellations against the zeros in $S$ due to poles in $G$. It then follows that $(WSV)_m$ is stable.

2) Introduce $f(s) = \max_{\|x_1\|_2=1,\,\|x_2\|_2=1} x_1^H (WSV)_m\, x_2$, then

$$\|WSV(s)\|_\infty = \|(WSV(s))_m\|_\infty \geq \|f(s)\|_\infty$$

3) Apply the maximum modulus theorem to $f(s)$ at the RHP-zeros $z_j$ of $G$

$$\|f(s)\|_\infty \geq |f(z_j)|$$

4) Resubstitute the factorization of RHP-zeros in $S$, i.e. use $\widetilde{S} = S\,\mathcal{B}_{po}^{-1}(G)$

$$\begin{aligned} f(z_j) &= \max_{\|x_1\|_2=1,\,\|x_2\|_2=1} x_1^H W_{mo}\, S\, \mathcal{B}_{po}^{-1}(G)\, (\mathcal{B}_{po}(G)\, V)_{mi}|_{s=z_j} x_2 \\ &= \max_{\|x_1\|_2=1,\,\|x_2\|_2=1} x_1^H W_{mo}\, S\, V\, \mathcal{B}_{zi}^{-1}(\mathcal{B}_{po}(G)\, V)|_{s=z_j} x_2 \end{aligned}$$

5) Use the interpolation constraint for RHP-zeros $z_j$ in $G$, i.e. use $y_{z_j}^H S(z_j) = y_{z_j}^H$

$$\begin{aligned} f(z_j) &= \max_{\|x_1\|_2=1,\,\|x_2\|_2=1} x_1^H W_{mo}\, S\, V \mathcal{B}_{zi}^{-1}(\mathcal{B}_{po}(G)\, V)|_{s=z_j} x_2 \\ &\geq \max_{\|x_1\|_2=1,\,\|x_2\|_2=1} x_1^H W_{mo}\, y_{z_j} y_{z_j}^H S\, V\, \mathcal{B}_{zi}^{-1}(\mathcal{B}_{po}(G)\, V)|_{s=z_j} x_2 \\ &= \max_{\|x_1\|_2=1,\,\|x_2\|_2=1} x_1^H W_{mo}\, y_{z_j} y_{z_j}^H V\, \mathcal{B}_{zi}^{-1}(\mathcal{B}_{po}(G)\, V)|_{s=z_j} x_2 \end{aligned}$$

6) Evaluate the lower bound

$$\|WSV(s)\|_\infty \geq |f(z_j)| \geq \|W_{mo}(z_j) y_{z_j}\|_2 \cdot \|y_{z_j}^H V \mathcal{B}_{zi}^{-1}(\mathcal{B}_{po}(G)\, V)|_{s=z_j}\|_2$$

Since these steps apply to all RHP-zeros in $G$, the bound (17) follows. □