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Full Papers

Effect of RHP zeros and poles on the sensitivity functions in multivariable systems

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This paper examines the implications of Right Half Plane (RHP) zeros and poles on performance of multivariable feedback systems. The results quantify the fundamental limitations imposed by RHP-zeros and poles in terms of lower bounds on the peaks in the weighted sensitivity and complementary sensitivity functions. © 1998 Elsevier Science Ltd. All rights reserved

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It is well known that the presence of RHP ('unstable') zeros and poles pose fundamental limitations on the achievable control performance. This was quantified for SISO systems by Bode[1] more than 50 years ago, and most control engineers have an intuitive feeling on the limitations for scalar systems. Rosenbrock[2,3] pointed out that multivariable RHP-zeros pose similar limitations. Nevertheless, the quantification of the effect of RHP-zeros and poles on closed-loop performance has been much more difficult for MIMO than for SISO systems. Important reasons are:

1. The definition of phase is difficult to generalize to MIMO-systems.
2. The directionality of zeros and poles in multivariable systems has not been well understood.

The goal of this paper is therefore to address the following questions:

1. How is closed-loop performance influenced by the location of the RHP-zeros and poles in MIMO-systems?
2. How is closed-loop performance influenced by the directionality of the RHP-zeros and poles in MIMO-systems?
3. How is closed-loop performance influenced by the combined effect of RHP-zeros and poles and their directions?

We will mainly quantify the fundamental limitations imposed by RHP-zeros and poles in terms of lower bounds on the peaks ($\mathcal{H}_\infty$– norm) in the closed-loop transfer functions S (sensitivity) and T (complementary sensitivity).

Why consider peaks in S and T?

Figure 1 shows a one degree-of-freedom feedback control configuration. The plant G and the controller K interconnection is driven by the reference commands r, disturbances d and measurement noise n. The outputs to be controlled are y, and u are the manipulated variables. We assume that the performance is measured at the output of the plant G in terms of the error signal $e \triangleq y - r$. For the closed-loop system we have the following important relationships:

$$y(s) = T(s)r(s) + S(s)d(s) - T(s)n(s) \tag{1}$$

$$e(s) = -S(s)r(s) + S(s)d(s) - T(s)n(s) \tag{2}$$

$$u(s) = K(s)S(s)(r(s) - n(s) - d(s)) \tag{3}$$

where sensitivity and complementary sensitivity functions are defined by

$$S(s) \triangleq (I + L(s))^{-1} \tag{4}$$

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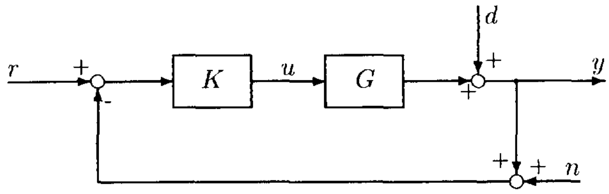

Figure 1 One degree-of-freedom feedback control configuration

$$T(s) \triangleq L(s)(I+L(s))^{-1} = L(s)S(s) = I - S(s) \quad (5)$$

and $L \triangleq GK$ is the loop transfer function. The relationships (1)–(3) imply several closed-loop objectives, in addition to the requirement the K should stabilize G[4]:

1. For *disturbance rejection* make $\bar{\sigma}(S)$ small.
2. For *noise attenuation* make $\bar{\sigma}(T)$ small.
3. For *reference tracking* make $\bar{\sigma}(T) \approx \underline{\sigma}(T) \approx 1$.
4. For *control energy reduction* make $\bar{\sigma}(KS)$ small.

If the unstructured uncertainty in the plant model G is represented by an additive perturbation, i.e. $G_p = G + \Delta$, then a further closed-loop objective is

5. For *robust stability* in the presence of an additive perturbation make $\bar{\sigma}(KS)$ small.

Alternatively, if the uncertainty is modelled by a multiplicative output perturbation such that $G_p = (I+\Delta)G$, then we have:

6. For *robust stability* in the presence of a multiplicative output perturbation make $\bar{\sigma}(T)$ small.

The condition $S + T = I$ holds for MIMO-systems, and it then follows that we cannot have both S and T small simultaneously, and that $\bar{\sigma}(S)$ is large if and only if $\bar{\sigma}(T)$ is large.

Typical plots of the maximum singular values $\bar{\sigma}(S(j\omega))$ and $\bar{\sigma}(T(j\omega))$ are shown in *Figure 2*. For those frequencies where $\bar{\sigma}(S(j\omega)) > 2$, we have more than 100% control error and for those frequencies where $\bar{\sigma}(T(j\omega)) > 2$, we have more than 100% amplification of the noise. The peaks $\| S(s) \|_\infty$ and $\| T(s) \|_\infty$ therefore tell us a great deal about the performance of the

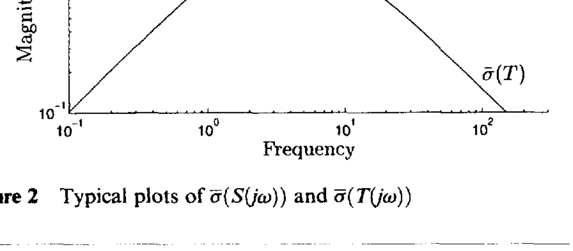

Figure 2 Typical plots of $\bar{\sigma}(S(j\omega))$ and $\bar{\sigma}(T(j\omega))$

*The term 'controllability' is here used in a wider sense than the meaning of state-controllability; see Ref 5, Definition 5.1, p. 160 and the discussion on p. 123.

feedback system for the worst case direction and the worst case frequency. Although S and T depend on the controller K, the lower bounds on $\| S(s) \|_\infty$ and $\| T(s) \|_\infty$ derived in this paper are independent of K. If the lower bounds are large (typically larger than 4) then the plant G is fundamentally difficult to control, i.e. the 'controllability'* of the plant G is poor. In this paper we look at the combined effect of RHP-zeros and poles and we show that the lower bounds on $\| S(s) \|_\infty$ and $\| T(s) \|_\infty$ can become quite large when the plant contains both RHP-zeros and poles. Finally, it should be noted that there are also other fundamental limitations on performance than those imposed by RHP-zeros and poles, (see e.g. Ref. 5).

Notation

We consider linear time invariant dynamical systems on state–space form

$$\dot{x} = Ax + Bu \quad (6)$$

$$y = Cx + Du \quad (7)$$

In Equations (6) and (7), u are the external inputs x are the states and y are the outputs. A, B, C and D are real matrices of dimensions $n \times n$, $n \times m$, $l \times n$ and $l \times m$ where n is the number of states, m is the number of inputs and l is the number of outputs. The short-hand notations

$$G \stackrel{s}{=} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ and } (A, B, C, D) \quad (8)$$

are frequently used to describe a linear state–space model of the continuous system G given by Equations (6) and (7). The rational transfer function matrix $G(s)$ (of size $l \times m$) defined by Equation (8) can be evaluated as a function of the complex variable s,

$$G(s) = C(sI - A)^{-1}B + D \quad (9)$$

We often omit to show the dependence on the complex variable s for transfer functions. We consider the feedback control configuration shown in *Figure 1* with the closed loop transfer functions given in Equations (1–3) where the sensitivity and the complementary sensitivity functions (S and T) are defined by Equations (4) and (5). With the term 'peak of a rational transfer function matrix' we mean its $\mathcal{H}_\infty$-norm, defined as (see also *Figure 2*)

$$\| M(s) \|_\infty \triangleq \sup_\omega \bar{\sigma}(M(j\omega)) \quad (10)$$

RHP-zeros z and poles p are in this paper defined to be in the closed RHP, denoted $\overline{\mathbb{C}}_+$, i.e. $z \in \overline{\mathbb{C}}_+$ implies $\mathrm{Re}\ z \geq 0$, and $p \in \overline{\mathbb{C}}_+$ implies $\mathrm{Re}\ p \geq 0$. However, for some of the results in this paper the location of some RHP-zeros or RHP-poles are restricted to be in the open RHP, denoted $\mathbb{C}_+$, i.e. $z \in \mathbb{C}_+$ implies $\text{Re}\, z > 0$, and $p \in \mathbb{C}_+$ implies $\text{Re}\, p > 0$.

Outline

The outline of the paper is as follows. First we give an literature overview and then we discuss zeros and poles of multivariable systems and their directions. We derive constraints on the sensitivity and the complementary sensitivity functions imposed by RHP-zeros and poles. Next, we consider the lower bounds on the peak in the weighted sensitivity and complementary sensitivity functions. At the end we give a multivariable example and a conclusion. All proofs are given in Appendix A.

Previous work on limitations imposed by RHP-zeros and poles

Bode[1], in his book on network analysis and feedback amplifiers, was probably the first to study *a priori* constraints on the achievable performance of SISO-systems. His analysis was focused on gain-phase relationships in the frequency domain which resulted in many useful interpretations applicable to feedback control. Horowitz[6] summarizes and generalizes Bode's work to control systems. The well-known *Bode sensitivity integral*[1] states that for stable SISO-systems with pole-zero excess of two or larger, the integral of the logarithmic magnitude of the sensitivity function over all frequencies must equal zero

$$\int_0^\infty \ln |S(j\omega)| d\omega = 0 \tag{11}$$

This implies that a peak in $|S|$ larger than 1 is unavoidable. Later, Bode's criterion has been extended to plants with RHP-zeros and poles by Freudenberg and Looze[7,8]. From these results it is clear that even larger peaks are expected when the plant contains RHP-zeros and/or RHP-poles.

A related result from optimal control theory is the *Kalman inequality*[9]

$$\bar{\sigma}(S_x(j\omega)) \leq 1, \forall \omega \tag{12}$$

where $S_x \triangleq (I + K(sI - A)^{-1}B)^{-1}$ and K is the optimal state feedback gain matrix. The Kalman inequality is valid for both stable and unstable MIMO-systems under optimal state feedback control, with diagonal weight on the manipulated variables in the performance objective (Ref. 5, pp. 357–358). This inequality is not in conflict with the Bode's sensitivity integral nor with the extended version valid for RHP-zeros. The reason for this is that optimal control with state feedback yields a loop transfer function with a pole-zero excess of one so Bode's sensitivity integral does not apply. Secondly, there are *no* RHP-*zeros* when all the states are measured so the extended Bode's sensitivity integral can not be applied.

The combination of no RHP-zeros when all the states are measured, and the introduction of optimal control theory (i.e. the Kalman inequality) may have had a misleading role in multivariable feedback design, which resulted in that very little attention where given to multivariable zeros during the 1960s and 1970s. As one example, in their book, Anderson and Moore[10] do not mention the effect of zeros on closed-loop performance for multivariable system at all. However, some quantification of the effect of RHP-zeros has been made during the 1970s. For MIMO systems Kwakernaak and Sivan (Ref. 11, pp. 306–307) state that perfect tracking with state feedback can be achieved if and only if the rational transfer function matrix from the inputs to the outputs has no RHP-zeros.

Zames and coworkers[12–16] consider minimizing the $\mathcal{H}_\infty$-norm of the sensitivity matrix multiplied by suitable weighting matrices. In Zames[12] it is shown how feedback can reduce the weighted sensitivity and in particular how the weighted sensitivity can be made arbitrarily small whenever the plant has no RHP-zeros. In Zames and Bensoussan[14] an alternative approach is developed which is not dependent on *a priori* parameterization, but specialized to diagonal feedback. Zames[12] derives a lower bound on the weighted sensitivity function (see Theorem 5 below), which is based on the interpolation constraint on the sensitivity function valid for RHP-zeros in G. The results in this paper are based on this and a similar interpolation constraint on the complementary sensitivity function valid for RHP-poles in G. We then follow much of the same approach made by Zames to derive the lower bounds.

Boyd and Desoer[17], Freudenberg and Looze[8], Boyd and Barratt[18] and Chen[19] have studied the limitations imposed by RHP-zeros and poles in terms of *sensitivity integral formulas* for MIMO-systems. A breakthrough was made by Boyd and Desoer who obtained inequality versions of the sensitivity and Poisson integral formulas, based on the recognition that the logarithm of the largest singular value of an analytic transfer function matrix is a subharmonic function. The work by Chen differs from the work by Boyd and Desoer in that Chen seeks equality versions of the sensitivity and Poisson integral formulas. Based on the results by Boyd and Desoer, Freudenberg and Looze and Boyd and Barratt generalize the integral constraints on the sensitivity (like Bode's sensitivity integral) to MIMO-systems. Although these integral relationships are interesting, it seems difficult to derive any concrete bounds on achievable performance from them. However, for the case when G has one RHP-zero z with output direction y_z and one RHP-pole p with output direction y_p, the following bound is given by Boyd and Desoer[17]

$$\| S(s) \|_\infty \geq \frac{|z + \bar{p}|}{|z - p|} \cos \angle(y_p, y_z) \tag{13}$$

where $\angle(y_p, y_z)$ is the angle between the pole and zero directions y_p and y_z.

The following similar but improved bound for the same case (one RHP-zero and one RHP-pole) is given by Chen[19,20].

$$\| S(s) \|_\infty \geq e^{Q(z)/\pi)} \sqrt{\sin^2 \angle(y_p, y_z) + \frac{|z+\bar{p}|^2}{|z-p|^2} \cos^2 \angle(y_p, y_z)} \quad (14)$$

where

$$Q(z) \triangleq \frac{1}{2} \int\int_{\bar{\mathbb{C}}_+} \log \left| \frac{x+jy+\bar{z}}{x+jy-z} \right| \nabla^2 \log \bar{\sigma}(S_m(x+jy)) dx dy \quad (15)$$

and $Q(z) \geq 0$ (see the text in Chen[19] following proof of Corollary 5.1 on p. 1712). Note that the factor $Q(z)$ can not be evaluated without knowledge about the controller K, and even when K is known it is hard to evaluate $Q(z)$. In any case, it appears[21] that $Q(z) = 0$ for the optimal controller minimizing $\| S(s) \|_\infty$. Using algebraic rather than integral constraints, we derive in this paper a tight bound which is similar to Equation (14) with $Q(z) = 0$. However, the bounds presented here extend Equation (14) to the case where the plant G has more than one RHP-pole (Theorem 7). Furthermore, we derive similar results in terms of the weighted complementary sensitivity $\| w_T T(s) \|_\infty$ for the case where the plant G has one or more RHP-poles and any number of RHP-zeros (Theorem 8).

Zeros and poles of multivariable systems

Zeros

Rosenbrock[3], Kailath[22] and Zhou *et al.*[23] all define the zeros as the roots of the non-zero numerator polynomials in the Smith–McMillan form. A slightly different approach which yield the same set of zeros is taken by Desoer and Schulman[24]. They consider a left coprime polynomial matrix factorization of $G(s)$, $G(s) = D_l^{-1}(s)N_l(s)$ and define the zeros as the complex numbers z where the rank of $N_l(z)$ is less than the normal rank of $N_l(s)$. This is similar to the definition we use, which is taken from MacFarlane and Karcanias[25]:

Definition 1 (Zeros): $z_i \epsilon \mathbb{C}$ is a zero of $G(s)$ if the rank of $G(z_i)$ is less than the normal rank of $G(s)$.

The normal rank of $G(s)$ is defined as the rank of $G(s)$ at all s except a finite number of singularities (which are the zeros). This definition of zeros is based on the transfer function matrix, corresponding to a minimal realization of a system. These zeros are sometimes called 'transmission zeros'[25], but we shall simply call them 'zeros'.

Definition 2 (Zero Directions): If $G(s)$ has a zero for $s = z \epsilon \mathbb{C}$ then there exist non-zero vectors labeled the output zero direction $y_z \epsilon \mathbb{C}^l$ and the input zero direction $u_z \epsilon \mathbb{C}^m$, such that $y_z^H y_z = 1$, $u_z^H u_z = 1$ and

$$G(z) u_z = 0; \quad y_z^H G(z) = 0 \quad (16)$$

The definitions of input and output zero directions can further be extended with the state input and output zero vectors through the use of generalized eigenvalues. For a system $G(s)$, the zeros z of the system, the input zero directions u_z and the state input zero vectors $\mathbf{x}_{zo} \epsilon \mathbb{C}^n$ can all be computed from the generalized eigenvalue problem

$$\begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{x}_{zi} \\ u_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (17)$$

In this setup we normalise the length of u_z, i.e. $u_z^H u_z = 1$. This imply that the length of $\mathbf{x}_{zi}$ is different from one.*

Similarly, one can compute the zeros z, the output zero directions y_z and the state output zero vectors $\mathbf{x}_{zo} \epsilon \mathbb{C}^n$ through the generalized eigenvalue problem

$$\begin{bmatrix} \mathbf{x}_{zo}^H & y_z^H \end{bmatrix} \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad (18)$$

with the length of y_z is normalised, so that $y_z^H y_z = 1$. Let (A, B, C, D) be a minimal realization of $G(s)$, computing the zeros from the eigenvalue problems (17) and (18) yields the 'transmission zeros'[25].

Poles

Rosenbrock[3], MacFarlane and Karcanias[25], Callier and Desoer[26] and Zhou *et al.*[23] all define the poles as the roots of the denominator polynomials in the Smith–McMillan form of $G(s)$. For a linear time invariant system with minimal state–space description Equations (6) and (7), these roots corresponds to the eigenvalues of the A matrix (Ref. 26, pp. 75–78). Thus, the poles are the roots of the characteristic equation

$$\phi(s) = \det(sI - A) = \prod_{i=1}^{n} (s - p_i) = 0 \quad (19)$$

Bode[1] states that the poles are the singular points at which the transfer function fails to be analytic. The singularities appear in the denominator so when the system G is evaluated† at $s = p$, $G(p)$ is infinite in some directions at the input and the output. This is the basis for the following definition of input and output pole directions.

Definition 3 (Pole Directions): Let $s = p \epsilon \mathbb{C}$ be a distinct pole of $G(s)$ then there exist unique input and output directions $u_p \epsilon \mathbb{C}^m$ and $y_p \epsilon \mathbb{C}^l$ such that

$$G(p) u_p = \infty; \quad y_p^H G(p) = \infty \quad (20)$$

More precisely $G(p) u_p = y_p \cdot \infty$ and $y_p^H G(p) = u_p^H \cdot \infty$.

*That $\| \mathbf{x}_{zi} \|_2$ is generally different from 1 is the primary reason why we denote $\mathbf{x}_{zi}$ vector and not a direction.

†Strictly speaking, the transfer function $G(s)$ cannot be evaluated at $s = p$, since $G(s)$ is not analytic at $s = p$.

The following result shows how to compute the pole directions for a system with state–space realization.

Lemma 4 (Pole Directions): For a system G with a minimal state–space realization (A, B, C, D), the pole directions associated with the distinct pole $p \in \mathbb{C}$ can be computed from

$$u_p = B^H x_{pi} / \| B^H x_{pi} \|_2; \quad y_p = C x_{po} / \| C x_{po} \|_2 \tag{21}$$

where $x_{pi} \in \mathbb{C}^n$ and $x_{po} \in \mathbb{C}^n$ are the eigenvectors corresponding to the two eigenvalue problems

$$x_{pi}^H A = x_{pi}^H p; \quad A x_{po} = p x_{po}$$

Constraints on S and T

To have internal stability, we cannot allow right half plane pole-zero cancellations between the plant and controller, and this may be formulated as 'interpolation constraints' on closed-loop transfer functions, such as S and T. For MIMO-systems these interpolation constraints have directions.

Constraint 1 (RHP-zero): If $G(s)$ has a RHP-zero at $s = z$ with output zero direction y_z, then for internal stability of the feedback system the following interpolation constraints must apply

$$y_z^H T(z) = 0; \quad y_z^H S(z) = y_z^H \tag{22}$$

In words, Equation (22) says that T must have a RHP-zero in the same direction as G and that $S(z)$ has an eigenvalue of 1 with corresponding left eigenvector y_z.

Constraint 2 (RHP-pole): If $G(s)$ has a RHP-pole at $s = p$ with output direction y_p, then for internal stability of the feedback system the following interpolation constraints must apply

$$S(p) y_p = 0; \quad T(p) y_p = y_p \tag{23}$$

Similar constraints apply to S_I and T_I, but these are in terms of the input zero and pole directions, u_z and u_p.

All-pass factorizations of RHP-zeros and poles

A transfer function matrix $B(s)$ is all-pass if $B(-s)^T B(s) = I$, which implies that all singular values of $B(j\omega)$ are equal to one.

A plant $G(s)$ with RHP-poles $p_i \in \mathbb{C}_+$ and RHP-zeros $z_j \in \mathbb{C}_+$, can be factorized at the *output* as follows*

*Note that the notation on the all-pass factorizations of RHP-zeros and poles used in this paper is reversed compared to the notation used in Refs 5, 27 and 28. The reason to this change of notation is to get consistent with what the literature generally defines as an all-pass filter.

$$G(s) = \mathcal{B}_{po}^{-1}(G(s)) G_{so}(s); \quad G(s) = \mathcal{B}_{zo}(G(s)) G_{mo}(s) \tag{24}$$

where G_{mo} is minimum phase, G_{so} is stable, and $\mathcal{B}_{po}(G)$ and $\mathcal{B}_{zo}(G)$ are stable all-pass rational transfer function matrices where $\mathcal{B}_{po}(G)$ contains the RHP-poles of G as RHP-zeros and $\mathcal{B}_{zo}(G)$ contains the RHP-zeros of G. $\mathcal{B}_{po}(G)$ is obtained by factorizing at the output one RHP-pole at a time, starting with $G(s) = \mathcal{B}_{p1}^{-1}(G) G_{p1}(s)$ where

$$\mathcal{B}_{p1}^{-1}(G) = I + \frac{2Re(p_1)}{s - p_1} \hat{y}_{p1} \hat{y}_{p_1}^H$$

and $\hat{y}_{p_1} = y_{p_1}$ is the output pole direction for p_1. This procedure may be continued to factor out p_2 from $G_{p1}(s)$ where $\hat{y}_{p2}$ is the output pole direction of G_{p1} (which need not coincide with y_{p2}, the pole direction of G), and so on. A similar procedure may be used for the RHP-zeros. We get[29]

$$\mathcal{B}_{po}(G) = \prod_{i=N_p}^{1} (I - \frac{2\mathrm{Re}(p_i)}{s + \bar{p}_i} \hat{y}_{pi} \hat{y}_{pi}^H); \qquad \mathcal{B}_{po}^{-1}(G) = \prod_{i=1}^{N_p} (I + \frac{2\mathrm{Re}(p_i)}{s - p_i} \hat{y}_{pi} \hat{y}_{pi}^H) \tag{25}$$

$$\mathcal{B}_{zo}(G) = \prod_{j=1}^{N_z} (I - \frac{2\mathrm{Re}(z_j)}{s + \bar{z}_j} \hat{y}_{zj} \hat{y}_{zj}^H); \qquad \mathcal{B}_{zo}^{-1}(G) = \prod_{j=N_z}^{1} (I + \frac{2\mathrm{Re}(z_j)}{s - z_j} \hat{y}_{zj} \hat{y}_{zj}^H) \tag{26}$$

If $N_z = 0$ we define $\mathcal{B}_{zo}(G) = I$ and if $N_p = 0$ define $\mathcal{B}_{po}(G) = I$.

For further details regarding the state–space realizations of the factorizations and properties of the all-pass filters, see Ref. 29. The output factorization of RHP-zeros is also given in Ref. 23, p. 145 and in Refs 19 and 20. It can be traced back to Wall *et al.*[30] We note that similar factorizations of RHP-zeros and poles apply at the plant input.

Alternative all-pass factorizations are in use, e.g. the inner-outer factorization used in Morari and Zafiriou[31] which is the same as Equation (26) except for the multiplication of a constant unitary matrix. Reasons for using the factorizations (25) and (26) are:

1. The factorization of RHP-zeros given here is analytic and in terms of the zeros and the zero directions, whereas the inner-outer factorization in Morari and Zafiriou[31] is given in terms of the solution to an algebraic Riccati equation.
2. To factorize RHP-poles using the inner-outer factorization one needs to assume that G^{-1} exit.

Lower bounds on $\| w_P S(s) \|_\infty$ and $\| w_T T(s) \|_\infty$

Limitations imposed by RHP-zeros

The following result is originally from Zames[12] and it is based on the interpolation constraints imposed by RHP-zeros in G.

Theorem 5 (RHP-zero and $\| w_P S(s) \|_\infty$): Suppose the plant $G(s)$ has an RHP-zero at $s = z$. Let $w_P(s)$ be a scalar stable transfer function. Then for closed-loop stability the weighted sensitivity function must satisfy

$$\| w_P S(s) \|_\infty \geq |w_P(z)| \tag{27}$$

The proof of Theorem 5 is given in Zames[17]. Condition Equation (27) shows that there are inherent performance limitations imposed by RHP-zeros. It involves the maximum singular value, $\| w_P S(s) \|_\infty = \sup_\omega \bar{\sigma}(w_P S(j\omega))$, which is the 'worst' direction, and the RHP-zero may therefore not be a limitation in the other directions.

Limitations imposed by RHP-poles

The following 'symmetric' result is based on the interpolation constraints imposed by RHP-poles in G. It extends the SISO result given in Doyle *et al.*[32]

Theorem 6 (RHP-pole and $\| w_T T(s) \|_\infty$): Suppose the plant $G(s)$ has an RHP-pole at $s = p$. Let $w_T(s)$ be a scalar stable transfer function. Then for closed-loop stability the weighted complementary sensitivity function must satisfy

$$\| w_T T(s) \|_\infty \geq |w_T(p)| \tag{28}$$

RHP-poles combined with RHP-zeros

By considering the effect of one RHP-zero and one RHP-pole separately we derived in Equations (27) and (28) the conditions

$$\| w_P S(s) \|_\infty \geq c_1 |w_P(z)| \tag{29}$$

$$\| w_T T(s) \|_\infty \geq c_2 |w_T(p)| \tag{30}$$

with $c_1 = c_2 = 1$. These conditions may be optimistic in that the lower bound may be too small, and indeed we show that $c_1 > 1$ and $c_2 > 1$ for the case when we have both an RHP-zero and an RHP-pole with some alignment in the same direction.

Theorem 7 (MIMO Sensitivity Peak): Suppose the plant $G(s)$ has $N_z \geq 1$ RHP-zeros z_j with output directions y_{zj} and $N_p \geq 0$ RHP-poles $p_i \epsilon \mathbb{C}_+$ with output directions y_{pi}. Let the performance weight w_P be a scalar stable minimum phase transfer function. Define the all-pass transfer function matrix in Equation (25). Then for closed-loop stability the weighted sensitivity function must satisfy

$$\begin{aligned} &\| w_P S(s) \|_\infty \geq \max_{zj} c_{1,j} |w_p(z_j)| \\ &\text{where } c_{1,j} = \| y_{zj}^H \mathcal{B}_{po}^{-1}(G)|_{s=zj} \|_2 \geq 1 \end{aligned} \tag{31}$$

Theorem 8 (MIMO Complementary Sensitivity Peak): Suppose the plant $G(s)$ has $N_z \geq 0$ RHP-zeros $z_j \epsilon \mathbb{C}_+$ with output directions y_{z_j} and $N_p \geq$ RHP-poles p_i with output directions y_{p_i}. Let the performance weight w_T be a scalar stable minimum phase transfer function. Define the all-pass transfer function matrix in Equation (26). Then for closed-loop stability the weighted complementary sensitivity function must satisfy

$$\begin{aligned} &\| w_T T(s) \|_\infty \geq \max_{pi} c_{2,i} |w_T(p_i)| \\ &\text{where } c_{2,i} = \| \mathcal{B}_{zo}^{-1}(G)|_{s=pi} y_{p_i} \|_2 \geq 1 \end{aligned} \tag{32}$$

Note that $c_{1,j}$ and $c_{2,i}$ are independent of the controller K and only depend on the location of RHP-zeros and poles and their directions. As we shall see in the examples the values of $c_{1,j}$ and $c_{2,i}$ can be much larger than one when the plant has both an RHP-zero and an RHP-pole located close to each other and with some alignment in their directions.

For the special case with one RHP-zero and one RHP-pole we have the following result.

Corollary 9 (One RHP-zero and one RHP-pole): Given the system $G(s)$ with one RHP-pole and one RHP-zero. In this case the constants c_1 and c_2 in Equations (31) and (32) are given by the equation

$$c = c_1 = c_2 = \sqrt{\sin^2(\phi) + \frac{|z+p|^2}{|z-p|^2}\cos^2(\phi)} \geq 1 \tag{33}$$

where $\phi = \cos^{-1}(|y_z^H y_p|)$.

For SISO-systems, Theorems 7 and 8 become:

Corollary 10 (SISO Sensitivity Peak): Let the $G(s)$ be a SISO-system with $N_z \geq 1$ RHP-zeros z_j and $N_p \geq 0$ RHP-poles $p_i \epsilon \mathbb{C}_+$. Let the performance weight w_P be a stable minimum phase transfer function. Then for closed-loop stability the weighted sensitivity function must satisfy

$$\begin{aligned} &\| w_P S(s) \|_\infty \geq \max_{zj} c_{1,j} |w_P(z_j)| \\ &\text{where } c_{1,j} = \prod_{i=1}^{N_p} \frac{|z_j + \bar{p}_i|}{|z_j - p_i|} \geq 1 \end{aligned} \tag{34}$$

Corollary 11 (SISO Complementary Sensitivity Peak): Let the $G(s)$ be a SISO-system with $N_z \geq 0$ RHP-zeros $z_j \epsilon \mathbb{C}_+$ and $N_p \geq 1$ RHP-poles p_i.

Let the performance weight w_T be a stable minimum phase transfer function. Then for closed-loop stability the weighted complementary sensitivity function must satisfy

$$\| w_T T(s) \|_\infty \geq \max_{pi} c_{2,i} |w_T(p_i)|$$

$$\text{where } c_{2,i} = \prod_{j=1}^{N_z} \frac{|\bar{z}_j + p_i|}{|z_j - p_i|} \geq 1 \tag{35}$$

Equations (34) and (35) follow easily from Theorems 7 and 8 by setting the zero and pole directions equal to 1 and assuming that all RHP-poles are observable and all RHP-zeros are 'transmission zeros'.

Peaks in S and T. From Theorem 7 we get by selecting $w_P(s) = 1$

$$\| S(s) \|_\infty \geq \max_{RHP\text{-}zeros,zj} c_{1,j} \tag{36}$$

and from Theorem 8 we get by selecting $w_T(s) = 1$

$$\| T(s) \|_\infty \geq \max_{RHP\text{-}poles,pi} c_{2,i} \tag{37}$$

Thus, a peak for $\bar{\sigma}(S(j\omega))$ and $\bar{\sigma}(T(j\omega))$ larger than 1 is unavoidable if the plant has both an RHP-zero and an RHP-pole (unless their relative angle ϕ is 90°).

Example

We consider the following plant

$$G(s) = \begin{bmatrix} \frac{1}{s-p} & 0 \\ 0 & \frac{1}{s+p} \end{bmatrix} \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}_{U_\alpha} \begin{bmatrix} \frac{s-z}{0.1s+1} & 0 \\ 0 & \frac{s+z}{0.1s+1} \end{bmatrix}; \quad z = 2, p = 3$$

which has an RHP-zero at $z = 2$ and an RHP-pole at $p = 3$. For $\alpha = 0°$ the rotation matrix $U_\alpha = I$, and the plant consists of two decoupled subsystems

$$G_0(s) = \begin{bmatrix} \frac{s-z}{(0.1s+1)(s-p)} & 0 \\ 0 & \frac{s+z}{(0.1s+1)(s+p)} \end{bmatrix}$$

The subsystem g_{11} has both an RHP-zero and an RHP-pole, and closed-loop performance is expected to be poor. On the other hand, there are no particular control problems related to subsystem g_{22}. With $\alpha = 90°$, $U_\alpha = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which gives

$$G_{90}(s) = \begin{bmatrix} 0 & -\frac{s+z}{(0.1s+1)(s-p)} \\ \frac{s-z}{(0.1s+1)(s+p)} & 0 \end{bmatrix}$$

we have again two decoupled subsystems, but this time in the off-diagonal elements. The main differences is that there is no interaction between the RHP-zero and the RHP-pole in this case, so we expect this plant to be easier to control. For other values of α we do not have decoupled subsystems, and there will be some interaction between the RHP-zero and the RHP-pole. Since the pole is located at the output of the plant, its output direction is fixed, we find $y_p = [1 \quad 0]^T$ for all values of α. On the other hand, the zero direction changes from $[1 \quad 0]^T$ for $\alpha = 0°$ to $[0 \quad 1]^T$ for $\alpha = 90°$. Thus, the angle between the pole and zero direction, ϕ, will also vary between 0° and 90° as α varies from 0° to 90°, as seen from *Table 1*, where we also give c_1 and c_2 for four rotation angles, $\alpha = 0°$, 30°, 60° and 90°. The table also shows the values of $\| S(s) \|_\infty$ and $\| T(s) \|_\infty$ using $\mathcal{H}_\infty$-optimal controllers minimizing

$$\min_K \left\| \begin{bmatrix} w_P S \\ w_u KS \end{bmatrix} \right\|_\infty \text{ with } w_u = 1; \; w_P = \left(\frac{s/M + \omega_B^*}{s}\right) I \tag{38}$$

where $M = 2$ and $\omega_B^* = 0.5$. The weight w_P for the weighted sensitivity means that we require $\| S(s) \|_\infty$ less than 2, and require tight control up to a frequency of about $\omega_B^* = 0.5$ rad s^{-1}. The minimum $\mathcal{H}_\infty$-norm for the stacked S/KS problem Equation (38), is given by the value of γ in *Table 1*. Plots of the sensitivity S and the complementary sensitivity T are given in *Figure 3*. The responses to the step change in the reference $r = [1 \; -1]^T$ are shown in *Figure 4*. Several things are worth noting:

1. We see from the simulation for $\phi = 0°$ in *Figure 4* that the response for y_1 is very poor. This is as expected because of the closeness of the RHP-zero and pole ($z = 2, p = 3$).
2. The bound c_1 on $\| S(s) \|_\infty$ in Equation (36) is tight in this case. This can be shown numerically by selecting $w_u = 0.01$, $\omega_B = 0.01$ and $M_s = 1$ (w_u and ω_B are small so the main objective is to minimize the peak of S). We find that the $\mathcal{H}_\infty$-designs for the four angles yield

α	0°	30°	60°	90°
$\| S(s) \|_\infty$	5.04	1.905	1.155	1.005
c_1	5.0	1.89	1.15	1.0

3. The angle ϕ between the zero and the pole directions, is quite different from the rotation angle α at intermediate values between 0° and 90°. The reason for this is the influence of the RHP-pole in output 1, which yields a strong gain in this

Table 1 Results

		α			
		0°	30°	60°	90°
	y_z	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.33 \\ -0.94 \end{bmatrix}$	$\begin{bmatrix} 0.11 \\ -0.99 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
	$\phi = \cos^{-1}\|y_z^H y_p\|$	0°	70.9°	83.4°	90°
	$c = c_1 = c_2$	5.0	1.89	1.15	1.0
$\mathcal{H}_\infty$–	$\|S(s)\|_\infty$	7.00	2.60	1.59	1.98
designs	$\|T(s)\|_\infty$	7.40	2.76	1.60	1.31
using	Stable K?	No	No	Yes	Yes
(38)	$\gamma(S/KS)$	9.55	3.53	2.01	1.59

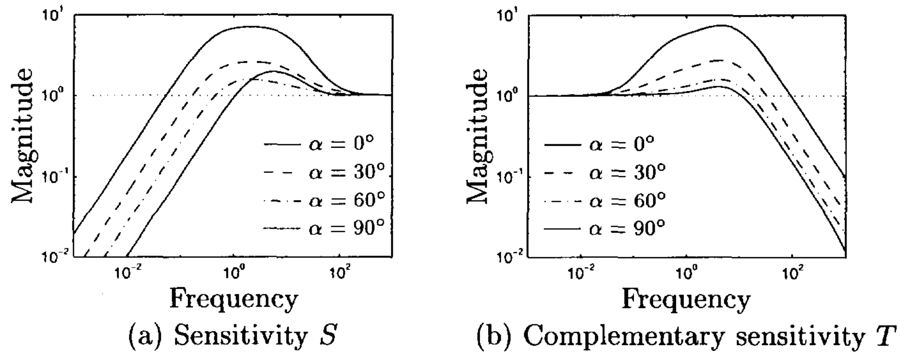

Figure 3 Sensitivity and complementary sensitivity functions for four angles α

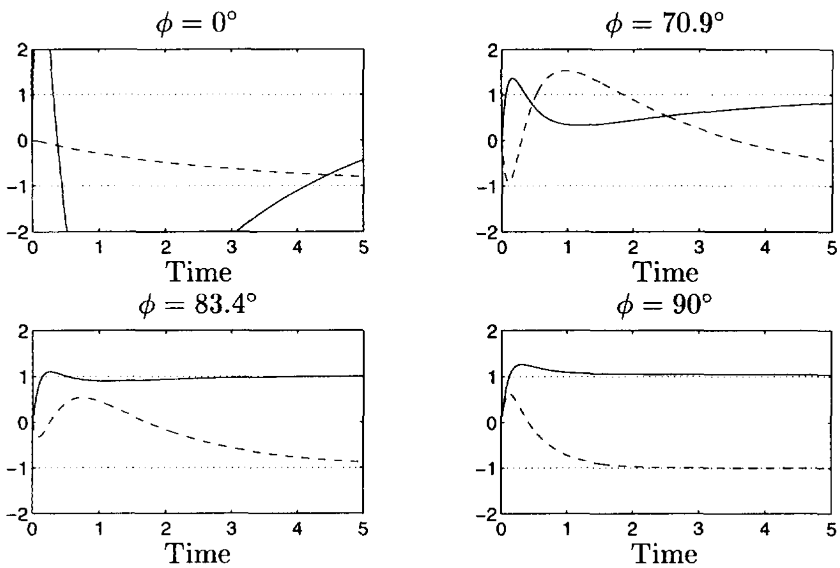

Figure 4 MIMO-plant with angle ϕ between the RHP-zero and the RHP-pole directions. Response to step in reference with $\mathcal{H}_\infty$-controller for four different values of ϕ. Solid line: y_1; dashed line: y_2

direction, and thus tends to push the zero direction toward output 2.

4. For $\alpha = 0°$ we have $c_1 = c_2 = 5$ so $\| S(s) \|_\infty \geq 5$ and $\| T(s) \|_\infty \geq 5$, so it is clearly impossible to get $\| S(s) \|_\infty$ less than 2, as required by the performance weight w_P.
5. The $\mathcal{H}_\infty$-optimal controller is unstable for $\alpha = 0°$ and 30°. This is not surprisingly, because for $\alpha = 0°$ the plant is two SISO-systems one of which needs an unstable controller to stabilize it, since $p > z$.

Conclusion

We have presented lower bounds on the peak in weighted sensitivity and complementary sensitivity functions for systems with RHP-zeros and poles. Peaks in the sensitivity and complementary sensitivity functions are unavoidable if the plant has both a RHP-zero and a RHP-pole with some alignment. These lower bounds on the sensitivity functions demonstrate the fundamental limitations imposed by open-loop characteristics as RHP-zeros and poles. The intentions with the derivation of these lower bounds are:

- To derive measures which quantify the effect of open-loop RHP-zeros and poles has on closed loop performance. These measures are independent of the controller and the control configuration and therefore reflects the controllability of the plant.
- To get better understanding of the directionality of RHP-zeros and poles.

We also expect that the derived bounds will be useful when selecting performance weights for controller design and analysis.

References

1. Bode, H. W., *Network Analysis and Feedback Amplifier Design*. Van Nostrand, New York, 1945.
2. Rosenbrock, H. H., On the design of linear multivariable systems. In Third IFAC World Congress, 1966, Paper 1A, London, UK.
3. Rosenbrock, H. H., *State-space and Multivariable Theory*. Nelson, London, 1970.

4. Doyle, J. C. and Stein, G., Multivariable feedback design: Concepts for a classical/modern synthesis. *IEEE Transactions on Automatic Control*, 1981, **AC-26**(1), 4–16.
5. Skogestad, S. and Postlethwaite, I., *Multivariable Feedback Control, Analysis and Design*. John Wiley & Sons, Chicester, UK, 1996.
6. Horowitz, I. M., *Synthesis of Feedback Systems*. Academic Press, London, 1963.
7. Freudenberg, J. S. and Looze, D. P., Right half plane poles and zeros and design tradeoffs in feedback systems. *IEEE Transactions on Automatic Control*, 1984, **AC-30**(6), 555–565.
8. Freudenberg, J. S. and Looze, D. P., *Frequency Domain Properties of Scalar and Multivariable Feedback Systems*. Lecture Notes in Control and Information Sciences, Vol. 104. Springer–Verlag, Berlin, 1988.
9. Kalman, R. E., When is a linear control system optimal? *Journal of Basic Engineering—Transaction on ASME—Series D*, 1964, **86**, 51–60.
10. Anderson B. D. O. and Moore, J. B., *Linear Optimal Control*. Prentice Hall, Englewood Cliffs, NJ, 1971.
11. Kwakernaak, H. and Sivan, R., *Linear Optimal Control Systems*. Wiley Interscience, New York, 1972.
12. Zames, G., Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Transactions on Automatic Control*, 1981, **AC-26**(2), 301–320.
13. Zames, G. and Francis, B. A., Feedback, minimax sensitivity, and optimal robustness. *IEEE Transactions on Automatic Control*, 1983, **AC-28**(5), 585–600.
14. Zames, G. and Bensoussan, D., Multivariable feedback, sensitivity and decentralized control. *IEEE Transactions on Automatic Control*, 1983, **AC-28**(11), 1030–1035.
15. Francis, B. A. and Zames, G., On $\mathcal{H}_\infty$ optimal sensitivity theory for SISO feedback systems. *IEEE Transactions on Automatic Control*, 1984, **AC-29**(1), 9–16.
16. O'Young, S. D. and Francis, B. A., Sensitivity tradeoffs for multivariable plants. *IEEE Transactions on Automatic Control*, 1985, **AC-30**(7), 625–632.
17. Boyd, S. and Desoer, C. A., Subharmonic functions and performance bounds in linear time-invariant feedback systems. *IMA Journal of Mathematical Control & Information*, 1985, **2**, 153–170.
18. Boyd, S. and Barratt, C., *Linear Controller Design—Limits of Performance*. Prentice Hall, Englewood Cliffs, NJ, 1991.
19. Chen, J., Sensitivity integral relations and design trade-offs in linear multivariable feedback systems. *IEEE Transactions on Automatic Control*, 1995, **AC-40**(10), 1700–1716.
20. Chen, J., Sensitivity Integral Relations and Design Trade-Offs in Linear Multivariable Feedback Systems Proceedings of American Control Conference, San Francisco, CA, June 1993, pp. 3160–3164.
21. Havre, K., Studies on controllability analysis and control structure design. *PhD thesis*, 1998, Chapter 5, Norwegian University of Science and Technology, Trondheim.
22. Kailath, T., *Linear Systems*. Prentice Hall, Englewood Cliffs, NJ, 1980.
23. Zhou, K., Doyle, J. C. and Glover, K., *Robust and Optimal Control*. Prentice Hall, Upper Saddle River, NJ, 1996.
24. Desoer, C. A. and Schulman, J. D., Zeros and poles of matrix transfer functions and their interpretation. *IEEE Transactions on Circuits and Systems*, 1974, **CAS-21**(1), 3–8.
25. MacFarlane, A. G. J. and Karcanias, N., Poles and zeros of linear multivariable systems: a survey of algebraic, geometric and complex variable theory. *International Journal of Control*, 1976, **24**, 33–74.
26. Callier, F. M. and Desoer, C. A., *Multivariable Feedback Systems*. Springer–Verlag, Berlin, 1982.
27. Green, M. and Limebeer, D. J. N., *Linear Robust Control*. Prentice Hall, Englewood Cliffs, NJ, 1995.
28. Havre, K. and Skogestad, S., Effect of RHP zeros and poles on performance in multivariable systems, In Proceedings from Control '96, University of Exeter, UK, September 1996. Institution of Electrical Engineers, London, UK, 1996.
29. Havre, K. and Skogestad, S., Directions and factorizations of zeros and poles in multivariable systems. Internal report, Norwegian University of Science and Technology, Trondheim, 1996.
30. Wall, J. E., Doyle, J. C. and Harvey, C. A., Tradeoffs in the design of multivariable feedback systems. In Proceedings of 18th Allerton Conference, pp. 715–725, 1980.
31. Morari, M. and Zafiriou, E., *Robust Process Control*, Prentice Hall, Englewood Cliffs, NJ, 1989.
32. Doyle, C. A. and Francis, B. and Tannenbaum, A., *Feedback Control Theory*. Macmillan, NY, 1992.

Appendix A: Proofs of the results

Proof of Lemma 4: We have for $s = p$, $G(p) = C(pI - A)^{-1}B + D$. Since p is an eigenvalue of A and x_{po} is the eigenvector corresponding to the pole p, $(pI - A)x_{po} = 0$. Therefore x_{po} is the output state direction with infinite gain for $(pI - A)^{-1}$. The normalized output pole direction becomes $y_p = Cx_{po}/\ \| Cx_{po} \|_2$ as long as $\| D \|$ is finite. The input pole direction u_p follows similarly as the conjugate of the output direction of the transposed system G^T.

Proof of Equation (22): The output direction is given by $y_z^H G(z) = 0$. For internal stability the controller cannot cancel the RHP-zero and it follows that $L = GK$ has a RHP-zero in the same direction, i.e. $y_z^H L(z) = 0$. Now, $S = (I + L)^{-1}$ is stable and thus has no RHP-pole at $s = z$. It then follows from $T = LS$ that $y_z^H T(z) = 0$ and $y_z^H(I - S(z)) = 0 \Longleftrightarrow y_z^H = y_z^H S(z)$.

Proof of Equation (23): The square matrix $L(s) = GK(s)$ has an RHP-pole at $s = p$, and if we assume that $L(s)$ has no RHP-zero at $s = p$, then $L^{-1}(p)$ exists and the output pole direction y_p is given by $L^{-1}(p)y_p = 0$. Since T is stable, it has no RHP-pole at $s = p$, so $T(p)$ is finite. It then follows from $S = TL^{-1}$ that $S(p)y_p = T(p)L^{-1}(p)y_p = 0$ and that $T(p)y_p = (I - S(p))y_p = y_p$.

Proof of Theorem 6. Introduce the scalar function

$$f(s) = y_p^H w_T(s) T(s) y_p$$

which is analytic in the RHP since $w_T T(s)$ is stable. We then have

$$\| w_T T(s) \|_\infty \geq \| f(s) \|_\infty \geq |f(p)| = |w_T(p)| \tag{A1}$$

The first inequality follows since the singular value measures the maximum gain of a matrix independent of direction and $\| y_p \|_2 = 1$. The second inequality follows from the maximum modulus theorem. The final equality follows since $w_T(s)$ is a scalar transfer function, and from the interpolation constraint $T(p)y_p = y_p$ we get $y_p^H T(p) y_p = y_p^H y_p = 1$.

Proof of c_{1j} in Theorem 7: We consider one RHP-zero z with output direction y_z at a time (the subscript j is omitted). Factorize the N_p RHP-poles p_i in $G(s) = \mathcal{B}_{po}^{-1}(G)G_{so}(s)$, where $\mathcal{B}_{po}^{-1}(G)$ is given by Equation (A25). It follows that $G_{so}(s)$ is stable, $\mathcal{B}_{po}(G)$ has all singular values and absolute value of all eigenvalues equal to one for $s = j\omega$ and $\bar{\sigma}(\mathcal{B}_{po}^{-1}(G(s))) \geq 1$ whenever $\mathrm{Re}(s) \geq 0$ (see Ref. 29, Lemma 2). The loop transfer function can then be written

$$L(s) = GK(s) = \mathcal{B}_{po}^{-1} G_{so} K(s) \triangleq \mathcal{B}_{po}^{-1}(G) L_m(s)$$

then

$$S = TL^{-1} = TL_m^{-1}(s)\mathcal{B}_{po}(G(s)) \triangleq S_m\mathcal{B}_{po}(G(s))$$

Introduce the scalar function $f(s) = y_z^H w_P(s) S_m(s) y$ which is analytic (stable) in RHP. We want to choose y so that $|f(s)|$ obtains maximum

$$J(s) = \max_{\|y\|_2=1} |f(s)\| = \max_{\|y\|_2=1} |y_z^H w_P(s) S_m(s) y|$$

We then get

$$\| w_P S(s) \|_\infty = \| w_P S_m(s) \|_\infty \geq \| J(s) \|_\infty \geq |J(z)| = \max_{\|y\|_2=1} |w_P(z)||y_z^H \mathcal{B}_{po}^{-1}(G)|_{s=z} y| = |w_P(z)| \, \| y_z^H \mathcal{B}_{po}^{-1}(G)|_{s=z} \|_2 \tag{A2}$$

The first equality follows since $\mathcal{B}_{po}^{-1}(G(s))$ is all-pass for $s = j\omega$. The first inequality follows since the singular value measures the maximum gain of a matrix independent of direction, so $\bar{\sigma}(A) \geq \| Aw \|_2$ and $\bar{\sigma}(A) \geq \| wA \|_2$ for any vector w with $\| w \|_2 = 1$. The second inequality follows from the maximum modulus theorem. The second equality follows from

$$y_z^H S_m(z) = y_z^H S(z) \mathcal{B}_{po}^{-1}(G)|_{s=z} = y_z^H \mathcal{B}_{po}^{-1}(G)|_{s=z}$$

and the fact that $w_P(s)$ is a scalar. The last equality follows from the fact that the largest singular value measures the largest gain and is equivalent to the two-norm. The fact that $c_{1,j} \geq 1$ follows from $\sigma_i(\mathcal{B}_{po}^{-1}(G(s))) \geq 1 \forall i$ when $\text{Re}(s) \geq 0$ (Ref. 29, Lemma 2).

Proof of $c_{2,i}$ in Theorem 8: We consider one RHP-pole p with output direction y_p at a time (the subscript i is omitted). Factorize the N_z RHP-zeros z_i in $G(s) = \mathcal{B}_{zo}(G) G_{mo}(s)$, where $\mathcal{B}_{zo}(G)$ is given by Equation (A26). It follows that $G_{mo}(s)$ is minimum phase, $\mathcal{B}_{zo}(G)$ has all singular values and absolute value of all eigenvalues equal to one for $s = j\omega$ (see Ref. 29, Lemma 2). The loop transfer function becomes

$$L(s) = GK(s) = \mathcal{B}_{zo}(G) G_{mo} K(s) \triangleq \mathcal{B}_{zo}(G) L_m(s)$$

Factorize $T = LS = \mathcal{B}_{zo}(G) L_m S \triangleq \mathcal{B}_{zo}(G) T_m$ and introduce the scalar function $f(s) = y^H w_T T_m(s) y_p$ which is analytic in RHP. We want to choose y so that $|f(s)|$ obtains maximum

$$J(s) = \max_{\|y\|_2=1} |f(s)| = \max_{\|y\|_2=1} |y^H w_T(s) T_m(s) y_p|$$

We then get

$$\| w_T T(s) \|_\infty = \| w_T T_m(s) \|_\infty \geq \| J(s) \|_\infty \geq |J(p)| = \max_{\|y\|_2=1} |w_T(p)||y^H \mathcal{B}_{zo}^{-1}(G)|_{s=p} y_p| = |w_T(p)| \, \| \mathcal{B}_{zo}^{-1}(G)|_{s=p} y_p \|_2 \tag{A3}$$

The first equality follows since $\mathcal{B}_{zo}(G(s))$ is all-pass for $s = j\omega$. The first inequality follows since the singular value measures the maximum gain of a matrix independent of direction, so $\bar{\sigma}(A) \geq \| Aw \|_2$ and $\bar{\sigma}(A) \geq \| wA \|_2$ for any vector w with $\| w \|_2 = 1$. The second inequality follows from the maximum modulus theorem. The second equality follows from $T_m(p) y_p = \mathcal{B}_{zo}^{-1}(G)|_{s=p} T(p) y_p = \mathcal{B}_{zo}^{-1}(G)|_{s=p} y_p$. The last equality follows from the fact that the largest singular value measure the largest gain and is equivalent to the two-norm. The fact that $c_{2,j} \geq 1$ follows from $\sigma_j(\mathcal{B}_{zo}^{-1}(G(s))) \geq 1 \forall j$ when $\text{Re}(s) \geq 0$.

Proof of $c = c_1 = c_2$ in Corollary 9: Note that when $N_z = N_p = 1$ both z and p are real and positive, so $\bar{z} = z$ and $\bar{p} = p$. Consider c_2

$$\begin{aligned} c_2 &= \| (I + \frac{2\text{Re}(z)}{p - z} y_z y_z^H) y_p \|_2 \\ &= \| \begin{bmatrix} U & y_z \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \frac{p+\bar{z}}{p-z} \end{bmatrix} \begin{bmatrix} U^H \\ y_z^H \end{bmatrix} y_p \|_2 \\ &= \| UU^H y_p + \frac{p + \bar{z}}{p - z} y_z y_z^H y_p \|_2 \\ &= \sqrt{\sin^2(\phi) + \frac{|\bar{z} + p|^2}{|z - p|^2} \cos^2(\phi)} \end{aligned} \tag{A4}$$

The matrix U contains a basis for the orthogonal subspace to y_z, $y_z^\perp$. The angle between y_p and $y_z^\perp$ is $90 - \phi$, $\cos(90 - \phi) = \sin(\phi)$ and Equation (A4) follows, We can interpret Equation (A4) as a weighted projection of y_p on the subspaces $y_z^\perp$, with weight 1, and y_z, with weight $\frac{|\bar{z}+p|^2}{|z-p|^2}$. In Ref. 17 (Equation (3.15), p. 164) it is the projection on the orthogonal subspace $y_z^\perp$ which lacks. By interchanging the roles of the pole and zero directions the bound the bound c_1 follows similarly.