LIMITATIONS IMPOSED BY RHP ZEROS/POLES IN MULTIVARIABLE SYSTEMS

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Abstract

This paper examines the limitations imposed by Right Half Plane (RHP) zeros and poles in multivariable feedback systems. The main result is to provide lower bounds on $\|WXV(s)\|_\infty$ where X is S, S_I, T or T_I (sensitivity and complementary sensitivity). Previously derived lower bounds on the $\mathcal{H}_\infty$-norm of S and T are thus generalized to the case with matrix-valued weights, including bounds for reference tracking and disturbance rejection. Furthermore, new bounds which quantify the minimum input usage for stabilization in the presence of measurement noise and disturbances are derived.

1 Introduction

It is well known that the presence of RHP-zeros and poles pose fundamental limitations on the achievable control performance. This was quantified for SISO systems by Bode (1945) more than 50 years ago, and most control engineers have an intuitive feeling on the limitations for scalar systems. Rosenbrock (1966; 1970) pointed out that multivariable RHP-zeros pose similar limitations.

In this paper we extend the work of Zames (1981), Skogestad and Postlethwaite (1996) and Havre and Skogestad (1996b), and quantify the fundamental limitations imposed by RHP-zeros and poles in terms of lower bounds on the peaks ($\mathcal{H}_\infty$-norm) of important closed-loop transfer functions.

Notation. We consider linear time invariant transfer function models on the form

$$y(s) = G(s)u(s) + G_d(s)d(s) \tag{1}$$

where u is the vector of manipulated inputs, d is the vector of disturbances and y is the vector of outputs. We often omit to show the dependence on the complex variable s for transfer functions. When we refer to zeros and poles and their directions we mean the zeros and poles of the plant G unless otherwise explicitly stated. In order to be able to stabilize the pair (G, G_d) we must require that all unstable modes in G_d is also a mode of G.

The $\mathcal{H}_\infty$-norm of a stable rational transfer function matrix $M(s)$ is defined as the peak value overall frequencies of the

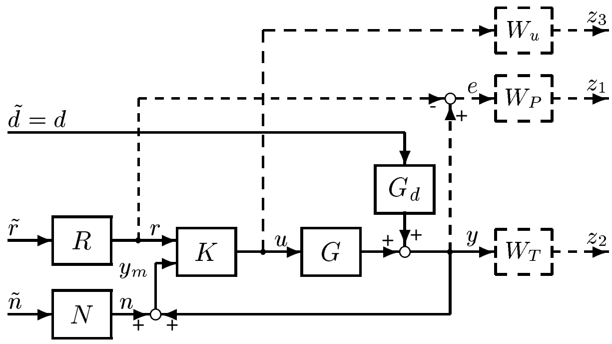

Figure 1: Two degrees-of-freedom control configuration

largest singular value of $M(j\omega)$

$$\|M(s)\|_\infty \triangleq \sup_\omega \bar{\sigma}(M(j\omega)) \tag{2}$$

Closing the loop. In this paper we consider the general two degrees-of-freedom (2-DOF) control configuration shown in Figure 1. In the figure the performance weights are given in dashed lines. We have included both references r and measurement noise n in addition to disturbances d as external inputs. The three matrices G_d, R and N can be viewed as weights on the inputs, and the inputs $\tilde{d}$, $\tilde{r}$ and $\tilde{n}$ are normalized in magnitude. Normally N is diagonal and $[N]_{ii}$ is the inverse of signal to noise ratio.

The controller can be divided into a positive feedback part from y (K_2) and a feed forward part from r (K_1)

$$u = K_1 r + K_2 y_m = K_1 r + K_2(y + n) \tag{3}$$

The closed-loop transfer function F from $[\,\tilde{r}^T \quad \tilde{d}^T \quad \tilde{n}^T\,]$ to z becomes

$$F(s) = \begin{bmatrix} W_P(SGK_1 - I)R & W_P S G_d & -W_P T N \\ W_T S G K_1 R & W_T S G_d & -W_T T N \\ W_u S_I K_1 R & W_u K_2 S G_d & W_u K_2 S N \end{bmatrix} \tag{4}$$

where the sensitivity S, the complementary sensitivity T and the input sensitivity S_I are defined by

$$S \triangleq (I - GK_2)^{-1} \tag{5}$$
$$T \triangleq I - S = -GK_2(I - GK_2)^{-1} \tag{6}$$
$$S_I \triangleq (I - K_2 G)^{-1} \tag{7}$$

with positive feedback controller K_2. We also define the input complementary sensitivity

$$T_I \triangleq I - S_I = -K_2 G (I - K_2 G)^{-1} \tag{8}$$

By setting $K_1 = -K_2$ in the above equations, the one degree-of-freedom (1-DOF) control configuration can be analyzed.

2 Zeros and poles in multivariable systems

2.1 Zero directions

DEFINITION 1 (ZEROS, MacFarlane and Karcanias, 1976). *$z_i \in \mathbb{C}$ is a zero of $G(s)$ if the rank of $G(z_i)$ is less than the normal rank of $G(s)$. The zero polynomial is defined as $z(s) = \prod_{i=1}^{N_z}(s - z_i)$ where N_z is the number of finite zeros of $G(s)$.*

The normal rank of $G(s)$ is defined as the rank of $G(s)$ at all s except a finite number of singularities (which are the zeros).

DEFINITION 2 (ZERO DIRECTIONS). *If $G(s)$ has a zero for $s = z \in \mathbb{C}$ then there exist non-zero vectors, denoted the output zero direction $y_z \in \mathbb{C}^l$ and the input zero direction $u_z \in \mathbb{C}^m$, such that $y_z^H y_z = 1$, $u_z^H u_z = 1$ and*

$$y_z^H G(z) = 0; \qquad G(z) u_z = 0 \tag{9}$$

For a system $G(s)$ with state-space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, the zeros z of the system, the zero input directions u_z and the state input zero directions $x_{zi} \in \mathbb{C}^n$ (n is the number of states) can all be computed from the generalized eigenvalue problem

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} x_{zi} \\ u_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10}$$

Similarly one can compute the zeros z and the output zero directions y_z from G^T.

2.2 Pole directions

Bode (1945) states that the poles are the singular points at which the transfer function fails to be analytic. This means that the system G evaluated[1] at $s = p$, $G(p)$, is infinite in some directions at the input and the output. This is the basis for the following definition of input and output pole directions.

DEFINITION 3 (POLE DIRECTIONS). *If $s = p \in \mathbb{C}$ is a distinct pole of $G(s)$ then there exist one output direction $y_p \in \mathbb{C}^l$ and one input direction $u_p \in \mathbb{C}^m$ with infinite gain for $s = p$.*

For a system $G(s)$ with state-space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ the pole directions y_p and u_p for a *distinct* pole p can be computed from (Havre and Skogestad, 1996c)

$$y_p = C x_R / \|C x_R\|; \quad u_p = B^H x_L / \|B^H x_L\| \tag{11}$$

[1]Strictly speaking, the transfer function $G(s)$ can *not* be evaluated at $s = p$, since $G(s)$ is not analytic in the point $s = p$.

where $x_R \in \mathbb{C}^n$ and $x_L \in \mathbb{C}^n$ are the eigenvectors corresponding to the two eigenvalue problems

$$A x_R = p x_R; \quad x_L^H A = x_L^H p$$

Note, that the pole directions are normalized, i.e. $\|y_p\| = 1$ and $\|u_p\| = 1$. We will only consider distinct poles in this paper, for the case where the pole p is not distinct refer to (Havre and Skogestad, 1996c).

2.3 Interpolation constraints on S, T, S_I and T_I

CONSTRAINT 1 (RHP-ZERO). *If $G(s)$ has a RHP-zero at $s = z$ with output zero direction y_z, then for internal stability of the feedback system the following interpolation constraints must apply*

$$y_z^H T(z) = 0; \qquad y_z^H S(z) = y_z^H \tag{12}$$
$$T_I(z) u_z = 0; \qquad S_I(z) u_z = u_z \tag{13}$$

In words, (12) says that T must have a RHP-zero in the same output direction as G and that $S(z)$ has an eigenvalue of 1 with corresponding left eigenvector y_z. In a similar way T_I has RHP-zero with the same input direction as G.

CONSTRAINT 2 (RHP-POLE). *If $G(s)$ has a RHP-pole at $s = p$ with output direction y_p, then for internal stability of the feedback system the following interpolation constraints must apply*

$$S(p) y_p = 0; \qquad T(p) y_p = y_p \tag{14}$$
$$u_p^H S_I(p) = 0; \qquad u_p^H T_I(p) = u_p^H \tag{15}$$

The constraint (12) was first proved by Zames (1981), the proof of (14) is given in (Skogestad and Postlethwaite, 1996). The proofs of (13) and (15) follows similarly.

3 Factorizations of RHP-zeros and poles

A rational transfer function matrix $M(s)$ with RHP-poles p_i can be factorized in *Blaschke products* either at the output (subscript o) or at the input (subscript i) of $M(s)$ as follows

$$M(s) = \mathcal{B}_{po}(M) M_{so}(s); \quad M(s) = M_{si} \mathcal{B}_{pi}(M(s)) \tag{16}$$

M_{so}, M_{si} – Stable (subscript s) versions of $M(s)$ with the RHP-poles mirrored across the imaginary axis.

$\mathcal{B}_{po}(M), \mathcal{B}_{pi}(M)$ – All-pass[2] rational transfer function matrices (all singular values are equal to 1 for $s = j\omega$) containing the RHP-poles (subscript p) of $M(s)$.

The all-pass filters are (Havre and Skogestad, 1996a)

$$\mathcal{B}_{po}(M(s)) \;=\; \prod_{i=1}^{N_p}\left(I + \frac{2\mathrm{Re}(p_i)}{s - p_i} \hat{y}_{p_i} \hat{y}_{p_i}^H\right) \tag{17}$$
$$\mathcal{B}_{pi}(M(s)) \;=\; \prod_{i=1}^{N_p}\left(I + \frac{2\mathrm{Re}(p_i)}{s - p_i} \hat{u}_{p_i} \hat{u}_{p_i}^H\right) \tag{18}$$

[2]Strictly speaking, our definitions of $\mathcal{B}_{po}(M)$ etc. give the inverse of what is usually called the *all-pass* transfer function.

$\mathcal{B}_{po}(M)$ is obtained by factorizing at the output one RHP-pole at a time, starting with $M = \mathcal{B}_{p_1 o}(M) M_{p_1 o}$ where

$$\mathcal{B}_{p_1 o}(M(s)) = I + \tfrac{2\mathrm{Re}(p_1)}{s - p_1} \hat{y}_{p_1} \hat{y}_{p_1}^H$$

and $\hat{y}_{p_1} = y_{p_1}$ is the output pole direction of M for p_1. This procedure may be continued to factor out p_2 from $M_{p_1 o}$ where $\hat{y}_{p_2}$ is the output pole direction of $M_{p_1 o}$ (which need not coincide with y_{p_2}, the pole direction[3] of M and so on. A similar procedure may be used to factorize the poles at the input of $M(s)$.

In a similar sequential manner, the RHP-zeros can be factorized either at the output or at the input of $M(s)$

$$M(s) = \mathcal{B}_{zo}^{-1}(M) M_{mo}(s); \quad M(s) = M_{mi} \mathcal{B}_{zi}^{-1}(M(s)) \quad (19)$$

M_{mo}, M_{mi} – Minimum phase versions of $M(s)$ with the RHP-zeros mirrored across the imaginary axis.

$\mathcal{B}_{zo}(M), \mathcal{B}_{zi}(M)$ – All-pass rational transfer function matrices (all singular values are equal to 1 for $s = j\omega$) containing the RHP-zeros (subscript z) as RHP-poles.

We get (Havre and Skogestad, 1996a)

$$\mathcal{B}_{zo}(M(s)) = \prod_{j=1}^{N_z} (I + \tfrac{2\mathrm{Re}(z_j)}{s - z_j} \hat{y}_{z_j} \hat{y}_{z_j}^H) \quad (20)$$

$$\mathcal{B}_{zi}(M(s)) = \prod_{j=1}^{N_z} (I + \tfrac{2\mathrm{Re}(z_j)}{s - z_j} \hat{u}_{z_j} \hat{u}_{z_j}^H) \quad (21)$$

3.1 Some useful properties

Assume that M^{-1} exists, then

$$\mathcal{B}_{zi}(M) = \mathcal{B}_{po}(M^{-1}); \qquad M_{mi} = (M^{-1})_{so}^{-1} \quad (22)$$

$$\mathcal{B}_{zo}(M) = \mathcal{B}_{pi}(M^{-1}); \qquad M_{mo} = (M^{-1})_{si}^{-1} \quad (23)$$

Proof of (22) *and* (23).

$$M = M_{mi} \mathcal{B}_{zi}^{-1}(M) \quad \text{and} \quad M^{-1} = \mathcal{B}_{po}(M^{-1})\,(M^{-1})_{so}$$

invert the latter to obtain $M = (M^{-1})_{so}^{-1}\, \mathcal{B}_{po}^{-1}(M^{-1})$, and (22) follows from inspection. (23) follows similarly from the output factorization of RHP-zeros and input factorization of poles for M^{-1}. □

Repeated factorization:

$$\mathcal{B}_{pi}(\mathcal{B}_{pi}(M)) = \mathcal{B}_{pi}(M) \quad \mathcal{B}_{po}(\mathcal{B}_{po}(M)) = \mathcal{B}_{po}(M) \quad (24)$$

$$\mathcal{B}_{zi}(\mathcal{B}_{zi}^{-1}(M)) = \mathcal{B}_{zi}(M) \quad \mathcal{B}_{zo}(\mathcal{B}_{zo}^{-1}(M)) = \mathcal{B}_{zo}(M) \quad (25)$$

$$\mathcal{B}_{zi}(\mathcal{B}_{po}^{-1}(M)) = \mathcal{B}_{po}(M) \quad \mathcal{B}_{zo}(\mathcal{B}_{pi}^{-1}(M)) = \mathcal{B}_{pi}(M) \quad (26)$$

$$\mathcal{B}_{pi}(\mathcal{B}_{zo}(M)) = \mathcal{B}_{zo}(M) \quad \mathcal{B}_{po}(\mathcal{B}_{zi}(M)) = \mathcal{B}_{zi}(M) \quad (27)$$

Proof of (24)–(27). The two first are obvious (repeating a factorization of the same type gives back the same all-pass filter). If we assume that M^{-1} exist then for $\mathcal{B}_{zi}(\mathcal{B}_{po}^{-1}(M))$ in (26) we have

$$\mathcal{B}_{zi}(\mathcal{B}_{po}^{-1}(M)) = \mathcal{B}_{zi}(\mathcal{B}_{zi}^{-1}(M^{-1})) = \mathcal{B}_{zi}(M^{-1}) = \mathcal{B}_{po}(M)$$

where (22) is used twice with M set to M^{-1}, the second part of (26) and (27) follows similarly by also using (23). When M is not invertible the proofs are more complicated (need to go into the state-space realizations of the factorizations to prove these relations). □

[3] In fact: $\hat{y}_{p_2} = \mathcal{B}_{p_1 o}^H(M)|_{s=p_2} y_{p_2}$. Here $B|_{s=s_0}$ means the rational transfer function matrix $B(s)$ evaluated at the complex number $s = s_0$. Thus, it provides an alternative to $B(s_0)$, and it will mainly be used to avoid double parenthesis. It will also be used in combination with the absolute value for SISO transfer functions, i.e. $|m|_{s=s_0} = |m(s_0)|$.

4 Main result

In this section we derive general lower bounds on the $\mathcal{H}_\infty$-norm of closed-loop transfer functions when the plant G has one or more RHP-zeros and/or poles, by combined use of the interpolation constraints and the maximum modulus theorem. The bounds are applicable to closed-loop transfer functions on the form

$$W(s)X(s)V(s) \quad (28)$$

where X may be S, T, S_I or T_I. The idea is to derive lower bounds on $\|WXV(s)\|_\infty$ which are independent of the controller K_2. In general, we assume that WXV is stable. The "weights" W and V must be independent of K_2. They may be unstable but it must be possible to stabilize all transfer functions through the outputs. This implies that the unstable modes of W and V must also appear in G. Otherwise, the system is not stabilizable.

THEOREM 1 (LOWER BOUNDS ON $\|WXV(s)\|_\infty$). *Consider a plant G with N_z RHP-zeros z_j with input directions u_{z_j}, output directions y_{z_j} and N_p RHP-poles p_i with input directions u_{p_i} and output directions y_{p_i}. Let the closed-loop transfer functions WSV, WTV, WS_IV and WT_IV be stable. For each RHP-zero z_j the following bounds apply*

$$\|WSV(s)\|_\infty \geq \|W_{mo}|_{s=z_j} y_{z_j}\|_2 \cdot \|y_{z_j}^H V \mathcal{B}_{zi}(\mathcal{B}_{po}^{-1}(G)\,V)|_{s=z_j}\|_2 \quad (29)$$

$$\|WS_IV(s)\|_\infty \geq \|\mathcal{B}_{zo}(W\mathcal{B}_{pi}^{-1}(G))W|_{s=z_j} u_{z_j}\|_2 \cdot \|u_{z_j}^H V_{mi}|_{s=z_j}\|_2 \quad (30)$$

and for each RHP-pole p_i the following bounds apply

$$\|WTV(s)\|_\infty \geq \|\mathcal{B}_{zo}(W\mathcal{B}_{zo}^{-1}(G))\,W|_{s=p_i} y_{p_i}\|_2 \cdot \|y_{p_i}^H V_{mi}|_{s=p_i}\|_2 \quad (31)$$

$$\|WT_IV(s)\|_\infty \geq \|W_{mo}|_{s=p_i} u_{p_i}\|_2 \cdot \|u_{p_i}^H V \mathcal{B}_{zi}(\mathcal{B}_{zi}^{-1}(G)V)|_{s=p_i}\|_2 \quad (32)$$

Proof: See Appendix A.

REMARK. The somewhat messy notation can easily be interpreted. As an example take the last factor of (29): Factorize the RHP-poles at the output of G into an all-pass filter $\mathcal{B}_{po}(G)$, take its inverse (yields RHP-zeros) and multiply on the right with V (may add RHP-zeros if V is non-minimum phase), then factorize at the input the RHP-zeros of the product into an all-pass transfer function, take its inverse, multiply on the left with $y_{z_j} V$ and finally evaluate the result for $s = z_j$.

4.1 Some important special cases

G has no RHP-poles. Then $\mathcal{B}_{po}^{-1}(G) = I$ and $\mathcal{B}_{pi}^{-1}(G) = I$, and (29) and (30) become

$$\|WSV(s)\|_\infty \geq \|W_{mo}|_{s=z_j} y_{z_j}\|_2 \cdot \|y_{z_j}^H V_{mi}|_{s=z_j}\|_2 \quad (33)$$

$$\|WS_IV(s)\|_\infty \geq \|W_{mo}|_{s=z_j} u_{z_j}\|_2 \cdot \|u_{z_j}^H V_{mi}|_{s=z_j}\|_2 \quad (34)$$

The bounds (31) and (32) do not apply in this case.

G has no RHP-zeros. Then $\mathcal{B}_{zo}^{-1}(G) = I$ and $\mathcal{B}_{zi}^{-1}(G) = I$, and (31) and (32) become

$$\|WTV(s)\|_\infty \geq \|W_{mo}|_{s=p_i} y_{p_i}\|_2 \cdot \|y_{p_i}^H V_{mi}|_{s=p_i}\|_2 \quad (35)$$

$$\|WT_IV(s)\|_\infty \geq \|W_{mo}|_{s=p_i} u_{p_i}\|_2 \cdot \|u_{p_i}^H V_{mi}|_{s=p_i}\|_2 \quad (36)$$

The bounds (29) and (30) do not apply in this case.

Scalar stable weights. For the case with a scalar stable input weight $V(s) = v(s) \cdot I$, then the last terms in (29) and (32) become

$$V\mathcal{B}_{zi}(\mathcal{B}_{po}^{-1}(G)\,V) = \overbrace{v\mathcal{B}_{zi}(v)}^{v_m}\mathcal{B}_{zi}(\mathcal{B}_{po}^{-1}(G)) = v_m\mathcal{B}_{po}(G) \quad (37)$$

$$V\mathcal{B}_{zi}(\mathcal{B}_{zi}^{-1}(G)\,V) = \underbrace{v\mathcal{B}_{zi}(v)}_{v_m}\mathcal{B}_{zi}(\mathcal{B}_{zi}^{-1}(G)) = v_m\mathcal{B}_{zi}(G) \quad (38)$$

Since $v(s)$ is stable, RHP-pole/zero cancellation between v and $\mathcal{B}_{po}^{-1}(G)$ and $\mathcal{B}_{zi}^{-1}(G)$ are avoided.

Similarly, with a scalar stable output weight $W = w(s) \cdot I$, the two first terms of (30) and (31) become

$$\mathcal{B}_{zo}(W\mathcal{B}_{pi}^{-1}(G))\,W = \overbrace{\mathcal{B}_{zo}(w)\,w}^{w_m}\mathcal{B}_{zo}(\mathcal{B}_{pi}^{-1}(G)) = w_m\mathcal{B}_{pi}(G) \quad (39)$$

$$\mathcal{B}_{zo}(W\mathcal{B}_{zo}^{-1}(G))\,W = \underbrace{\mathcal{B}_{zo}(w)\,w}_{w_m}\mathcal{B}_{zo}(\mathcal{B}_{zo}^{-1}(G)) = v_m\mathcal{B}_{zo}(G) \quad (40)$$

Thus, for the case where both weights are scalar and stable, the bounds (29)–(32) become

$$\|wSv(s)\|_\infty \ge |w_m(z_j)| \cdot |v_m(z_j)| \cdot \|y_{z_j}^H\mathcal{B}_{po}(G)|_{s=z_j}\|_2 \quad (41)$$

$$\|wS_Iv(s)\|_\infty \ge |w_m(z_j)| \cdot |v_m(z_j)| \cdot \|\mathcal{B}_{pi}(G)|_{s=z_j}u_{z_j}\|_2 \quad (42)$$

$$\|wTv(s)\|_\infty \ge |w_m(p_i)| \cdot |v_m(p_i)| \cdot \|\mathcal{B}_{zo}(G)|_{s=p_i}y_{p_i}\|_2 \quad (43)$$

$$\|wT_Iv(s)\|_\infty \ge |w_m(p_i)| \cdot |v_m(p_i)| \cdot \|u_{p_i}^H\mathcal{B}_{zi}(G)|_{s=p_i}\|_2 \quad (44)$$

SISO-case. For the SISO-case where all transfer functions are scalars, some very nice and simple bounds can be derived, for further details see Section 6.

5 Applications of lower bounds

The lower bounds on $\|WXV(s)\|_\infty$ in Theorem 1 can be used to derive a large number of interesting and useful bounds.

5.1 Output performance

The previously derived bounds in terms of the $\mathcal{H}_\infty$-norms of S and T given in (Zames, 1981; Skogestad and Postlethwaite, 1996; Havre and Skogestad, 1996b) follow easily, and further generalizations involving output performance can be derived.

1. Weighted sensitivity, W_PS. Select $W = W_P$, $X = S$ and $V = I$. Note, this generalizes the previously found bounds to the case with a matrix valued weight. For the special case $W_P(s) = I$ we derive from (29)

$$\|S(s)\|_\infty \ge \max_{\text{RHP-zeros, } z_j} \|y_{z_j}^H\mathcal{B}_{po}(G)|_{s=z_j}\|_2 \ge 1 \quad (45)$$

2. Weighted complementary sensitivity, W_TT. Select $W = W_T$, $X = T$ and $V = I$. For $W_T(s) = I$ we derive from (31)

$$\|T(s)\|_\infty \ge \max_{\text{RHP-poles, } p_i} \|\mathcal{B}_{zo}(G)|_{s=p_i}y_{p_i}\|_2 \ge 1 \quad (46)$$

3. Closed-loop transfer function from disturbances to weighted outputs, W_PSG_d (see Figure 1). Select $W = W_P$, $X = S$ and $V = G_d$, and apply the bound (29).

4. Closed-loop transfer function from references of magnitude R to weighted outputs for 1-DOF controller, $W_P(-SGK_2 - I)R = -W_PSR$. Select $W = W_P$, $X = S$ and $V = R$, and apply the bound (29).

Note that we can also look at the combined effect of the two above by selecting $V = [\,G_d \quad R\,]$.

5. Closed-loop transfer function from measurement noise of magnitude N to weighted outputs, $-W_PTN$. Select $W = W_P$, $X = T$ and $V = N$, and apply the bound (31).

5.2 Input usage

The above provide generalizations of previous results, but we can also from Theorem 1 derive some new bounds in terms of input usage. These new bounds provide very interesting insights, for example, into the possibility of stabilizing an unstable plant with inputs of bounded magnitude.

The basis of these new bounds is to note that the transfer function from the outputs to the inputs, K_2S, can be rewritten as $-K_2S = T_IG^{-1}$. When G is unstable, G^{-1} has one or more RHP-zeros, it is important that the bounds in Theorem 1 can handle the case when $V = G^{-1}$ has RHP-zeros. Otherwise, G^{-1} evaluated at the pole of G would be zero in a certain direction, and we would not derive any useful bounds.

Some examples of bounds involving the input magnitude are (see Figure 1):

1. Closed-loop transfer function from disturbances d to weighted inputs u, $W_uK_2SG_d = -W_uT_IG^{-1}G_d$. Select $W = W_u$, $X = T_I$ and $V = G^{-1}G_d$, and apply the bound (32).

2. Closed-loop transfer function from references r of magnitude R to weighted inputs u, $W_uK_2SR = -W_uT_IG^{-1}R$. Select $W = W_u$, $X = T_I$ and $V = G^{-1}R$, and apply (32).

3. Closed-loop transfer function from measurement noise n of magnitude N to weighted inputs u, $W_uK_2SN = -W_uT_IG^{-1}N$. Select $W = W_u$, $X = T_I$ and $V = G^{-1}N$, and apply the bound (32).

Note that we can also look at the combined effect of the three above by selecting $V = [\,G_d \quad R \quad N\,]$.

For further implications of these bounds see the SISO case discussed in Section 6.

5.3 Tightness of bounds

Numerical results indicate that the bounds in Theorem 1 are tight, that is, in most cases there exists a controller which achieves the lower bound under consideration. For example, consider the problem:

- *Does there exists a controller which can stabilize a given unstable plant in the presence of disturbances, with $\|u\|_2 \le 1$ for all $\|d\|_2 \le 1$?*

The answer is "yes" if and only if there exists a controller for which $\|K_2SG_d\|_\infty < 1$. To test this bound, we can solve the $\mathcal{H}_\infty$-problem

$$\min_{K_2} \|K_2SG_d(s)\|_\infty$$

Alternatively, we can use

$$\min_{K_2} \ \|K_2 S G_d(s)\|_\infty = \min_{K_2} \ \|T_I G^{-1} G_d(s)\|_\infty$$
$$\geq \max_{p_i} \ \|u_{pi}^H G^{-1} G_d \mathcal{B}_{zi} (\mathcal{B}_{zi}^{-1}(G) G^{-1} G_d)|_{s=p_i}\|_2$$

Numerical results indicate that the inequality is always an equality when the number of disturbances are fewer or equal to the number of outputs.

5.4 2-DOF controller

For a 2-DOF controller the closed-loop transfer function from references to outputs is bounded as follows

$$\|W_P(SGK_1 - I)R\|_\infty \geq \|(W_P)_{mo}|_{s=z_j} y_{z_j}\|_2 \cdot \|y_{z_j}^H R_{mi}(z_j)\|_2 \quad (47)$$

Proof: See Appendix A; note that this bound does not follow directly from Theorem 1. The bound in (47) should be compared to the following bound for a 1-DOF controller (which follows from Theorem 1)

$$\|W_P S R(s)\|_\infty \geq \|(W_P)_{mo}|_{s=z_j} y_{z_j}\|_2 \cdot \|y_{z_j}^H R \mathcal{B}_{zi}(\mathcal{B}_{po}^{-1}(G)\, R)|_{s=z_j}\|_2 \quad (48)$$

The bounds in (47) and (48) apply to all RHP-zeros z_j in G. We see that for the 2-DOF controller only the RHP-zeros pose limitations.

6 Implications of the bounds (SISO-case)

Some of the important implications of the bounds are most easily understood by studying the SISO case. We consider a plant with a single RHP-pole p and/or a single RHP-zero z. Then

$$\mathcal{B}_p(G) \triangleq \mathcal{B}_{pi}(G) = \mathcal{B}_{po}(G) = 1 + \frac{2p}{s-p} = \frac{s+p}{s-p} \quad \left\{ \begin{array}{l} \text{if } G \text{ has a} \\ \text{RHP-pole} \end{array} \right.$$

$$\mathcal{B}_z(G) \triangleq \mathcal{B}_{zi}(G) = \mathcal{B}_{zo}(G) = 1 + \frac{2z}{s-z} = \frac{s+z}{s-z} \quad \left\{ \begin{array}{l} \text{if } G \text{ has a} \\ \text{RHP-zero} \end{array} \right.$$

Also note[4] that $|\mathcal{B}_p(G)|_{s=z} = |\mathcal{B}_z(G)|_{s=p} = \frac{|z+p|}{|z-p|}$. We define

$$M_{pz} \overset{\text{def}}{=} \left\{ \begin{array}{ll} \frac{|z+p|}{|z-p|} & \text{if } G \text{ has both a RHP-zero and pole} \\ 1 & \text{otherwise} \end{array} \right.$$

6.1 Combined RHP-zeros and poles

- *Closely located* RHP-*poles and zeros imply large sensitivity peak.*

From (45) and (46) we get

$$\|S(s)\|_\infty \geq M_{pz} \quad \text{and} \quad \|T(s)\|_\infty \geq M_{pz} \quad (49)$$

which is large if $|z-p|$ is small. If the RHP-zero and pole are different by a factor of 10 or more, then the interaction between them are small, for example, $p/z = 10$ gives $\frac{|z+p|}{|z-p|} = 1.22$.

[4] $|\mathcal{B}_p(G)|_{s=z_j} \triangleq |\mathcal{B}_p(G)|_{s=z_j}|$.

6.2 RHP-poles and stabilization

- *Input saturation combined with disturbances and noise may make stabilization impossible.*

The input signal for a 1-DOF controller is (positive feedback)

$$u = K_2 S(G_d d + N\tilde{n} - R\tilde{r}) \quad (50)$$

Disturbances. The input magnitude in response to a "worst-case" sinusoidal disturbance $|d| = 1$, has the following lower bound

$$|u_{\max}| = \|K_2 S G_d(s)\|_\infty \geq \left|(G^{-1}G_d)_{ms}\right|_{s=p} \cdot M_{pz} \quad (51)$$

where "ms" means that both the RHP-poles and zeros are mirrored into the left half plane.

Proof of (51). Follows directly from (44) with $v = G^{-1}G_d$ and $w = 1$ when v is stable. For the case with v unstable (i.e. when G has RHP-zeros) some more careful analysis is needed. □

EXAMPLE 1 *Let*

$$G(s) = \frac{5}{(10s+1)(s-1)} \quad \textit{and} \quad G_d(s) = \frac{k_d(s-2)}{(s+1)(0.2s+1)(s+2)}$$

Then $(G^{-1}G_d)_{ms} = \frac{10s+1}{5}\frac{k_d}{0.2s+1}$ *and we get* $|u_{\max}| \geq |(G^{-1}G_d)_{ms}|_{s=1} = \frac{11}{6}k_d$. *If we require* $|u| \leq 1$ *then we must have* $k_d \leq \frac{6}{11} \approx 0.55$. *In other words, we may encounter problems stabilizing the plant if* $k_d > 0.55$. *Indeed, this is confirmed by a particular nonlinear simulation example (see, Skogestad and Postlethwaite, 1996, page 192).*

References. For reference changes of magnitude R we find the same bound (51), but with $G_d(s)$ replaced by $R(s)$. However, the implications for stabilization are less since we always may choose *not* to follow the references (e.g. set $R = 0$).

Measurement noise. More importantly, the same bound (51), but with $G_d(s)$ replaced by $N(s)$, applies to measurement noise of magnitude $N(s)$, since $u = K_2 S N\tilde{n}$. More specifically, we have

$$|u_{\max}| = \|K_2 S N(s)\|_\infty \geq \left|(G^{-1}N)_{ms}\right|_{s=p} \cdot M_{pz} \quad (52)$$

Thus, if we require $|u| \leq 1$, then we can expect problems with stabilization[5] if $|G_{ms}(p)| < |N(p)| \cdot M_{pz}$ (we have here assumed that $N(s)$ is minimum-phase). That is:

- *We expect problems if the plant gain is less than the measurement noise at frequencies corresponding to the unstable pole.*

EXAMPLE 2 *Consider the plant* $G(s) = 1/(s-10)$ *with the unstable mode* $p = 10$. *With measurement noise we have* $(M_{pz} = 1)$

$$|u_{\max}| = \|K_2 S N(s)\|_\infty \geq \left|(G^{-1}N)_{ms}\right|_{s=p} = |N(p)| \cdot \left|\frac{(p+10)}{1}\right| = 20 \cdot |N(p)|$$

Thus, if we require $|u| \leq 1$, *then we can expect problem with stabilization if the noise is such that* $|N(p)| > 0.05$. *Simulations with a proportional controller* $K_2 = -20$ *(positive feedback) and input saturation* ± 1 *confirms this bound. For a higher controller gain the noise may need to be increased by a factor of two to get instabilities.*

[5] We cannot rigorously determine nonlinear stability from a linear analysis, but if the required linear inputs exceed the saturation bounds around the frequency of the RHP-pole, then nonlinear instability is highly likely.

6.3 RHP-zeros and output performance

From (41) with $w = 1$ and $v = G_d$ we find for a plant with RHP-zero z

$$|e_{\max}| = \|SG_d(s)\|_\infty \geq |(G_d)_{ms}|_{s=z} \cdot M_{pz} \qquad (53)$$

and we must require $|(G_d)_{ms}|_{s=z} \cdot M_{pz} < 1$ to guarantee $|e| < 1$ for $|d| = 1$. Similarly, we have from (48) for reference changes with a 1-DOF controller

$$|e_{\max}| = \|SR(s)\|_\infty \geq |R(z)| \cdot M_{pz} \qquad (54)$$

where we have assumed that R is stable and minimum-phase. The factor M_{pz} does not apply to the 2-DOF control configuration, since (47) gives

$$|e_{\max}| = \|(SGK_1 - 1)R(s)\|_\infty \geq |R(z)| \qquad (55)$$

i.e. only the RHP-zeros pose fundamental limitations. Thus, the effect of the RHP-pole may be counteracted with a 2-DOF controller, at least if we neglect the presence of model uncertainty.

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Internal reports and MATLAB-software to compute the bounds in Theorem 1 are available on the internet, see `http://www.kjemi.unit.no/~skoge`.

A Proofs of the results

Proof of Theorem 1. Due to limited space we will only prove (29), the proofs of (30)–(32) follows similarly. To prove (29), factorize RHP-zeros in W at the output, i.e. $W = \mathcal{B}_{zo}^{-1}(W)W_{mo}$. Define[6] $-S_1 \triangleq T(G_{so}K_2)^{-1}$, it then follows that $S = S_1\mathcal{B}_{po}^{-1}(G)$. We get[7]

$$\begin{aligned} WSV &= \mathcal{B}_{zo}^{-1}(W)\, W_{mo}S_1\mathcal{B}_{po}^{-1}(G)\, V \\ &= \mathcal{B}_{zo}^{-1}(W)\, \underbrace{W_{mo}S_1\, (\mathcal{B}_{po}^{-1}(G)\, V)_{m,i}}_{\triangleq (WS_1V)_m}\, \mathcal{B}_{zi}^{-1}(\mathcal{B}_{po}^{-1}(G)\, V) \end{aligned}$$

Introduce the scalar transfer function $f(s) = x_1^H(WS_1V)_m x_2$, which is analytic (stable) in RHP. We want to choose x_1 and x_2 so that $|f(s)|$ obtains maximum $J(s) = \max_{\|x_1\|=1,\, \|x_2\|=1} |f(s)|$. Then the following relation holds for all $z_j \in$ RHP

$$\|WSV(s)\|_\infty = \|(WS_1V(s))_m\|_\infty \geq \|J(s)\|_\infty \geq J(z_j)$$

The first equality follows since $(WS_1V)_m$ only differ from WSV with all-pass filters on the input and output. The first inequality follows since the largest singular value measures the maximum gain of a matrix independent of direction, i.e. $\bar{\sigma}(A) \geq \|Aw\|_2$ and $\bar{\sigma}(A) \geq \|w^H A\|_2$ for any vector w with $\|w\|_2 = 1$. The second inequality follows from the maximum modulus theorem. We get

$$\begin{aligned} J(z_j) &= \max_{\|x_1\|=1,\, \|x_2\|=1} x_1^H W_{mo}S_1\, (\mathcal{B}_{po}^{-1}(G)\, V)_{mi}|_{s=z_j} x_2 \\ &\geq \max_{\|x_1\|=1,\, \|x_2\|=1} x_1^H W_{mo} y_{z_j} y_{z_j}^H S_1\, (\mathcal{B}_{po}^{-1}(G)\, V)_{mi}|_{s=z_j} x_2 \\ &= \|W_{mo}|_{s=z_j} y_{z_j}\|_2 \cdot \|y_{z_j}^H\, (\mathcal{B}_{po}^{-1}(G)\, V)_{mi}|_{s=z_j}\|_2 \end{aligned}$$

□

Proof of (47). Factorize RHP-zeros of R and W_P at the input and the output, i.e. $R = R_{mi}\mathcal{B}_{zi}^{-1}(R)$ and $W_P = \mathcal{B}_{zo}^{-1}(W_P)\,(W_P)_{mo}$. Then

$$\|W_P(SGK_1 - I)R\|_\infty = \|(W_P)_{mo}(SGK_1 - I)R_{mi}\|_\infty$$

Introduce the scalar function $f(s) = x_1^H W_P(SGK_1 - I)Rx_2$ which is analytic (stable) in RHP since the closed-loop system is stable. We want to choose x_1 and x_2 so that $|f(s)|$ obtains maximum $J(s) = \max_{\|x_1\|=1,\, \|x_2\|=1} |f(s)|$. We get

$$\|(W_P)_{mo}(SGK_1 - I)R_{mi}\|_\infty \geq \|J(s)\|_\infty \geq J(z_j)$$

The first inequality follows since the largest singular value measures the maximum gain of a matrix independent of direction, i.e. $\bar{\sigma}(A) \geq \|Aw\|_2$ and $\bar{\sigma}(A) \geq \|w^H A\|_2$ for any vector w with $\|w\|_2 = 1$. The second inequality follows from the maximum modulus theorem. We get

$$\begin{aligned} J(z_j) &= \max_{\|x_1\|=1,\, \|x_2\|=1} x_1^H W_P(SGK_1 - I)R(z_j)x_2 \\ &\geq \max_{\|x_1\|=1,\, \|x_2\|=1} x_1^H W_P y_{z_j} y_{z_j}^H (SGK_1 - I)R(z_j)x_2 \\ &= \|W_P(z_j)y_{z_j}\|_2 \cdot \|(\underbrace{y_{z_j}^H G(z_j)}_{0} K_1(z_j) - y_{z_j}^H)R(z_j)\|_2 \\ &= \|W_P(z_j)y_{z_j}\|_2 \cdot \|y_{z_j}^H R(z_j)\|_2 \end{aligned}$$

□

[6] Note that generally $S_1 \neq S_{mi}$ since only RHP-zeros in S due to RHP-poles in G has been factorized in S_1, S may have additional RHP-zeros due to RHP-poles in K_2

[7] The notation $(WS_1V)_m$ means the minimum phase version of WS_1V with the RHP-zeros of W factorized on the output and the RHP-zeros of S_1V on the input.