SVD Controllers for H_2-, H_∞-, and μ-Optimal Control

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Plant structure is utilized for the simplification of system analysis and controller synthesis. For plants where the directionality is independent of frequency, the singular value decomposition (SVD) is used to decouple the system into nominally independent subsystems of lower dimension. In H_2- and H_∞-optimal control, the controller synthesis can thereafter be performed for each of these subsystems independently, and the resulting overall SVD controller will be optimal (the same will hold for any norm which is invariant under unitary transformations). In H_∞-optimal control the resulting controller is also *super-optimal*, as a controller of dimension $n \times n$ will minimize the norm in n directions. For robust control in terms of the structured singular value, μ, the SVD controller is optimal for a practically relevant class of block diagonal structures and uncertainty and performance weights.

Key words: Linear Control Systems, Robust Control, Large-scale Systems, Singular Value Decomposition, Structured Singular Value, Symmetric Circulant Plants

1 Introduction

In this paper we study SVD controllers which we define to have the form

$$K(s) = V\Sigma_K(s)U^H, \tag{1}$$

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where $\Sigma_K(s)$ is a diagonal matrix with real rational transfer functions on the diagonal, and U and V are real unitary (e.g., orthogonal) singular vector matrices which are derived from a singular value decomposition (SVD) of the plant $G(s)$. Here H denotes Hermitian (complex conjugate transpose) which for real matrices is equal to the transpose, i.e., $U^H = U^T$.

SVD controllers have been studied previously by Hung and MacFarlane (1982) and Lau *et al.* (1985). In both these references the SVD structure is essentially used to counteract interactions at one given frequency, as the problems considered are such that U and V change with frequency. However, in this paper we consider a class of problems for which U and V are constant at all frequencies and can be chosen to be real. Restricting our attention to these cases allows us to address the optimality of the SVD controller for H_2-, H_∞- and μ-optimal control. To be more specific, we consider plants $G(s)$ of dimension $n \times n$ which can be decomposed into

$$G(s) = U\Sigma_G(s)V^H; \quad \Sigma_G(s) = \mathrm{diag}\{\sigma_{Gi}(s)\} \tag{2}$$

where the output and input rotation matrices, U and V, are constant real unitary matrices, and $\Sigma_G(s)$ is a diagonal matrix with real rational transfer functions on the diagonal. The requirement for $\Sigma_G(s)$ to have rational transfer function elements arises because we use state-space based controller synthesis methods, and need the elements to be realizable. Restricting U and V to be real implies that the controller $K(s)$ will always be realizable provided $\Sigma_K(s)$ is realizable.

Equation (2) is the singular value decomposition of the plant $G(s)$ with the slight modification that the diagonal elements of $\Sigma_G(s)$, which we will refer to as *singular values*, have *phase*, and without necessarily requiring that the singular values in $\Sigma_G(s)$ are ordered according to their magnitudes. At a given frequency any transfer function can be decomposed into its singular value decomposition, but we are here assuming that the rotation matrices U and V are independent of frequency. In this case the singular value decomposition can be used to decompose the plant into n "subplants" $\sigma_{Gi}(s)$ (the diagonal elements of $\Sigma_G(s)$). To simplify the presentation, we consider in this paper only SISO subplants, but it is straightforward to generalize the results to cases where unitary transformations decompose the plant into MIMO subplants, that is, $\Sigma_G(s)$ is block-diagonal (see Hovd and Skogestad, 1994*a*, for details).

The two main contributions of this paper are to show that under certain mild conditions on the control problem weights there exists an optimal SVD controller for a plant of the form in (2), and that the controller design can be simplified for such problems. The basis for these results is that the H_2- and H_∞-norms

$$\|M(s)\|_2 \equiv \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left(M^H(j\omega)M(j\omega)\right) d\omega};$$

$$\|M(s)\|_\infty \equiv \sup_\omega \; \overline{\sigma}(M(j\omega))$$

are invariant to unitary scalings. To make use of this property we need that not only the plant, but the control problem as a whole (including the weights) can be "diagonalized" by unitary matrices. For the diagonalized problem we show that there exists a diagonal controller that is optimal, from which an optimal SVD controller for the original problem can be constructed. Furthermore, controller design is simplified since the elements of the diagonal controller can be obtained by performing controller synthesis on *n independent* subsystems involving $\Sigma_G(s)$. We show that in the H_∞ case the resulting controller is *super-optimal*, as the norm is minimized in the worst direction for each of these subsystems.

These results have not to our knowledge been presented before in the literature, at least not in this general form. This is somewhat surprising since plants of the form in (2) are common in practical applications. This paper extends the results of Hovd and Skogestad (1994*a*), by proving that under certain conditions an SVD controller is optimal when we consider robust H_∞ performance (i.e., μ-optimal control) and have model uncertainty which allows for plants which may not be of the form in (2) (although the nominal plant is of this form). In particular, we find that with some mild conditions on the weights the result holds for any combination of full-block (unstructured) and repeated diagonal complex uncertainty.

2 Examples of Plants Described by SVD

In this section we provide examples of plants which can be expressed in the form given in (2). The multivariable directionality of these plants, as expressed by the two singular vector matrices U and V, does not change with frequency, and U and V are real. The following two classes of plants are of special interest in applications:

A. Plants with scalar dynamics multiplied by a constant matrix. Let

$$G(s) = k(s)A \tag{3}$$

where A is a constant real matrix. Plant models of this form occur frequently in practice, at least in the chemical process industries, where the control engineer often chooses to work with very crude models.

B. Symmetric circulant plants. Plants with symmetric circulant transfer matrices are common in practice, and include a large number of processes with some symmetric spatial arrangement. Examples include paper machines where edge effects are neglected (Laughlin *et al.*, 1993; Wilhelm and Fjeld, 1983), dies for plastic films (Martino, 1991), and multizone crystal growth furnaces (Abraham and Lunze, 1991). In general, all symmetric circulant matrices can be diagonalized by the same unitary matrix, that is,

$$C(s) = R^T \Lambda_C(s) R \tag{4}$$

where R is the real Fourier matrix (see Hovd and Skogestad, 1994*a*, for details). This is of the form in (2) with $U = V = R^T$.

A subset of circulant symmetric matrices are called *parallel* matrices, and are described by $P = aI + bE$, where I is the identity matrix, E is a matrix with each element equal to one, and a and b are real scalars. Parallel transfer function matrices occur frequently in the process industries, and arise whenever there are identical units in parallel which interact with each other. Examples are found in distribution networks, when there are parallel units (e.g., reactors, compressors, pumps, heat exchangers) in a chemical plant (Shinskey, 1979; Shinskey, 1984; Hovd and Skogestad, 1994*a*) for electric power systems (Lunze, 1986; Lunze, 1991), for adhesive coating processes (Braatz *et al.*, 1992), or for communication between ships (Hazewinkel and Martin, 1983).

Remark: The set of plants given by (2) is more general than the two classes A and B given above, since the first class only includes plants for which the diagonal elements of $\Sigma_G(s)$ have the same dynamic behavior, and the second class only includes plants for which $U = V$.

3 SVD Control Problem

In this section we consider plants which can be decomposed into $G(s) = U\Sigma_G(s)V^H$ (as shown in Eq. 2) and define more exactly the class of control problems covered by the results of this paper. For a general control problem, $M(s)$ is the closed loop transfer function between external input signals (e.g., disturbances, noise, references) and external output signals (e.g., control error, error signals) which we want to keep small. The closed loop transfer function $M(s)$ depends on the controller $K(s)$, and the controller synthesis problem is to minimize $\|M\|$ over the set of all stabilizing controllers K. Typical choices of norm include the H_2- or the H_∞-norm (this is generalized to the structured singular value for the case with model uncertainty). The general class of *SVD problems* which are covered by the results of this paper are described below.

Definition 1 (SVD problem) *Consider an* $n \times n$ *plant* $G(s) = U\Sigma_G(s)V^H$*, where* U *and* V *are real orthogonal matrices and* $\Sigma_G(s)$ *is a diagonal transfer function matrix. Consider a control problem where the objective is to design a feedback controller* $K(s)$ *which minimizes a unitary invariant norm of*

$$M(s) = W_O(s)M_0(s)W_I(s),$$

where

$$M(s) = F_l(N(s), K(s)) = N_{11}(s) + N_{12}(s)K(s)\left[I - N_{22}(s)K(s)\right]^{-1} N_{21}(s). \quad (5)$$

The interconnection matrix $N(s)$ *is a function of the plant model and the weights, but is independent of the controller* K*.*

The weighting matrices $W_O(s)$ *and* $W_I(s)$ *are defined to be block-diagonal with each block having dimensions compatible with the dimensions of the subblocks containing* $G(s)$ *and* $K(s)$ *in* $M_0(s)$*:*

$$W_O(s) = \mathrm{diag}\{W_{Oi}(s)\}; \quad W_{Oi}(s) = U_{Oi}\Sigma_{W_{Oi}}(s)V_{Oi}^H$$

$$W_I(s) = \mathrm{diag}\{W_{Ii}(s)\}; \quad W_{Ii}(s) = U_{Ii}\Sigma_{W_{Ii}}(s)V_{Ii}^H,$$

and V_{Oi} *and* U_{Ii} *satisfying*

- $V_{Oi} = U$ *when* $W_{Oi}(s)$ *premultiplies* $G(s)$ *in subblocks of* $M_0(s)$*;*
- $V_{Oi} = V$ *when* $W_{Oi}(s)$ *premultiplies* $K(s)$ *in subblocks of* $M_0(s)$*;*
- $U_{Ii} = V$ *when* $W_{Ii}(s)$ *postmultiplies* $G(s)$ *in subblocks of* $M_0(s)$*;*
- $U_{Ii} = U$ *when* $W_{Ii}(s)$ *postmultiplies* $K(s)$ *in subblocks of* $M_0(s)$*.*

The terms "premultiply" and "postmultiply" are used in a general sense, for instance, in the formula $W_O(I + GK)^{-1}W_I$, the weight W_O premultiplies G and W_I postmultiplies K. There are no requirements on the other matrices in the weights, other than U_{Oi} and V_{Ii} being unitary and $\Sigma_{W_{Ii}}(s)$ and $\Sigma_{W_{Oi}}(s)$ being diagonal.

Remark 1. The definition of an SVD control problem may seem restrictive and complicated, but the conditions on the weights are satisfied for most problems with a plant on the form $G(s) = U\Sigma_G(s)V^H$.

Remark 2. Essentially, the weights must be consistent with the plant $G(s)$, such that, after substituting $G(s) = U\Sigma_G(s)V^H$ and $K(s) = V\Sigma_K(s)U^H$ into $M_0(s)$, the unitary matrices U and V are canceled by the weights when forming $M(s)$, in the sense that we can write $M(s) = U_O\tilde{M}(s)V_I^H$ where all the blocks of $\tilde{M}(s)$ are diagonal. A similar transformation may be used to obtain a block diagonal $\tilde{N}(s)$, but since $N(s)$ is independent of the controller we do not need to assume an SVD-controller to achieve this. This is important when proving that the SVD-controller is actually optimal (see next section).

Remark 3. Scalar times identity weights, $W_i(s) = w_i(s)I$ always satisfy the conditions of an SVD problem since $w_i(s)I = Uw_i(s)U^H = Vw_i(s)V^H$.

4 H_2- and H_∞-Optimal Control

For an SVD problem, $N(s)$ will be such that there exists block-diagonal unitary matrices

$$U_W = \text{diag}\{\text{diag}\{U_{Oi}\}, U\}; \qquad V_W = \text{diag}\{\text{diag}\{V_{Ii}\}, V\} \tag{6}$$

such that

$$\tilde{N}(s) = U_W^H N(s) V_W \tag{7}$$

is a matrix consisting of diagonal subblocks. The rows and columns of $\tilde{N}(s)$ can be rearranged to give a block-diagonal matrix, for which an optimal controller $\tilde{K}$ can be constructed by solving n SISO independent optimal controller synthesis problems for each subblock (this is proved in Hovd *et al.*, 1996). An optimal SVD controller for the original N is constructed from $K = V\tilde{K}U^H$. The result also applies to any other norm which is invariant under unitary transformations.

Theorem 1 (H_2- and H_∞-Optimality) *Consider an SVD problem (Definition 1). Then*

(i) There exists an SVD controller that is $H_2(H_\infty)$-optimal.
(ii) The optimal controller can be computed by designing n independent SISO $H_2(H_\infty)$-optimal controllers, one for each of the SISO subplants of the plant.
(iii) For H_∞-optimal control, this controller is super-optimal, that is, the H_∞-objective is optimized in n directions.

Proof: See Hovd *et al.* (1996) for a detailed proof and Hovd and Skogestad (1994*a*) for a more general but less detailed proof. □

Remark 1. The number of states of the controller computed via Thm. 1 is equal to the number of states of a controller based on regular H_∞ synthesis

(Hovd *et al.*, 1996). That is, for this class of problems super-optimality does not require a controller with a higher number of states.

Remark 2. In general we solve n independent synthesis subproblems of low dimension. In some cases the problem is even further reduced in size since some of these subproblems are identical. For example, for the case of symmetric circulant systems we need only solve $(n+1)/2$ SISO problems for odd n and $n/2+1$ problem for even n. For the case of parallel processes we need only solve two independent subproblems (since $n-1$ subproblems are identical). For details see Hovd and Skogestad (1994*a*).

Remark 3. The theorem may be generalized to cases where the subplants $\sigma_{Gi}(s)$ are matrices. For example, see Hovd and Skogestad (1994*a*) which considered the special case of symmetric circulant plants.

5 μ-Optimal Control

In this section we shall generalize the H_∞-problem studied above to the design of robust optimal controllers. This control problem results when we introduce model uncertainty and want to minimize the H_∞-norm for robust performance, or alternatively want to optimize robust stability.

5.1 The Structured Singular Value

The structured singular value, μ, takes uncertainty in a feedback system explicitly into account. Readers not familiar with the structured singular value are referred to Doyle (1982); only a very brief introduction will be given here. The uncertainties in the system are modeled with H_∞ norm-bounded perturbation blocks with weights to normalize the maximum singular value of each perturbation block to unity. The block diagram for the feedback system is then rearranged to give an interconnection matrix $M(s)$ and a block-diagonal matrix Δ with the perturbation blocks along the diagonal (see Fig. 1). If Δ is a full matrix (i.e., Δ has no structure), the controller synthesis problem is a

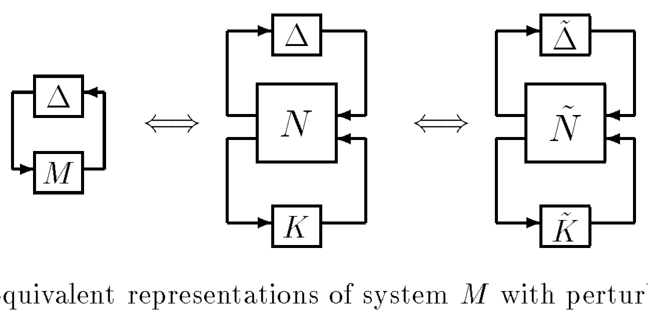

Fig. 1. Equivalent representations of system M with perturbation Δ.

H_∞ problem, and is covered by the results of the previous section. Otherwise, the structured singular value is needed to account for the uncertainty in a nonconservative manner.

The structured singular value with respect to the uncertainty structure Δ is defined as

$$\mu(M) \equiv \begin{cases} 0 \text{ if there does not exist } \Delta \text{ such that } \det(I+M\Delta)=0 \\ \left[\min_{\Delta}\{\bar{\sigma}(\Delta)\,|\det(I+M\Delta)=0\}\right]^{-1} \quad \text{otherwise} \end{cases} \tag{8}$$

Currently no simple computational method exists for exactly calculating μ in general, and recent work suggests that an efficient exact method is most likely not possible (Braatz *et al.*, 1994). This motivates the common practice which is to compute instead the upper bound

$$\mu(M) \leq \inf_D \; \bar{\sigma}(DMD^{-1}) \tag{9}$$

where D is an invertible matrix with a structure such that $D^{-1}\Delta D = \Delta$. For example, $D = dI$ if Δ is a full matrix, and D is a full matrix if Δ is repeated diagonal ($\Delta = \delta I$). For complex uncertainties the upper bound (9) is equal to μ for three or fewer full blocks (Doyle, 1982), and usually within 1-2% when there are no repeated blocks (Balas *et al.*, 1991). A controller which minimizes the upper bound for μ in Eq. (9) will be said to be DMD^{-1}*-optimal.*

Another reason for using the upper bound is that the goal of the most popular procedure for designing robust controllers, called DK-iteration, is to minimize the upper bound. Also, when all the uncertainties are full and complex, the upper bound is a necessary and sufficient condition for robustness to *arbitrarily-slow time-varying linear uncertainty* (see Poolla and Tikku, 1995, for details). It can be argued that this uncertainty description may be more useful for practical control problems.

The standard DK-iteration procedure (Doyle and Chu, 1985) attempts to find the DMD^{-1}*-optimal* controller. DK-iteration involves alternating between the following two steps until the upper bound is no longer minimized.

D Step: Find $D(s)$ to minimize frequency-by-frequency the upper bound on μ in (9).

K Step: Scale the controller design problem with $D(s)$, and design an H_∞-optimal controller for the scaled design problem DMD^{-1}.

Although convergence to the global optimum is not guaranteed, DK-iteration appears to work well for processes of low dimensionality (Doyle and Chu, 1985).

5.2 Structure of the uncertainty

It is important to note that Δ often has two levels of structure. First, Δ is often composed of *subblocks* Δ_i *of the same size as* G

$$\Delta = \text{diag}\{\Delta_i\} \tag{10}$$

These subblocks may represent different *sources* of uncertainty in the system. For example, actuator uncertainty is located at the input of the plant and is commonly modeled as multiplicative input uncertainty, i.e., $\Delta_i = \Delta_I$. Second, each subblock Δ_i may have structure to reduce conservatism. For example, actuators may not influence each other, so uncertainty associated with these actuators would be described by a *diagonal* Δ_i. The most common (and useful) structures for the subblocks Δ_i are:

- Full block uncertainty: Δ_i is a full matrix.
- Independent diagonal uncertainty: $\Delta_i = \text{diag}\{\delta_{ij}\},\ j = 1, \ldots, n$, is a diagonal matrix.
- Repeated diagonal uncertainty: $\Delta_i = \delta_i I$, i.e., a scalar uncertainty δ_i multiplied with an identity matrix.

5.3 μ-Optimality of SVD controllers

The uncertainty weights must satisfy certain conditions to ensure μ-optimality of the SVD-controllers.

Definition 2 Robust SVD Problems. *Consider an SVD problem with* $M(s) = W_O(s)M_0(s)W_I(s)$ *as in Definition 1, and multiple sources of uncertainty* $\Delta = diag\{\Delta_i\}$, *as illustrated in Fig. 1. In addition to the requirements of Definition 1, the weights* $W_{Oi} = U_{Oi}\Sigma_{W_{Oi}}(s)V_{Oi}^H(s)$ *and* $W_{Ii} = U_{Ii}\Sigma_{W_{Ii}}(s)V_{Ii}^H(s)$ *related to each* Δ_i *should fulfill the following:*

(i) $U_{Oi} = V_{Ii}$ *for all repeated diagonal uncertainty,* $\Delta_i = \delta_i I$
(ii) $U_{Oi} = V_{Ii} = I$ *for all independent diagonal uncertainty,* $\Delta_i = \text{diag}\{\delta_{ik}\}, k = 1, \ldots, n$

Remark 1. For a "full" (unstructured) uncertainty block Δ_i no further requirements on the weights are needed.

Remark 2. Requirement 1 for repeated diagonal blocks holds regardless of the uncertainty's location when the plant is described by a normal transfer function matrix (e.g., symmetric circulant plants) and the weights are repeated diagonal.

Remark 3. Requirement 2 on the weights for independent diagonal uncertainty is very restrictive. For example, it allows for scalar times identity weights only for cases when U or V are equal to the identity matrix (that is, the inputs or outputs to the plant are naturally aligned in the direction of the singular values).

Now we show that for this class of problems the interconnection matrix N can be pre- and postmultiplied by block-diagonal unitary matrices to arrive at an equivalent interconnection matrix $\tilde{N}$ which consists of diagonal subblocks.

Lemma 1 *Let $\tilde{N}$ be defined as in Eqs. (6) and (7). For μ-optimality and DMD^{-1}-optimality of Robust SVD problems (Definition 2), the "diagonalized" control problem is* equivalent *to the original problem, in the sense that*

$$\min_K \mu\left(F_l(N,K)\right) = \min_{\tilde{K}} \mu\left(F_l(\tilde{N},\tilde{K})\right) \tag{11}$$

$$\min_K \inf_D \left(DF_l(N,K)D^{-1}\right) = \min_{\tilde{K}} \inf_D \left(DF_l(\tilde{N},\tilde{K})D^{-1}\right) \tag{12}$$

where both μ problems are with respect to the uncertainty in the original control problem, and the structure of the D matrices in both DMD^{-1}-problems is compatible with this uncertainty.

Proof: In the block diagram for the system, replace G with $U\Sigma_G(s)V^H$, and substitute in the weights $W_{Ii}(s)$ and $W_{Oi}(s)$. Rearranging the block diagram (see Fig. 1) gives $\tilde{N}$ with diagonal subblocks with the subblocks of $\tilde{\Delta}$ given by $\tilde{\Delta}_i = V_{Ii}^H \Delta_i U_{Oi}$. Note that under the assumptions on U_{Oi} and V_{Ii} in Definition 2

(i) $\tilde{\Delta}_i$ is full if and only if Δ_i is full;
(ii) $\tilde{\Delta}_i$ is repeated diagonal if and only if Δ_i repeated diagonal;
(iii) $\tilde{\Delta}_i$ is independent diagonal if and only if Δ_i independent diagonal.

Thus in Fig. 1 the middle block diagram is equivalent to the rightmost block diagram.

A similar argument holds with regard to the upper bound of μ. Under the assumptions on U_{Ii} and V_{Oi}, for each diagonal or full block Δ_i the corresponding D_i and its inverse commute with U_{Ii} and V_{Oi}. For repeated diagonal blocks the U_{Ii} and V_{Oi} can be absorbed into the D_i. □

The following results on the optimality of the SVD controller follow from Thm. 1 and Lemma 1.

Theorem 2 (**DMD^{-1}-Optimality**) *Consider a SVD problem where the objective is to minimize $\sup_\omega \min_D \| DMD^{-1} \|_\infty$ where $D^{-1}\Delta D = \Delta$. Assume* *that all uncertainty blocks Δ_i are full blocks. Then*

(i) There exists an SVD controller which is optimal.

(ii) If DK-iteration is used to obtain the optimal controller, the **K** *step (with fixed D) consists of n independent SISO H_∞-optimal control problems, one for each of the SISO subplants σ_{Gi} of $G(s)$.*

Proof: Let N denote the interconnection matrix corresponding to M, where N has a block structure corresponding to the uncertainties Δ_i. With fixed D-scales we may absorb D and D^{-1} into N to get

$$N_D = \hat{D} N \hat{D}^{-1}; \quad \hat{D} = \mathrm{diag}\{D, I\}$$

We are then left with an H_∞-problem in terms of N_D. Since all uncertainty blocks Δ_i are full, the D-scales are of the form $D = \mathrm{diag}\{D_i\},\ D_i = d_i I_i$. Then the only difference between N and N_D will be that the offdiagonal blocks are multiplied by scalars. Thus, the structure of each block in N_D will be the same as in N, and we can use the transformation $\tilde{N}_D(s) = U_W^H N(s) V_W$ to obtain a $\tilde{N}_D$ with diagonal blocks. Subsequent permutations yield a block-diagonal $\tilde{N}_D$. This implies that for a fixed D there exists an optimal SVD-controller which can be obtained by solving n independent SISO H_∞-problems. Since an SVD-controller is optimal for any fixed D this structure must also be optimal for the optimal D. □

Theorem 3 (μ-Optimality) *Consider a Robust SVD problem where the objective is to minimize $\sup_\omega \mu(M)$. Assume that all uncertainty blocks Δ_i are diagonal (repeated or independent) except possibly one full block. Then*

(i) There exists an SVD controller which is optimal.

(ii) When all the uncertainty blocks are diagonal, then the μ-optimal control problem decouples into n independent SISO μ-optimal control problems, one for each of the SISO subplants of the plant.

Proof: If all uncertainty blocks Δ_i are diagonal (including repeated diagonal uncertainty), then the system consists of independent subsystems. If one uncertainty block is full, then the diagonal uncertainty blocks can be absorbed into the interconnection matrix to get a "reduced" $\tilde{N}$ which still consist of diagonal subblocks after absorbing the diagonal uncertainty blocks. Whatever the values of the diagonal blocks, Thm. 1 implies that an SVD controller is optimal for this "reduced" control problem. Thus an SVD controller is optimal for the original μ problem. □

The above results complement each other in that Thm. 2 handles one form of uncertainty (full) and Thm. 3 handles the other (diagonal). Both types of uncertainty can be handled by assuming μ is equal to its upper bound.

Theorem 4 (μ- and DMD^{-1}-Optimality) *Consider a Robust SVD control problem (Definition 2), and assume that μ is equal to its upper bound (9). Then*

(i) There exists an SVD controller which is μ-optimal.
(ii) For the DK-iteration procedure the **K** *step consists of n independent SISO H_∞-optimal control problems, one for each of the SISO subplants of the plant.*
(iii) For repeated diagonal uncertainty: D_i can be taken to be diagonal *rather than full in the* **D** *step.*

Proof: All diagonal blocks (repeated or independent) can be absorbed into the interconnection matrix $\tilde{N}$ without changing its structure. By Thm. 2 an SVD controller is optimal for this "reduced" control problem for all values of the diagonal blocks. Thus an SVD controller is optimal for the original μ problem.

For independent diagonal and full block Δ_i, D_i is diagonal and cannot induce interaction between individual subproblems. This also holds for D_i corresponding to repeated diagonal $\Delta_i = \delta_i I$. To see this, again consider the "reduced" control problem. If the D_i corresponding to the repeated diagonal blocks introduced interaction between subproblems, they would effectively allow for a larger class of uncertainty than the original uncertainty description.

Scalings D_i which do not cause interactions between subproblems are parameterized by unitary times diagonal matrices. The unitary matrices do not affect the value of the H_∞-norm, so can be ignored. $\square$

The assumption that μ is equal to its upper bound is not restrictive (see Section 5.1).

Thm. 1-4 state that there exists an optimal SVD controller for classes of problems of engineering interest. Hovd *et al.* (1996) consider other classes of problems for which an SVD controller may not be optimal, but where a substantial simplification in system analysis and controller synthesis results in *selecting* the controller to be of SVD form. In particular, conditions are given on the weights and on the source of uncertainty for which an H_∞-optimal controller is μ-optimal *irrespective of the structure of the uncertainty block.*

5.4 DK-Iteration: Reduction of Computational Effort

The above results can be used to reduce the computational effort involved in the **K** step of the DK-iteration procedure in two ways. First, instead of solving one large H_∞-synthesis problem, one may solve n smaller H_∞-synthesis subproblems. Second, some of these n subproblems may be repeated (identical), for example, this occurs for the important case when both the plant and weights are symmetric circulant (or parallel). In general, the computational effort is *not* reduced in the **D** step where the upper bound to μ is computed, since for the case of full block uncertainty we have $D = dI$ so d should be the same for all subproblems. This restriction is difficult to incorporate unless a simultaneous approach is used. However, all *repeated* subproblems need only be considered once in computing the D_i (see item 3 in Theorem 4). Thus repeated subproblems can be deleted before starting the DK-iteration design procedure, and for a large number of subsystems the size of the DK-iteration and μ-analysis problems can be reduced dramatically.

Table 1
Algorithm for μ-optimal SVD Controllers using DK-iteration

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1. Test whether the problem is a Robust SVD-problem as given by Definitions 1 and 2. If the structure of an uncertainty Δ_i and its corresponding weights $W_i(s)$ do not satisfy Definition 2, then an SVD controller may not be optimal. To use the design procedure, treat the uncertainty as a full block, realizing that this is potentially conservative.

2. Form $\tilde{N}(s)$ as given by Eq. 7 and rearrange it such that it is block-diagonal.

3. Delete all identical subproblems in $\tilde{N}$.

4. **K step:** Design an H_∞-optimal controller for each independent unique subproblem, and collect the optimal $\tilde{K}_i(s)$ (without repetitions) into a diagonal matrix.

5. **D step:** Calculate the tight upper bound on μ in (9) and obtain $D(s)$. Return to step 4 until DK-iteration converges.

6. Collect the optimal $\tilde{K}_i(s)$ (including repetitions for identical subproblems) into a diagonal matrix $\Sigma_K(s)$. Form $K(s) = V\Sigma_K(s)U^H$.

7. If the DK-iteration procedure converged to the global minimum, then this would be the μ-optimal controller under the assumptions of Thm. 4, for the uncertainty assumed in Step 1 of this algorithm.

When all uncertainty blocks are diagonal except possibly one full block, and the weights for the diagonal blocks satisfy Definition 2, the subproblems can be considered independently for the **D** step, since the D_i corresponding to the full block can be normalized to be the identity matrix. The general DK-iteration procedure for designing SVD controllers for SVD problems is summarized in Table 1. Performing DK-iteration on the transformed system will converge faster and is numerically better conditioned than on the original system. This is both because the H_∞ subproblems are smaller than the original problem, and because the algorithm will be initialized with a controller which has the correct (optimal) directionality. This will be illustrated in the examples.

6 Examples

The following examples illustrate the computational usefulness of the results of this paper.

6.1 Example 1: Distillation Column

Consider the robust controller design problem for the simplified distillation column introduced by Skogestad *et al.* (1988) that has been used as a benchmark problem for comparing methods for robust controller design (Freudenberg, 1989; Chen and Freudenberg, 1990; Yaniv and Barlev, 1990; Limebeer, 1991; Lundström *et al.*, 1991; Lin, 1992). Thm. 4 implies that there exists an SVD controller that is μ-optimal for this design problem (see Hovd *et al.*, 1996, for details). This knowledge can be used to design an SVD controller which is nearly μ-optimal that *has only four states* (Hovd *et al.*, 1996).

6.2 Example 2: Parallel Reactors With Combined Precooling

A simplified model $G(s)$ of four parallel reactors with combined precooling (Skogestad *et al.*, 1989) is

$$G(s) = \frac{1}{100s+1}\begin{bmatrix} 1 & 0.7 & 0.7 & 0.7 \\ 0.7 & 1 & 0.7 & 0.7 \\ 0.7 & 0.7 & 1 & 0.7 \\ 0.7 & 0.7 & 0.7 & 1 \end{bmatrix} \qquad (13)$$

The real Fourier matrix diagonalizes the plant (Hovd *et al.*, 1996), that is, $G(s) = R^H \Sigma_G(s) R$ where the plant singular values are

$$\sigma_{G1}(s) = \frac{3.1}{100s+1}; \qquad \sigma_{G2}(s) = \sigma_{G3}(s) = \sigma_{G4}(s) = \frac{0.3}{100s+1}$$

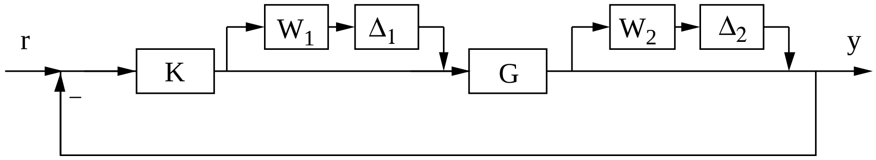

Fig. 2. Block diagram for plant with uncertainties in Example 2.

Consider the process with input and output uncertainty as shown in Fig. 2. The input uncertainty Δ_1 and output uncertainty Δ_2 are both assumed to be independent diagonal, with uncertainty weights $W_1(s) = \text{diag}\{0.2\frac{5s+1}{0.5s+1}\}$ and $W_2(s) = \text{diag}\{0.2\frac{2.5s+1}{0.25s+1}\}$. To reject disturbances at the plant output, we include the performance specification $\bar{\sigma}(W_3 S_p) < 1, \forall\omega$, with $W_3(s) = \text{diag}\{0.5\frac{10s+1}{10s}\}$. The overall problem (before SVD reduction) has two diagonal 4×4 uncertainty blocks and one full 4×4 performance block, and we get a 12×12 μ interconnection matrix:

$$M = \begin{bmatrix} -W_1 KG(I+KG)^{-1} & -W_1 K(I+GK)^{-1} & W_1 K(I+GK)^{-1} \\ W_2 G(I+KG)^{-1} & -W_2 GK(I+GK)^{-1} & W_2 GK(I+GK)^{-1} \\ -W_3 G(I+KG)^{-1} & -W_3(I+GK)^{-1} & W_3(I+GK)^{-1} \end{bmatrix} \quad (14)$$

In order to make this a Robust SVD problem (see Definition 2) we need to assume that all the uncertainty blocks are full. Then there are three full blocks and μ is equal to its upper bound. Thm. 4 implies that there exists an SVD controller that is μ-optimal for this potentially-conservative case. In addition, three of the four subproblems in $\tilde{N}$ are identical in the algorithm for for μ-optimal SVD controllers (Table 1). Thus in DK-iteration we may solve two 3×3 independent H_∞ problems in the **K** step, and obtain the scalings $d_1(s)$ and $d_2(s)$ from a 6×6 μ matrix M in the **D** step. Using this procedure we were able find a controller resulting in a μ-value of 0.93. For brevity, the state space representation of the eigenvalues of this controller is given elsewhere (Hovd *et al.*, 1996).

Thereafter we attempted to use DK-iteration to improve the controller design by using the true diagonal structure for the uncertainties Δ_1 and Δ_2, the original 12×12 M-matrix (14), and the above controller as a starting point. However, we found that this increased the complexity of the controller synthesis problem so much that we were unable to improve the design using DK-iteration. The best controller the software was able to obtain had a μ-value of 0.96, which is larger than the μ-value for the controller the algorithm was initialized with. This result shows that there are numerical inaccuracies with the off-the-shelf software. It also demonstrates the important advantage of reduced problem size which results from applying our method.

7 Discussion

The SVD structure can be used for designing controllers with a low number of states. Using V as a pre-compensator and U^H as a post-compensator, we are left with n SISO controllers to design for a plant of dimension $n \times n$. This design problem is similar to the conventional decentralized control problem (e.g. Hovd and Skogestad, 1994*b*), and may be solved by sequential design, independent design, or simultaneous design (parameter optimization). The last approach was used in Example 1.

Several authors have proposed to design controllers with the SVD structure based on the SVD of the plant at some important frequency (Lau *et al.*, 1985; Hung and MacFarlane, 1982). The results of this paper imply that the SVD structure is optimal at that fixed frequency (with some restrictions on the structures of the perturbation blocks given in Definition 2), thus providing a theoretical justification for such design procedures. These design methods should perform well for process control problems which do not have U and V varying rapidly as a function frequency, but may perform poorly for processes such as flexible structures.

A simple lower bound on the achievable value for the upper bound to μ can be computed by applying the algorithm to minimize the upper bound at each frequency (Lee *et al.*, 1995). Because each design subproblem at a fixed frequency only involves finding one complex scalar, the synthesis part is very simple (the state-space algorithm need not be used). This frequency-by-frequency approach will not yield a realizable controller, since issues such as causality and phase-gain relationships are ignored. However this is a valid lower bound on the achievable performance by a realizable controller, and can be a useful controllability measure (Lee *et al.*, 1995).

The results of this paper are easily generalized to cases with multivariable, possibly nonsquare subplants. Synthesis problems similar to class B in Section 2 arise naturally whenever identical multivariable plants are arranged in parallel or in a symmetric manner.

8 Conclusions

For plants where the directionality is independent of frequency, the singular value decomposition (SVD) is used to decouple the system into nominally independent subsystems of lower dimension. In H_2- and H_∞-optimal control, the controller synthesis can thereafter be performed for each of these subsystems independently, and the resulting overall SVD controller will be optimal (the same will hold for any norm which is invariant under unitary transformations). In H_∞-optimal control the resulting controller is also *super-optimal*, as a controller of dimension $n \times n$ will minimize the norm in n directions.

For robust control in terms of the structured singular value, μ, the SVD controller is optimal for a wide class of systems with full block and repeated diagonal complex uncertainty. Substantial computational savings can be achieved in the controller synthesis step of the DK-iteration scheme. The results of this manuscript provide a theoretical justification of controller synthesis methods proposed by other authors. Other applications of the SVD controller were described, including its use for low order controller design, and in efficiently computing controllability measures.

An extended version of this paper is available as a technical report (Hovd *et al.*, 1996).

Nomenclature

D - Block diagonal scaling matrix
F_l - Lower linear fractional transformation (see Eq. (5))
$G(s) = U\Sigma_G(s)V^H$ - Transfer function matrix for the plant
$K(s)$ - Transfer function matrix for the controller
$M(s)$ - Matrix whose norm is to be minimized in the controller synthesis
$M_0(s)$ - The matrix $M(s)$ with the weights removed
n - Plant dimension ($n \times n$)
$N(s)$ - Interconnection matrix for the synthesis problem
R - "Real Fourier matrix"; real, unitary eigenvector matrix for symmetric circulant matrices
s - Laplace variable
U - Output singular vector matrix of the plant $G(s)$
U_{Oi} - Output singular vector matrix for output weight i
V - Input singular vector matrix of the plant $G(s)$
V_{Ii} - Input singular vector matrix for input weight i
$W_I(s) = \text{diag}\{W_{Ii}(s)\}$ - Block diagonal matrix of weights for the inputs to $M(s)$
$W_O(s) = \text{diag}\{W_{2i}(s)\}$ - Block diagonal matrix of weights for the outputs from $M(s)$
Δ - Block diagonal matrix of perturbations
Δ_i - i'th block on the diagonal of Δ (of the same size as $G(s)$)
μ - Structured singular value
σ - Singular value
$\bar{\sigma}$ - Largest singular value
$\Sigma(s)$ - Matrix of singular values

ω - Frequency

Subscripts
I - Input to synthesis problem
O - Output from synthesis problem
i - Block i on the diagonal of a block diagonal matrix j - Singular value j

Superscripts
H - Hermitian (complex conjugate transpose)
$\tilde{}$ - Denotes that the matrix has been transformed by pre- and postmultiplication with unitary matrices.

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