Two Degree of Freedom Controller Design for an Ill-conditioned Distillation Process Using μ-synthesis

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Abstract

The structured singular value framework is applied to a distillation benchmark problem formulated for the 1991 IEEE Conference on decision and control (CDC). A two degree of freedom controller, which satisfies all control objectives of the CDC problem, is designed using μ-synthesis. The design methodology is presented and special attention is paid to the approximation of given control objectives into frequency domain weights.

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1 Introduction

The purpose of this paper is to demonstrate, by way of an example, how the structured singular value (SSV, μ) framework, Doyle [4], may be used to design a robust controller for a given control problem. The problem involves an uncertain system and control objectives that cannot be directly incorporated into the μ-framework. In particular, we consider how to approximate the given problem as a μ-problem by deriving suitable frequency dependent weights. These define the model uncertainty and control objectives in the μ-framework.

The control problem studied in this paper was introduced by Limebeer [8] as a benchmark problem at the 1991 CDC, where it formed the basis for a design case study aimed at investigating the advantages and disadvantages of various controller design methods for ill-conditioned systems.

The problem originates from Skogestad *et al.* [17] where a simple model of a high purity distillation column was used to demonstrate that ill-conditioned plants are potentially extremely sensitive to model uncertainty. In [17] uncertainty and performance specifications were given as frequency dependent weights, *i.e.* the problem was *defined* to suit the μ-framework and therefore a μ-optimal controller yields the optimal solution to that problem.

However, in the CDC benchmark problem [8] uncertainty is defined in terms of parametric gain and delay uncertainty and the control objectives are a mixture of time domain and frequency domain specifications. These specifications cannot be directly transformed into frequency dependent weights, but have to be approximated to fit into the μ-framework.

The distillation problem in [17] and variants of this problem, like the CDC problem [8], has been studied by several authors, *e.g* Freudenberg [6], Yaniv and Barlev [21], Lundström *et al.* [10], Hoyle *et al.* [7], Postlethwaite *et al.* [14], Yaniv and Horowitz [22] and Zhou and Kimura [24]. In three recent studies; Limebeer *et al.* [9], van Diggelen and Glover [3] and Whidborne *et al.* [20], two degree of freedom controllers are designed for the CDC problem. The three latter papers are all based on the loop shaping design procedure of McFarlane and Glover [13], where uncertainties are modeled as $\mathcal{H}_\infty$-bounded perturbations in the normalized coprime factors of the plant. To obtain the desired performance, [9] use a reference model design approach, [3] use the Hadamard weighted $\mathcal{H}_\infty$-Frobenius formulation from [2], while [20] use the method of inequalities (Zakian and Al-Naib [23]) where the performance requirements are explicitly expressed as a set of algebraic inequalities.

The two degree of freedom design in this paper differs from [9], [3] and [20] in that we use μ-synthesis for our design. With this method uncertainty is modeled as linear fractional uncertainty and performance is specified as in a standard $\mathcal{H}_\infty$-control problem. Like [9], we specify some of the control objectives as a model-matching problem.

The paper is organized as follows: A brief introduction to the μ-framework is presented in Section 2. The benchmark problem is defined in Section 3. In Section 4 we outline the design method used in this paper. In Section 5 we gradually transform (approximate) the given problem into a μ-problem and demonstrate the effect of different weight adjustments. The final controller designed in this section demonstrates that the control objectives defined by Limebeer [8] are achievable. Finally the results are discussed and summarized.

All of the results and simulations presented in this paper were computed using the MATLAB "μ-Analysis and Synthesis Toolbox" [1].

2 CDC problem definition

The plant model and design specifications for the CDC benchmark problem [8] are presented in this section.

2.1 Plant model

The process to be controlled is a distillation column with reflux flow and boilup flow as manipulated inputs and product compositions as outputs. The resulting model is ill-conditioned, as is here given by

$$\hat{G}(s) = \frac{1}{75s+1}\begin{bmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{bmatrix}\begin{bmatrix} k_1 e^{-\theta_1 s} & 0 \\ 0 & k_2 e^{-\theta_2 s} \end{bmatrix} \tag{1}$$

$$k_i \in [0.8 \;\; 1.2] \;\; ; \;\; \theta_i \in [0.0 \;\; 1.0] \tag{2}$$

In physical terms this is equivalent to a gain uncertainty of $\pm 20\%$ and a delay of up to 1 min in each input channel. The set of possible plants defined by Eq.1-2 is denoted Π in the sequel.

2.2 Design specifications

Specifications **S1** to **S4** should be fulfilled for *every* plant $\hat{G} \in \Pi$:

S1 Closed loop stability.

S2 For a unit step demand in channel 1 at $t = 0$ the plant outputs y_1 (tracking) and y_2 (interaction) should satisfy:

- $y_1(t) \geq 0.9$ for all $t \geq 30$ min.
- $y_1(t) \leq 1.1$ for all t
- $0.99 \leq y_1(\infty) \leq 1.01$
- $y_2(t) \leq 0.5$ for all t
- $-0.01 \leq y_2(\infty) \leq 0.01$

Corresponding requirements hold for a unit step demand in channel 2.

S3 $\bar{\sigma}(K_y \hat{S}) < 316, \;\; \forall \omega.$

S4 Alt.1: $\bar{\sigma}(\hat{G} K_y) < 1$ for $\omega \geq 150$
Alt.2: $\bar{\sigma}(K_y \hat{S}) < 1$ for $\omega \geq 150$

Here K_y denotes the feedback part of the controller and $\hat{S} = (I + \hat{G} K_y)^{-1}$ the sensitivity function for the worst case $\hat{G}$.

Specifications **S3** and **S4** are not explicitly stated in [8], but formulated as "the closed loop transfer function between output disturbance and plant input $[K_y \hat{S}]$ be gain limited to about 50 dB [≈ 316 (**S3**)] and the unity gain cross over frequency of the largest singular value should be below 150 rad/min [(**S4**)]." Different researchers have given the latter specification

Figure 1: Block diagram without weight functions.

different interpretations, *e.g.* [3] use Alt.1. while [20] use Alt.2. For the purpose of this paper, this diversity is advantageous, since it gives us the opportunity to start with the easier alternative (Alt.1) and then show how to refine the μ-problem to achieve the tougher requirement (Alt.2).

Note that **S4** Alt.1 is implied by **S1** in practice which in turn is implied by **S2**, so the actual performance requirements are **S2** and **S3** (and **S4** Alt.2).

Most of the specifications in this paper may be viewed as bounds on transfer functions from some inputs to some outputs. The notation for these transfer functions is defined by Fig. 1 and the matrices in Eq. 4 - 5. The controller K in Fig. 1 may be a One Degree of Freedom controller (ODF) or a Two Degree of Freedom controller (TDF). A TDF controller may be partitioned into two parts

$$K = [K_r \ K_y] = \left[\begin{array}{c|cc} A_K & B_{Kr} & B_{Ky} \\ \hline C_K & D_{Kr} & D_{Ky} \end{array}\right] \tag{3}$$

where K_y is the feedback part of the controller and K_r is the prefilter part.

For an ODF controller $K_r = K_y$, which yields the following transfer functions:

$$\begin{bmatrix} e \\ y \\ u \end{bmatrix} = \begin{bmatrix} S & T \Leftrightarrow T_{yr,id} & \Leftrightarrow T \\ S & T & \Leftrightarrow T \\ \Leftrightarrow K_y S & K_y S & \Leftrightarrow K_y S \end{bmatrix} \begin{bmatrix} d \\ r \\ n \end{bmatrix} \tag{4}$$

where $S = (I + GK_y)^{-1}$ is the sensitivity function, $T = (I + GK_y)^{-1}GK_y$ is the complementary sensitivity function and $T_{yr,id}$ is the reference model for the setpoint change. Note that if $T_{yr,id} = I$, then the transfer function from r to e is $T \Leftrightarrow T_{yr,id} = \Leftrightarrow S$.

For a TDF controller $K_r \neq K_y$, which yields the following transfer functions:

$$\begin{bmatrix} e \\ y \\ u \end{bmatrix} = \begin{bmatrix} S & SGK_r \Leftrightarrow T_{yr,id} & \Leftrightarrow T \\ S & SGK_r & \Leftrightarrow T \\ \Leftrightarrow K_y S & (I + K_y G)^{-1} K_r & \Leftrightarrow K_y S \end{bmatrix} \begin{bmatrix} d \\ r \\ n \end{bmatrix} \tag{5}$$

In this case, the transfer function from r to e is not equal to $\Leftrightarrow S$ if $T_{yr,id} = I$.

3 The μ-framework

This section gives a very brief introduction to μ-analysis and synthesis and defines some of the nomenclature used in the rest of the paper. For further details, the interested reader may consult for example [17], [18] and [1].

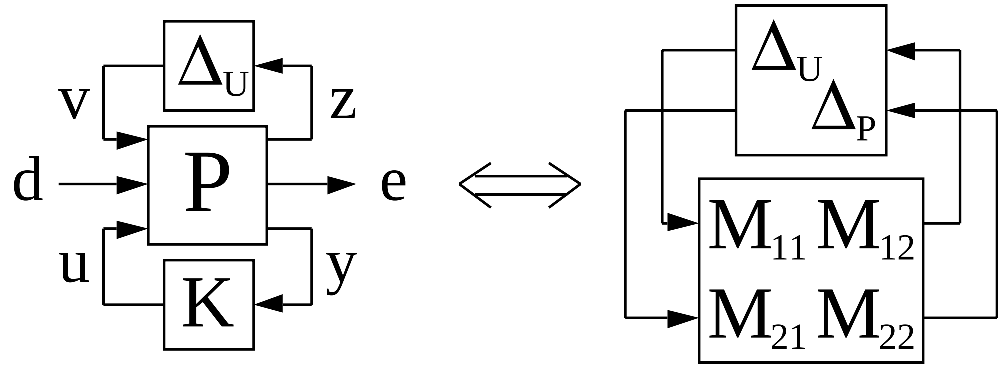

Figure 2: General problem description

The $\mathcal{H}_\infty$-norm of a transfer function $M(s)$ is the peak value of the maximum singular value over all frequencies.

$$||M(s)||_\infty \equiv \sup_\omega \bar{\sigma}(M(j\omega)) \tag{6}$$

The left block diagram in Fig.2 shows the general problem formulation in the μ-framework. It consists of an augmented plant P (including a nominal model and weighting functions), a controller K and a (block-diagonal) perturbation matrix $\Delta_U = \mathrm{diag}\{\Delta_1, \cdots, \Delta_n\}$ representing the uncertainty.

Uncertainties are modeled by the perturbations (Δ_i's) and uncertainty weights included in P. These weights are chosen such that $\|\Delta_U\|_\infty \leq 1$ generates the family of all possible plants to be considered. In principle Δ_U may contain both real and complex perturbations, but in this paper only complex perturbations are used.

The performance is specified by weights in P which normalize $\mathbf{d}$ and $\mathbf{e}$ such that a closed-loop $\mathcal{H}_\infty$-norm from $\mathbf{d}$ to $\mathbf{e}$ of less than 1 (for the worst case Δ_U) means that the control objectives are achieved [1].

The framework in Fig.2 may be used for both one degree of freedom (ODF) and two degree of freedom (TDF) controller design. In the ODF case the controller input $\mathbf{y}$ is the difference between set-points and measured plant outputs, $\mathbf{y} = r \Leftrightarrow y_m$, while in the TDF case $\mathbf{y} = \begin{bmatrix} r^T & \Leftrightarrow y_m^T \end{bmatrix}^T$.

The right block diagram in Fig.2 is used for robustness analysis. M is a function of P and K, and Δ_P ($\|\Delta_P\|_\infty \leq 1$) is a fictitious "performance perturbation" connecting $\mathbf{e}$ to $\mathbf{d}$. Provided that the closed loop system is nominally stable the condition for Robust Performance (RP) is:

$$RP \quad \Leftrightarrow \quad \mu_{RP} = \sup_\omega \mu_\Delta(M(j\omega)) < 1 \tag{7}$$

where $\Delta = \mathrm{diag}\{\Delta_U, \Delta_P\}$.

μ is computed frequency-by-frequency through upper and lower bounds. Here we only consider the upper bound

$$\mu_\Delta(M(j\omega)) \leq \inf_{D \in \mathbf{D}} \bar{\sigma}(DMD^{-1}) \tag{8}$$

[1] Note that $\mathbf{d}$ and $\mathbf{e}$ in Fig. 2 are not equivalent to d and e in Fig. 1, but may *contain* d and e among other signals.

where $\mathbf{D} = \{D | D\Delta = \Delta D\}$.

At present there is no direct method to synthesize a μ-optimal controller, however, μ-synthesis (DK-iteration) which combines μ-analysis and $\mathcal{H}_\infty$-synthesis often yields good results. This iterative procedure was first proposed in [5] and [15]. The idea is to attempt to solve

$$\min_K \inf_{D \in \mathbf{D}} \sup_\omega \bar{\sigma}(DMD^{-1}) \tag{9}$$

(where M is a function of K) by alternating between minimizing $\sup_\omega \bar{\sigma}(DMD^{-1})$ for either K or D while holding the other fixed. The iteration steps are:

DK1 Scale the interconnection matrix M with a stable and minimum phase rational transfer matrix $D(s)$ with appropriate structure (an identity matrix with right dimensions is a common initial choice).

DK2 Synthesize an $\mathcal{H}_\infty$-controller for the scaled problem, $\min_K \sup_\omega \bar{\sigma}(DMD^{-1})$.

DK3 Stop iterating if the performance is satisfactory or if the $\mathcal{H}_\infty$-norm does not decrease, else continue.

DK4 Compute the upper bound on μ (Eq.8) to obtain new D-scales as a function of frequency $D(j\omega)$.

DK5 Fit the magnitude of each element of $D(j\omega)$ to a stable and minimum phase rational transfer function and go to **DK1**.

Each of the minimizations (steps **DK2** and **DK4**) are convex, but joint convexity is not guaranteed.

The $\mathcal{H}_\infty$-controller synthesized in step **DK2** has the same number of states as the augmented plant P plus twice the number of states of D, hence it is desirable to keep the order of P and the D-scales as low as possible whilst satisfying the controller specification criteria.

4 Design procedure

The CDC specifications in Section 2 cannot be directly applied in the μ-framework. The reasons for this are: 1) The gain-delay uncertainty in Eq. 1-2 has to be approximated into linear fractional uncertainty (Fig.2); 2) Specification **S2** needs to be approximated since it is defined in the time domain; 3) In the μ-framework, it is not possible to directly bound the four SISO transfer functions associated with **S2** and the 2×2 transfer function associated with **S3** (and **S4 Alt.2**). Instead these control objectives must be reflected in the $\mathcal{H}_\infty$-norm of the transfer function from $\mathbf{d}$ to $\mathbf{e}$ (Fig.2).

The following approach makes it possible to apply μ-synthesis to this kind of problem:

1 Approximate the given problem into a μ-problem.

2 Synthesize a robust controller for the μ-problem.

3 Verify that the controller satisfies the original specifications (**S1**-**S4**) for the original set of plants (Π).

Step 1 is our major concern in this paper. Several approaches may be used to obtain the μ-problem, however, the following are general guidelines: A) Choose $\mathbf{d}$ and $\mathbf{e}$ such that all essential control objectives are reflected in the $\mathcal{H}_\infty$-norm of the transfer function between these signals. At the same time keep the dimension of $\mathbf{d}$ and $\mathbf{e}$ as small as possible. B) Use low order uncertainty and performance weights to keep the order of P and thereby the order of the controller low. The complexity and order of these weights may be increased later, if required. C) Use weighting parameters with physical meaning, since these parameters are the 'tuning knobs' during the design stage. The derivation of such weighting functions for the CDC problem is treated in detail in the next section.

Step 2 is fairly straightforward using DK-iteration and the available software (*e.g.* [1]). Experience with this iterative scheme shows that, for the first few iterations, it is best if the controller synthesized in step **DK2** is slightly sub-optimal ($\mathcal{H}_\infty$-norm 5-10% larger than the optimal) and that the D-scale fit in step **DK5** is of low order. In subsequent iterations, controllers that are close to optimality and higher order D-scales may be used if required. However, it is also recommended that the final controller is slightly sub-optimal since this yields a blend of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimality with generally better high frequency roll-off than the optimal $\mathcal{H}_\infty$-controller.

Step 3 is, in this paper, performed using time simulations with the four extreme combinations of gain uncertainty (Eq.2) and a 1 min delay (approximated as a second order Padé approximation).

5 Controller design

In this section we design controllers for the benchmark problem, using the design procedure outlined above. Actually, we start with a controller designed for the "original" problem defined in Skogestad *et al.* [17] and check the performance of this controller with respect to the CDC specifications defined in Section 2. We then gradually refine the μ-formulation by adding further input and output signals to $\mathbf{d}$ and $\mathbf{e}$ and by adjusting the uncertainty and performance weighting functions.

This gradual approach clearly demonstrates the effect of the weighting function refinements, and thereby is of tutorial value. Moreover, it is also a good approach for "real" problems, since one should not put more effort into the μ-formulation than required, that is, one should start with a simple problem formulation, and refine the problem formulation if the specifications are not met.

5.1 ODF-controller for "original" specifications

The "original" problem presented in Skogestad et al. (1988) [17] is defined by Fig. 3 and the following transfer function matrices.

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{bmatrix} \tag{10}$$

$$W_\Delta(s) = \frac{(s+0.2)}{(0.5s+1)} I_{2\times 2} \tag{11}$$

$$W_e(s) = \frac{1}{2}\frac{(20s+2)}{20s} I_{2\times 2} \tag{12}$$

Figure 3: Original ODF-problem formulation.

Table 1: Control performance for ODF-original with gain uncertainty and a second order Padé approximation of a 1 min delay. (See also Fig.4)

step	gain unc.		set-point tracking			interaction	
ch.	k_1	k_2	$t = 30$	max	$t = 100$	max	$t = 100$
1	1.2	1.2	0.989	1.008	1.000	**0.856**	0.000
1	1.2	0.8	0.934	1.001	1.001	**1.047**	0.000
1	0.8	1.2	0.941	1.006	1.000	0.427	-0.001
1	0.8	0.8	**0.889**	1.000	1.000	**0.625**	0.000
2	1.2	1.2	0.993	1.095	1.000	**0.859**	0.001
2	1.2	0.8	0.964	1.007	1.000	**0.536**	-0.001
2	0.8	1.2	0.956	**1.198**	1.001	**0.934**	0.000
2	0.8	0.8	0.929	1.000	1.000	**0.627**	0.000

Table 2: Control performance for TDF-original with gain uncertainty and a second order Padé approximation of a 1 min delay. (See also Fig.7)

step	gain unc.		set-point tracking			interaction	
ch.	k_1	k_2	$t = 30$	max	$t = 100$	max	$t = 100$
1	1.2	1.2	**0.889**	1.008	1.003	0.175	0.004
1	1.2	0.8	0.913	1.000	1.000	0.497	0.000
1	0.8	1.2	0.902	1.000	1.000	0.257	0.000
1	0.8	0.8	0.905	1.000	1.000	0.156	0.000
2	1.2	1.2	**0.891**	1.014	1.005	0.175	0.004
2	1.2	0.8	0.917	1.000	1.000	0.126	0.000
2	0.8	1.2	0.928	1.000	1.000	0.368	0.000
2	0.8	0.8	0.921	1.000	1.000	0.156	0.000

In conclusion, the one degree-of-freedom controller designed for the "original problem", almost satisfies the tracking requirements for the CDC-problem, but the closed-loop suffers from strong interactions and excessive use of manipulated inputs, in particular at very high frequencies ($\omega > 10$ rad/min). We next see of a two degrees-of-freedom (TDF) design can alleviate these problems.

5.2 TDF-controller for "original" specifications

Strictly speaking, the original problem formulation of Skogestad et al. [17] cannot take advantage of a TDF controller, because the specification is on the sensitivity function $S = (I + GK_y)^{-1}$, which depends only on the feedback part of the controller, K_y. However, if instead, we interpret the specification in terms of the transfer function from references r to errors e, $SGK_r \Leftrightarrow I$, then robust performance can be improved. Lundström et al. [10] interpreted the specifications in this way and were able to reduce μ_{RP} from 0.978 to 0.926 with a TDF controller. We denote this design "TDF-original".

Simulations and tabulated data for the TDF-original design are shown in Fig. 7 and Table 2. The setpoint tracking specification is still not quite satisfied, but the interactions have almost disappeared compared to the ODF-original response. However, there are unpleasant high frequency oscillations in all responses. This oscillation also shows up as a "ringing peak" in the transfer function from r to e ($SGK_r \Leftrightarrow I$) as illustrated in Fig. 8. This peak could have been eliminated if a better uncertainty weight had been used, *i.e.* an uncertainty weight that covers a 1 min delay (rather than only 0.9 min). Furthermore, as illustrated in Fig. 9, the TDF-original design suffers from a very high peak (about 100) on the sensitivity function. This is unacceptable for a practical design, but is not unexpected since there is no explicit bound on S.

We conclude that we are not able to meet the CDC-specifications by designing a μ-optimal controller using the "original" uncertainty and performance weights. We therefore need to consider the CDC-specifications explicitly and modify the weights. We also need to include a weight which explicitly bounds the transfer function KS.

5.3 Weight selection for CDC specifications

In this section we approximate the CDC specifications as frequency dependent weights.

5.3.1 Uncertainty weights

The gain-delay uncertainty in Eq.2 is not quite covered by the uncertainty weight defined in Eq.11. A better weight is presented in [12]:

$$W_\Delta(s) = \frac{(1+\frac{k_r}{2})\theta_{max}s + k_r}{\frac{\theta_{max}}{2}s+1} I_{2\times 2} = \frac{1.1s+0.2}{0.5s+1} I_{2\times 2} \tag{16}$$

where $k_r = 0.2$ is the relative gain uncertainty and $\theta_{max} = 1$ is the maximum delay. This weight has the same low order as that of Eq. 11 and *almost* cover the gain and delay uncertainty. A slight modification to Eq.16 yields a weight that *completely* covers the uncertainty ([12]), but is of higher order:

$$W_\Delta(s) = \frac{1.1s+0.2}{0.5s+1} * \frac{\left(\frac{s}{2.363}\right)^2 + 2*0.838\frac{s}{2.363}+1}{\left(\frac{s}{2.363}\right)^2 + 2*0.685\frac{s}{2.363}+1} I_{2\times 2} \tag{17}$$

It is often fruitful to start with the simpler weight (Eq. 16) and if the performance verification (step 3 of the design procedure) shows that this uncertainty model does not yield a robust controller for the set of plants Π, then the more rigorous uncertainty model (Eq. 17) should be used. This is the approach taken here.

5.3.2 ODF performance weights

A simple way to approximate the performance specifications **S2** and **S3** into a μ-problem is shown in Fig.10, where K_y is an ODF-controller.

Here, the weight W_{S2} on the sensitivity function is used to represent specification S2 and weight W_{S3} represents specification S3. A reasonable choice for W_{S2} is the following which is taken from the original formulation.

$$W_{S2}(s) = \frac{1}{M_S}\frac{\tau_{cl}s + M_S}{\tau_{cl}s + A} I_{2\times 2}. \tag{18}$$

For $||W_{S2}\hat{S}||_\infty < 1$ this weight yields: 1) Steady-state error less than A; 2) Closed-loop bandwidth higher than $\omega_B = 1/\tau_{cl}$; and 3) Amplification of high-frequency output disturbances less than a factor M_S. The values used in [17] were $M_s = 2$, $A = 0$ and $\tau_{cl} = 20$.

To satisfy specification S2 ($\|K_y\hat{S}\|_\infty < 316$), we choose the weight

$$W_{S3} = \frac{1}{M_{KS}} I_{2\times 2} \tag{19}$$

As a starting point we may choose $M_{KS} = 316$; the value given in **S3**. However, it is likeley that this value is somewhat small. The reason is that the formulation in Fig. 10 lumps the four SISO requirements of **S2** and the 2×2 requirement of **S3** into a bound on the entire 4×2 transfer function from $\hat{r}$ to $\begin{bmatrix}\hat{e}^T & \hat{u}^T\end{bmatrix}^T$. From the relation

$$\max\{\bar{\sigma}(A), \bar{\sigma}(B)\} \leq \bar{\sigma}([A\ B]^T) \leq \sqrt{2}\max\{\bar{\sigma}(A), \bar{\sigma}(B)\} \tag{20}$$

it is clear that the physical interpretation of the weights results in performance requirements that are slightly too tight.

In accordance with the results in Section 5.1, we found that a ODF-controller did not yield the required performance; thus in the following we focus on the TDF-design.

5.3.3 TDF performance weights

For the TDF-design we use the block diagram in Fig.11. The objective for the μ-synthesis is to minimize the worst-case weighted transfer function from references r and noise n to control error e and input signals u (the hats used in the figure indicate that the signals have been weighted). Note that the noise signal n and input signal u were not included in our μ-optimal design for the original problem, but these are needed to satisfy the CDC-specifications.

Figure 11 gives

$$\begin{bmatrix}\hat{e} \\ \hat{u}\end{bmatrix} = \begin{bmatrix} W_e N_{11} W_r & -W_e N_{12} W_n \\ W_u N_{21} W_r & -W_u N_{22} W_n \end{bmatrix} \begin{bmatrix}\hat{r} \\ \hat{n}\end{bmatrix} \tag{21}$$

Here

$$N_{11} = T_{yr} - T_{yr,id} = SGK_r - T_{yr,id}; \quad N_{12} = GK_yS$$

$$N_{21} = K_rS; \quad N_{22} = K_yS$$

where $S = (I + GK_y)^{-1}$. (Strictly speaking G and S should have a subscript p to denote the perturbed plant, $G_p = G(I + \Delta_U W_U)$, but this has been omitted to simplify the notation.) We now need to select the four performance weights, W_e, W_r, W_n, W_u and the ideal tracking response $T_{yr,id}$.

For simplicity, we use scalar times identity weights for the four weights, that is, $W_i = w_i I_{2\times 2}$. To determine the weights we should first consider the resulting bounds on the four closed-loop transfer functions N_{11}, N_{12}, N_{21} and N_{22}. We note that W_eW_r forms a bound on N_{11}, which is closely related to specification S2. Furthermore, W_uW_n forms a bound on N_{22}, which is directly related to specifications S3 (and to specification S4, alt.2). The following should be considered when selecting the four weights.

1. Since the weights are scalar, we may choose one of them freely. Thus we choose $W_r = I$ at all frequencies.

2. In order to penalize the difference between the actual and ideal tracking the combined weight $W_e W_r$ may be chosen similar to $W_{s2}(s)$ in Eq. 18, i.e, we choose $W_e = W_{S2}$.

3. Specification S3 limits the peak value of $K_y S$, which is the transfer function from output disturbances (noise) to inputs. In practice, the peak occurs at higher frequencies just beyond the closed-loop bandwidth. Thus, we must make sure that $W_u W_n = 1/M_{KS}$ at frequencies where $K_y S$ has its peak. For simplicity, we select $W_u = 1/M_{KS}$ (a constant). It then follows that W_n should approach 1 at high frequencies, and one should make sure that it reaches this value around the bandwidth (which is approximately equal to $1/\tau_{cl}$ with the selected weight for W_e).

4. The inverse of $W_e W_n$ forms an upper bound on $N_{12} = T$, the complimentary sensitivity. Since W_e is large at low frequency, its inverse is small at these frequencies. However, the magnitude of T is greater or equal to 1 at low frequency, so it follows that W_n must be small at low frequencies. To be specific, let M_T denote the maximum value of T (i.e, the infinity norm of T) at low frequencies, then W_n should be selected such that $W_e W_n = 1/M_T$ at low frequencies. (Note that simply selecting $W_n(s) = (M_T W_e(s))^{-1}$ may not satisfy the above requirement that W_n should approach 1 at high frequencies.)

5. Note that $W_u W_r = W_u = 1/M_{KS}$ forms a bound on $N_{21} = K_r S$ which is the transfer function from references to inputs. Although, there is no specification on this transfer function, is seems reasonable that it should be limited in a way similar to $K_y S$.

The following weights satisfy the above requirements:

$$W_r(s) = I_{2\times 2} \tag{22}$$

$$W_e(s) = \frac{1}{M_S}\frac{\tau_{cl}s + M_S}{\tau_{cl}s + A} I_{2\times 2} \tag{23}$$

$$W_u(s) = \frac{1}{M_{KS}} I_{2\times 2} \tag{24}$$

$$W_n(s) = \frac{\tau_{cl}s + A}{\tau_{cl}s + M_T} I_{2\times 2} \tag{25}$$

5.4 TDF-controller for CDC specifications; Alt.1

In this section we synthesize a TDF controller for CDC specifications S1, S2, **S4**, Alt.1, by adjusting the parameters in the above weights. Since all of the parameters have physical significance it is easy to find reasonable values, and almost all of them were determined directly from the original specifications in Section 2.

Note that the perturbation matrix Δ_U in Eq. 13 is diagonal. However, to simplify the numerical calculations we use an unstructured perturbation matrix Δ_U which yields a very simple D-scale for the μ-synthesis, $D(s) = \mathrm{diag}\{d(s), d(s), I_{4\times 4}\}$. In any case, it also seems that for this problem the structure of the uncertainty does not matter.

Initially $d(s)$ is set to 0.01, obtained from a natural physical scaling ('logarithmic compositions' [16]). This simple scaling substantially reduces the number of iterations required to obtain 'good' D-scales.

The initial weight parameters were chosen to: 1) Yield an ideal response which satisfies **S2** with some margin without too large an overshoot ($\tau_{id} = 8, \zeta_{id} = 0.71$); 2) Require a close fit to the ideal response at low frequencies ($A = 10^{-4}$) and a looser fit at high frequencies ($\tau_{cl} = 10, M_S = 3$); 3) Yield a loose requirement on $K_y S_p$ to be increased if required ($M_T = 3, M_{KS} = 630$ (56dB)).

Table 3: Final weight parameters and D-scales

Weight parameters						
τ_{id}	ζ_{id}	τ_{cl}	A	M_S	M_T	M_{KS}
8.0	0.71	9.5	10^{-4}	3.5	2.0	630

$$D(s) = \mathrm{diag}\{d(s), d(s), I_{4\times 4}\}$$

$$d(s) = 0.00299 \frac{(s + 5.70)}{(s + 0.0144)} \frac{(s^2 + 2 * 0.6645 * 0.112s + 0.112^2)}{(s^2 + 2 * 0.622 * 0.568s + 0.568^2)}$$

Table 4: Control performance for TDF-Alt.1 with gain uncertainty and second order Padé approximation of a 1 min delay. (See also Fig.12)

step	gain unc.		set-point tracking			interaction	
ch.	k_1	k_2	$t = 30$	max	$t = 100$	max	$t = 100$
1	1.2	1.2	1.066	1.092	0.998	0.051	0.001
1	1.2	0.8	0.984	1.036	0.999	0.471	-0.001
1	0.8	1.2	0.969	1.030	1.000	0.426	0.001
1	0.8	0.8	0.906	1.000	1.000	0.138	0.000
2	1.2	1.2	1.052	1.074	0.999	0.051	0.001
2	1.2	0.8	0.987	1.030	1.000	0.265	0.001
2	0.8	1.2	1.002	1.038	0.999	0.310	0.000
2	0.8	0.8	0.950	1.002	1.000	0.138	0.000

Only two DK-iterations were needed to ensure $\mu_{RP} < 1$, however, the **S2** and **S3** performance specifications were not satisfied. M_S, M_T and τ_{cl} were adjusted to 3.5, 2.0 and 9.5, respectively. After two more DK-iterations a controller which satisfied **S1**-**S4** was obtained. The controller has 24 states, yields a closed loop $\mathcal{H}_\infty$-norm of 1.015 and may be synthesized using the final weights and D-scales given in Table 3.

The performance of the TDF controller is demonstrated in Fig.12 where time responses for the four extreme combinations of uncertainty are shown. The simulation results are also summarized in Table 4 and are seen to satisfy specification **S2**. The maximum peak of $\bar{\sigma}(K_y\hat{S})$ is 306 (Fig. 13), which is less than 316 (50 dB), as required in **S3**, and the unit gain cross over frequency, $\bar{\sigma}(\hat{G}K_y) = 1$, is at 1 rad/min, well below 150 rad/min, as required in **S4** Alt.1. Specification **S4** Alt.2 is not satisfied as shown in Fig. 13.

The transfer functions N_{12} and N_{21}, which are not part of the CDC problem, have peak values of 3.4 and 420, respectively.

5.5 TDF-controller for CDC specifications; Alt.2

Recall that there were two alternative interpretations of specification **S4**. In this section we show that we can also satisfy specification **S4** Alt.2, which was used in [20], using the design procedure presented in this paper. We again use the problem formulation in Fig.11, but the signal weights W_u and W_n need to be modified. In addition we need to use the tighter

Table 5: Final weight parameters and D-scales

Weight parameters							
τ_{id}	ζ_{id}	τ_{cl}	A	M_S	M_T	M_{KS}	ω_0
8.0	0.71	9.5	10^{-4}	3.0	2.5	1000	200

$$D(s) = \mathrm{diag}\{d(s), d(s), I_{4\times 4}\}$$

$$d(s) = 7.3 * 10^{-4} \frac{(s+23.1)(s^2 + 2 * 0.637 * 0.116s + 0.116^2)}{(s+0.0213)(s+0.372)(s+0.673)}$$

uncertainty weight from Eq. 17.

Specifications **S3** and **S4** Alt.2 require:

$$\bar{\sigma}(K_y \hat{S}(j\omega)) < \begin{cases} 50 \text{ dB} & \omega < 150 \text{ rad/min} \\ 0 \text{ dB} & \omega \geq 150 \text{ rad/min} \end{cases} \tag{26}$$

which is more more difficult to satisfy than in Alt.1. We use the same procedure as in the previous design; first approximating Eq. 26 by a rational transfer function (W_{S34}), whose inverse forms an upper bound on $K_y\hat{S}$, and then deriving W_u and W_n such that $W_u W_n \approx W_{S34}$ at high frequency. Let

$$W_{S34} = \frac{1}{M_{KS}} \left(\frac{\frac{M_{KS}^{1/n}}{\omega_0} s + 1}{\frac{1}{c\omega_0} s + 1} \right)^n I_{2\times 2} \tag{27}$$

The weight is equal to $1/M_{KS}$ at low frequencies, and then starts increasing sharply and crosses 1 at about the frequency ω_0 (which should then be about 150 rad/min). It levels off at the value c^n at high frequency. The parameter n is an integer. By increasing n, a tighter approximation of Eq. 26 is achieved, but on the other hand the complexity of the control problem increases.

We decided to select $n = 3$ and $c = 5$. Following the procedure in Section 5.3 we selected W_u and W_n as follows

$$W_u(s) = \frac{1}{M_{KS}} \left(\frac{\frac{M_{KS}^{1/n}}{\omega_0} s + 1}{\frac{1}{c\omega_0} s + 1} \right)^{(n-1)} I_{2\times 2} \tag{28}$$

$$W_n(s) = \frac{\tau_{cl} s + A}{\tau_{cl} s + M_T} \left(\frac{\frac{M_{KS}^{1/n}}{\omega_0} s + 1}{\frac{1}{c\omega_0} s + 1} \right) I_{2\times 2} \tag{29}$$

After a few iterations and parameter adjustments a controller which satisfies **S1**, **S2**, **S3** and **S4** Alt.2 was obtained. The final weight parameters and D-scales are given in Table 5. The controller yields a closed loop $\mathcal{H}_\infty$-norm of 1.0 and has 34 states. The number of states was reduced to 22 using optimal Hankel norm approximation, without violating the control objectives. The performance of the 22 state controller is shown in Fig.14. The simulation results are also summarized in Table 6 and are seen to satisfy specification **S2**.

Table 6: Control performance for TDF-Alt.2 with gain uncertainty and second order Padé approximation of a 1 min delay. (See also Fig.12)

step	gain unc.		set-point tracking			interaction	
ch.	k_1	k_2	$t = 30$	max	$t = 100$	max	$t = 100$
1	1.2	1.2	1.063	1.082	0.991	0.036	0.008
1	1.2	0.8	0.976	1.013	0.990	0.464	-0.001
1	0.8	1.2	0.977	1.031	0.999	0.424	0.010
1	0.8	0.8	0.908	0.998	0.998	0.130	0.002
2	1.2	1.2	1.050	1.067	0.994	0.036	0.008
2	1.2	0.8	0.995	1.036	1.001	0.264	0.008
2	0.8	1.2	0.994	1.019	0.992	0.305	0.001
2	0.8	0.8	0.951	0.999	0.999	0.130	0.002

Fig. 15 shows that the maximum peak of $\bar{\sigma}(K_y\hat{S})$ is 313, which is less than 316 (50 dB), as required in **S3**, and the unit gain cross over frequency, $\bar{\sigma}(K_y\hat{S}) = 1$, is below 150 rad/min, as required in **S4** Alt.2.

We obtained this reduction in controller gain with almost no deterioration in performance. Compared to TDC-Alt.1 the peak value of $N_{12} = \hat{T}$ was reduced from 3.4 to 2.6 (which is an advantage), whereas the peak value of $N_{21} = K_r\hat{S}$ increased from 420 to 435.

6 Discussion

The inability to independently penalize separate elements of the closed loop transfer function complicates the performance weight selection in the μ-framework. The Hadamard weighted approach [3] does not exhibit this problem and will therefore yield better performance with respect to the specifications in the CDC problem, **S1** - **S4**. However, for a practical engineering problem the transfer functions N_{12} and N_{21} in Fig.11 *are* of importance, so it seems reasonable to include them in the control problem.

7 Conclusions

The paper has shown how a demanding design problem, involving parametric gain-delay uncertainty and a mixture of time domain and frequency domain performance specifications, can be reformulated and solved using the structured singular value framework. A two degrees-of-freedom controller was needeed to satisfy the specifications. The results, in terms of meeting the specifications, are comparable or better than those given in Limebeer *et al.* [9], van Diggelen and Glover [3] and Whidborne *et al.* [20],

Acknowledgments. Support from NTNF is gratefully acknowledged.

References

[1] Balas, G.J, Doyle, J.C., Glover, K., Packard, A.K. and Smith, R. (1991). "The μ-Analysis and Synthesis Toolbox", The MathWorks Inc., Natick, MA.

[2] van Diggelen, F. and Glover, K. (1991). "Element-by-element weighted $\mathcal{H}_\infty$-Frobenius and $\mathcal{H}_2$ norm problems", *Proc. IEEE Conf. Decision Contr.*, Brighton, England, 923-924.

[3] van Diggelen, F. and Glover, K. (1992). "A Hadamard weighted loop shaping design procedure", *Proc. IEEE Conf. Decision Contr.*, Tucson, Arizona, 2193-2198.

[4] Doyle, J.C. (1982). "Analysis of Feedback Systems with Structured Uncertainties", *IEE Proc.*, **129**, Part D, 242-250.

[5] Doyle, J.C. (1983). "Synthesis of robust controllers and filters", *Proc. IEEE Conf. Decision Contr.*, San Antonio, TX.

[6] Freudenberg, J.S. (1989). "Analysis and design for ill-conditioned plants, Part 2. Directionally uniform weightings and an example", *Int. J. Control*, **49**, 3, 873-903.

[7] Hoyle, D.J., Hyde, R.A. and Limebeer, D.J.N, (1991). "An $\mathcal{H}_\infty$ approach to two degree of freedom design", *Proc. IEEE Conf. Decision Contr.*, Brighton, England, 1581-1585.

[8] Limebeer, D.J.N. (1991). "The specification and purpose of a controller design case study", *Proc. IEEE Conf. Decision Contr.*, Brighton, England, 1579-1580.

[9] Limebeer, D.J.N., Kasenally, E.M. and Perkins, J.D. (1993). "On the design of robust two degree of freedom controllers", *Automatica*, **29**, 1, 157-168.

[10] Lundström, P., Skogestad, S. and Wang, Z.-Q. (1991). "Performance weight selection for $\mathcal{H}_\infty$ and μ-control methods", *Trans. Inst. MC*, **13**, 5, 241-252.

[11] Lundström, P., Skogestad, S. and Wang Z.-Q. (1991). "Uncertainty weight selection for $\mathcal{H}_\infty$ and μ-control methods", *Proc. IEEE Conf. Decision Contr.*, Brighton, England, 1537-1542.

[12] Lundström, P., Wang Z.-Q. and Skogestad, S. (1993). "Modelling of gain and delay uncertainty in the structured singular value framework", *Preprints IFAC World Congress on Automatic Control*, Sydney, Australia.

[13] McFarlane, D.C. and Glover, K. (1992). "A loop shaping design procedure using $\mathcal{H}_\infty$ synthesis", *IEEE Trans. Autom. Control*, **AC-37**, 6, 759-769.

[14] Postlethwaite, I., Lin, J.-L., and Gu, D.-W. (1991). "Robust control of a high purity distillation column using $\mu \Leftrightarrow K$ iteration", *Proc. IEEE Conf. Decision Contr.*, Brighton, England, 1586-1590.

[15] Safonov, M.G. (1983). "$\mathcal{L}_\infty$ optimal sensitivity v.s. stability margin", *Proc. IEEE Conf. Decision Contr.*, San Antonio, TX.

[16] Skogestad, S. and Morari, M. (1988). "Understanding the Dynamic Behavior of Distillation Columns", *Ind. Eng. Chem. Res.*, **27**, 10, 1848-1862.

[17] Skogestad, S., Morari, M. and Doyle, J.C (1988). "Robust Control of Ill-conditioned Plants: High-purity Distillation", *IEEE Trans. Autom. Control*, **33**, 12, 1092-1105. (Also see correction to μ-optimal controller in **34**, 6, 672).

[18] Stein, G., Doyle, J.C. (1991). "Beyond singular values and loop shapes", *J. of Guidance Control and Dynamics*, **14**, 1, 5-16.

[19] Wang, Z.-Q., Lundström, P. and Skogestad, S. (1994). "Representation of uncertain time delays in the H_∞ framework", *Int. J. Control*, **59**, 3, 627-638.

[20] Whidborne, J.F., Postlethwaite, I. and Gu, D.W. (1992). "Robust controller design using $\mathcal{H}_\infty$ loop-shaping and the method of inequalities", Report 92-33, Department of Engineering, University of Leicester.

[21] Yaniv, O. and Barlev, N. (1990). "Robust non iterative synthesis of ill-conditioned plants", *Proc. Am. Control Conf.*, San Diego, CA, 3065-3066.

[22] Yaniv, O. and Horowitz, I. (1991). "Ill-conditioned plants: A Case study", *Proc. IEEE Conf. Decision Contr.*, Brighton, England, 1596-1600.

[23] Zakian, V. and Al-Naib, U. (1973). "Design of dynamical and control systems by the method of inequalities", *IEE Proc.*, **120**, 11, 1421-1427.

[24] Zhou, T. and Kimura, H. (1991). "Controller design of an ill-conditioned plant using robust stability degree assignment", *Proc. IEEE Conf. Decision Contr.*, Brighton, England, 1591-1595.

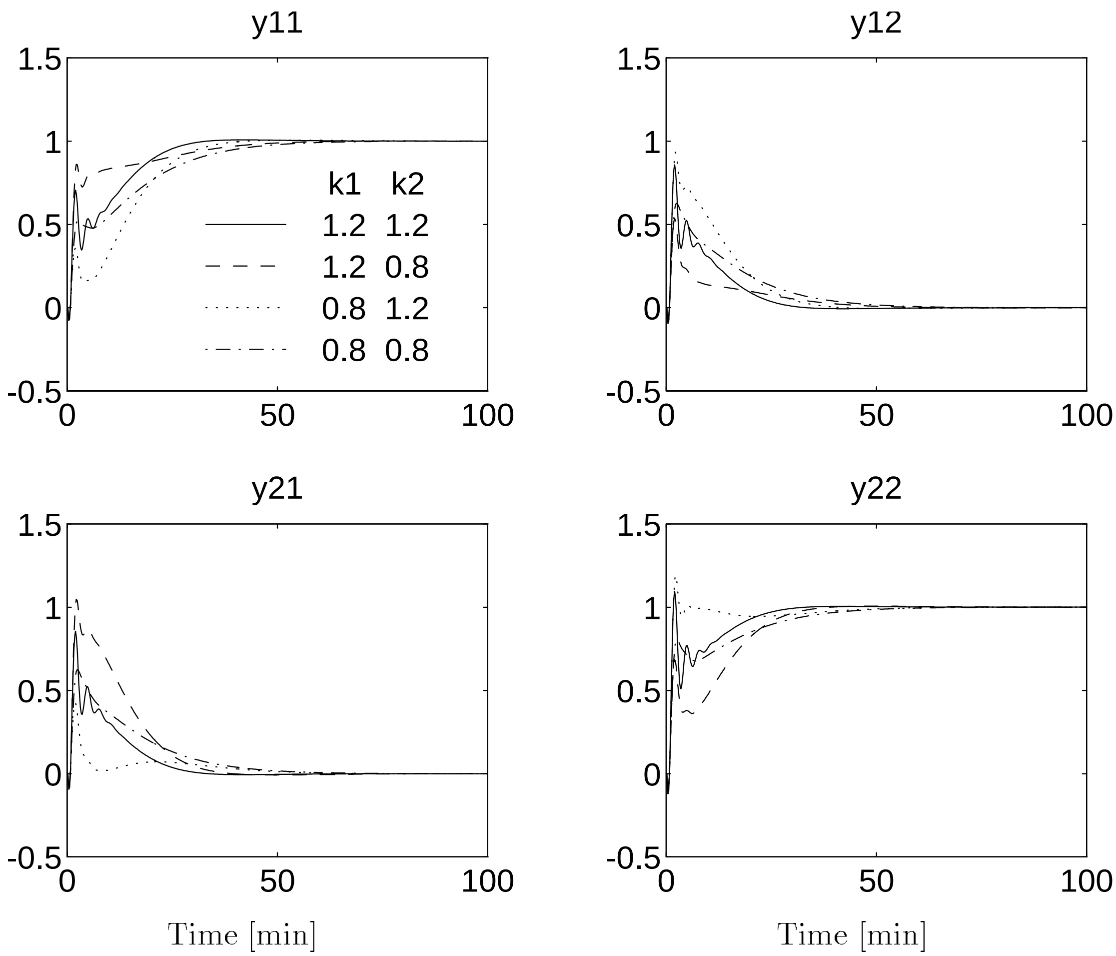

Figure 4: Output responses for ODF-original controller with plant-model mismatch. y_{ij} shows response in output i for step change of set-point j at $t = 0$. All responses with 1 min delay (2nd order Padé).

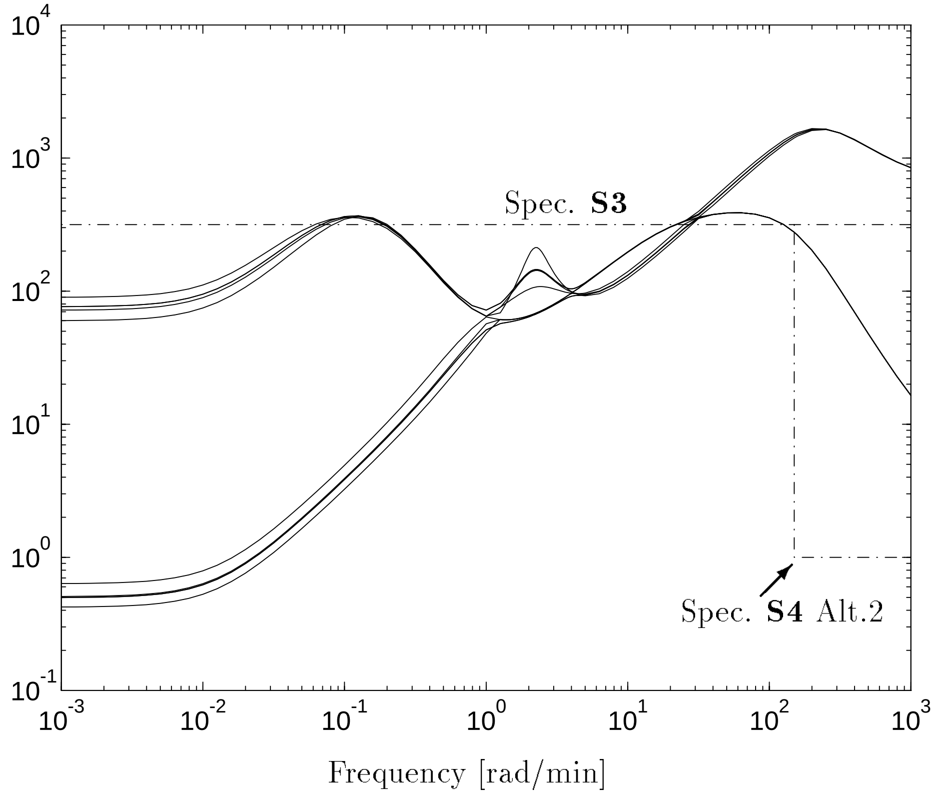

Figure 5: Maximum and minimum singular values of $K_y\hat{S}$ for ODF-original controller for four different plants.

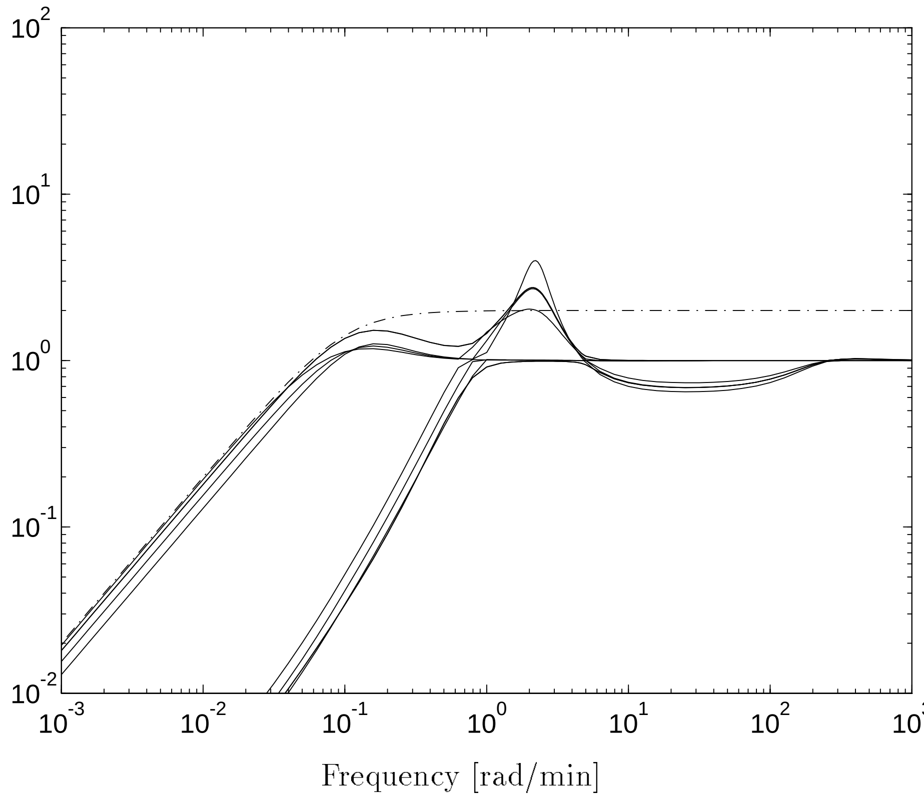

Figure 6: Maximum and minimum singular values of $\hat{S}$ for ODF-original controller. Dashed: Original upper bound on S ($1/W_e$, Eq.12).

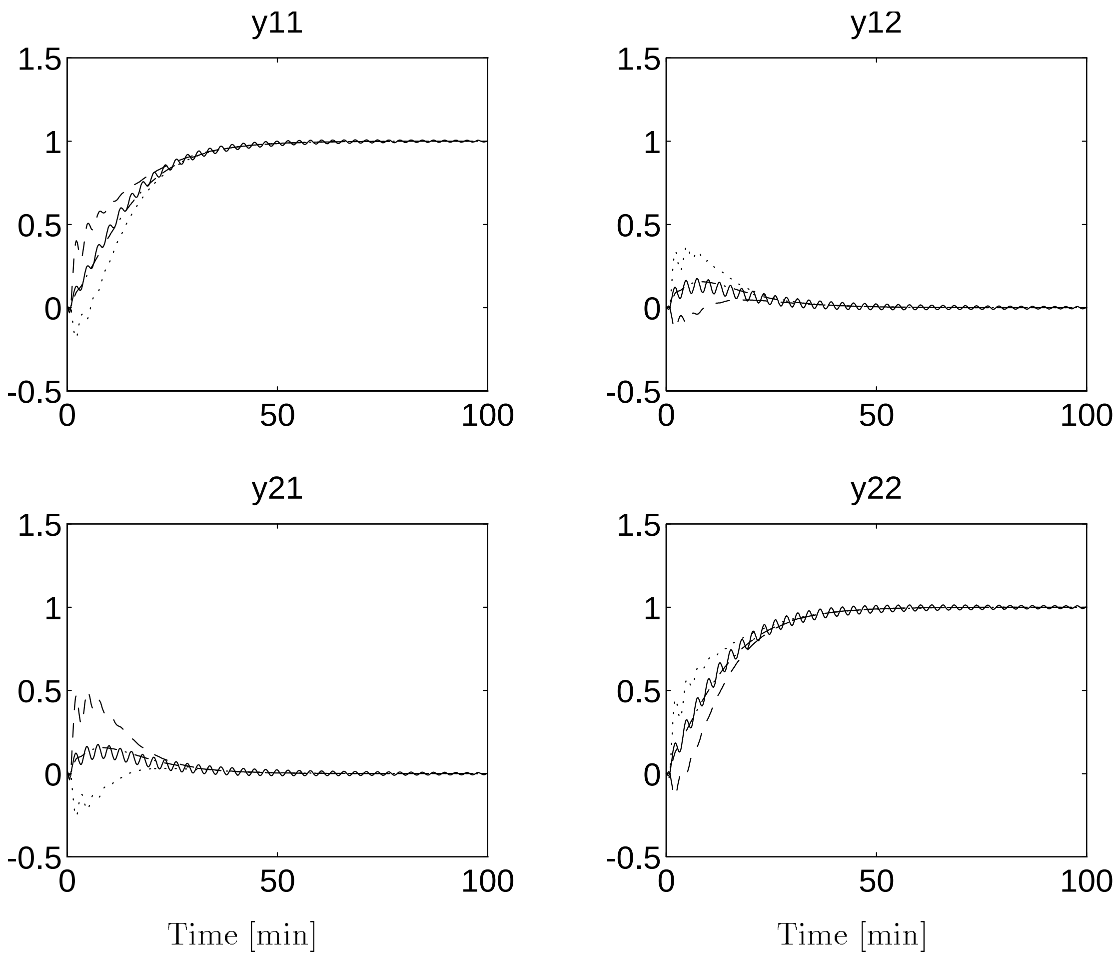

Figure 7: Output responses for TDF-original controller with plant-model mismatch. y_{ij} shows response in output i for step change of set-point j at $t = 0$. All responses with 1 min delay (2nd order Padé).

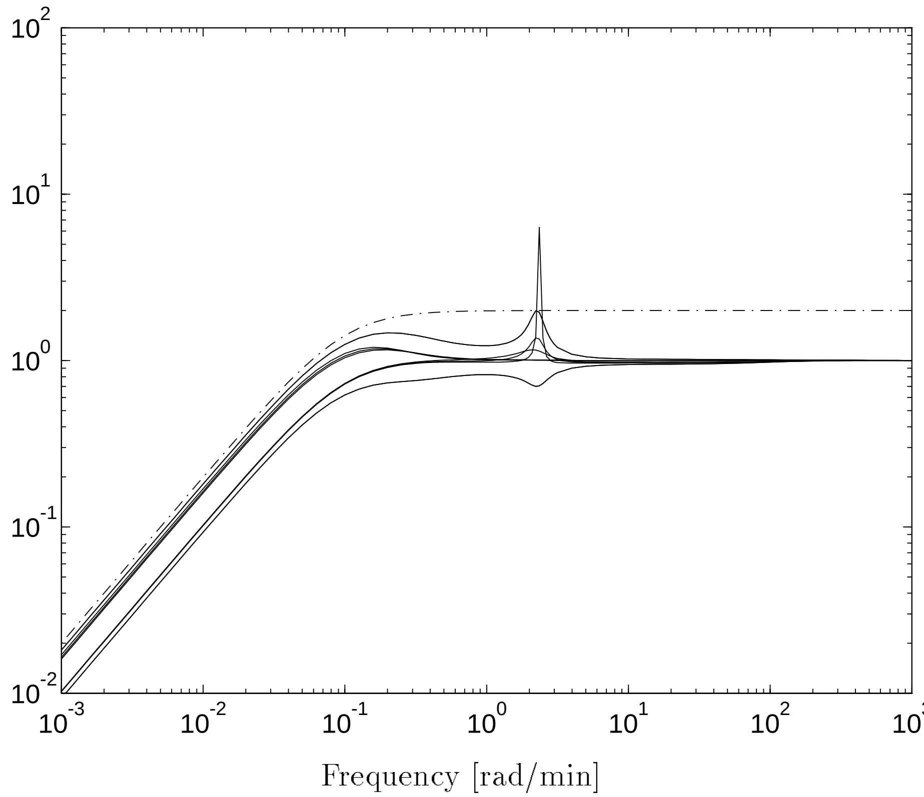

Figure 8: Maximum and minimum singular values of $\hat{S}\hat{G}K_r - I$ for TDF-original controller. Dashed: Original upper bound on S ($1/W_e$, Eq.12).

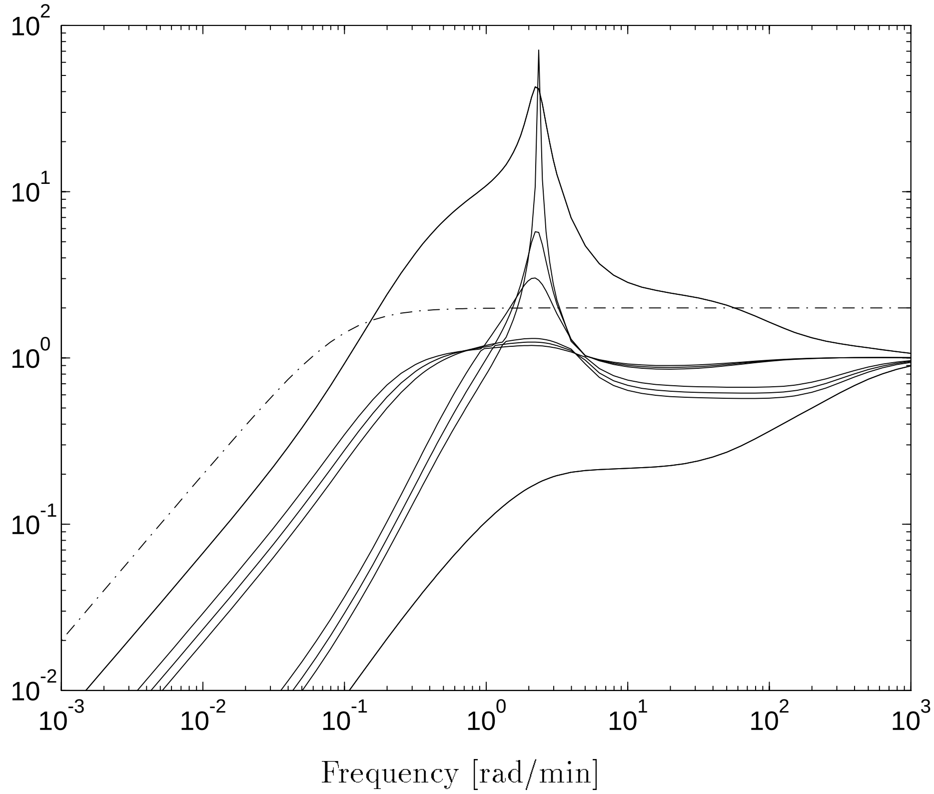

Figure 9: Maximum and minimum singular values of $\hat{S}$ for TDF-original controller. Dashed: Original upper bound on S ($1/W_e$, Eq.12).

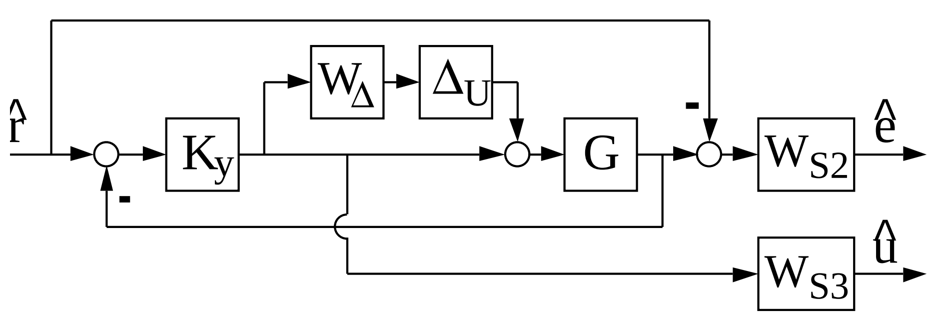

Figure 10: Block diagram for one degree of freedom controller.

Figure 11: Block diagram for two degree of freedom controller.

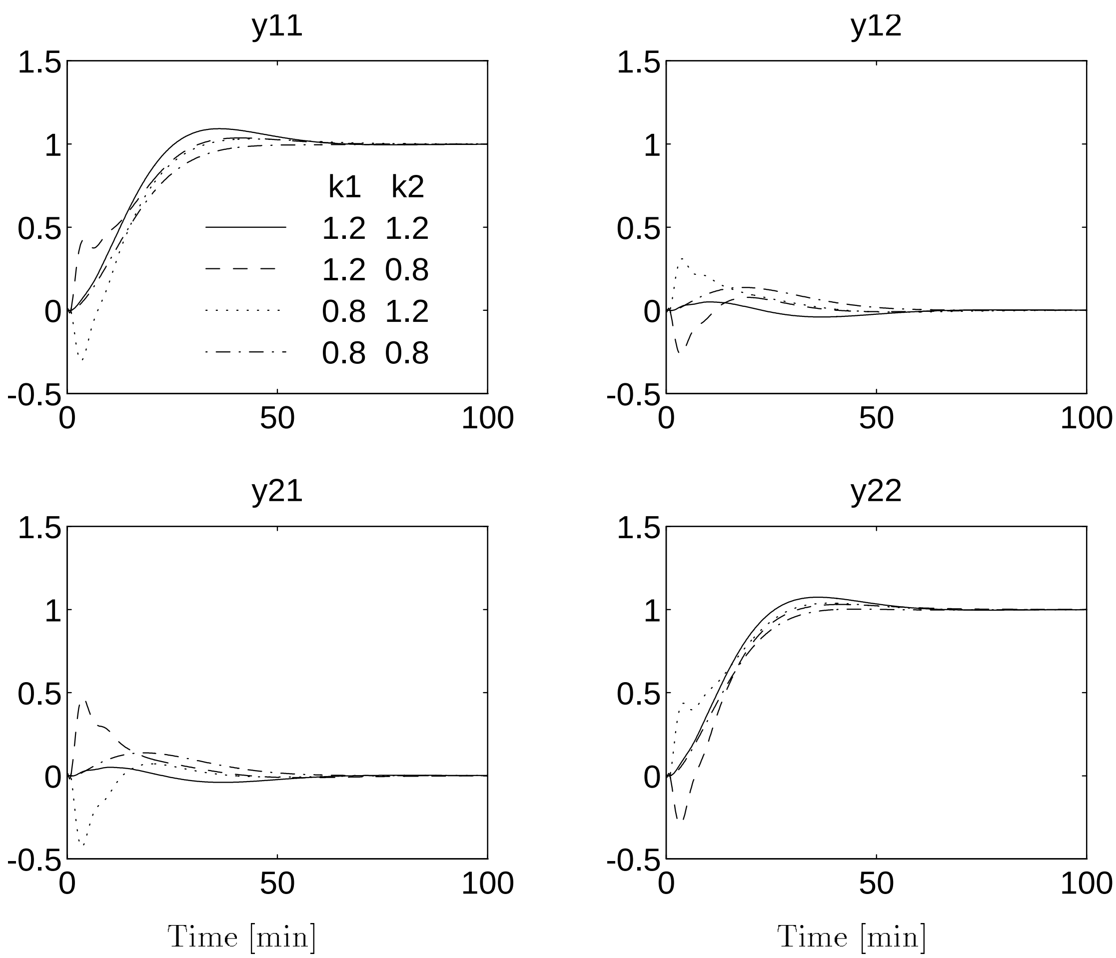

Figure 12: Output responses for TDF-Alt.1 controller with plant-model mismatch. y_{ij} shows response in output i for step change of set-point j at $t = 0$. All responses with 1 min delay (2nd order Padé).

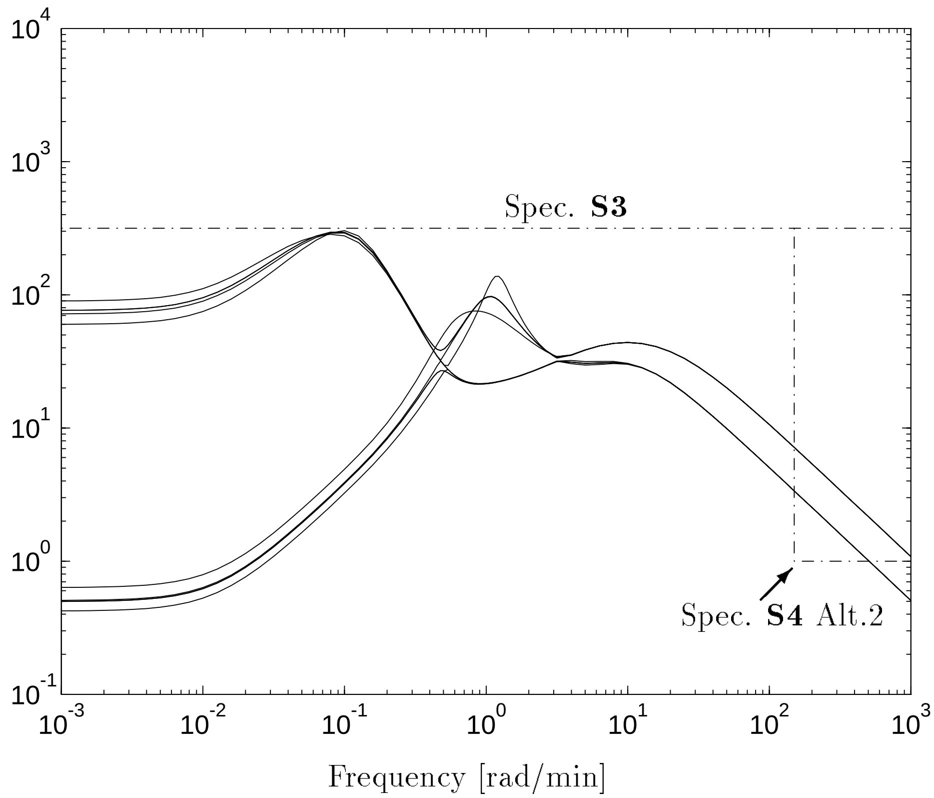

Figure 13: Maximum and minimum singular values of $K_y\hat{S}$ for TDF-Alt.1 controller for four different plants.

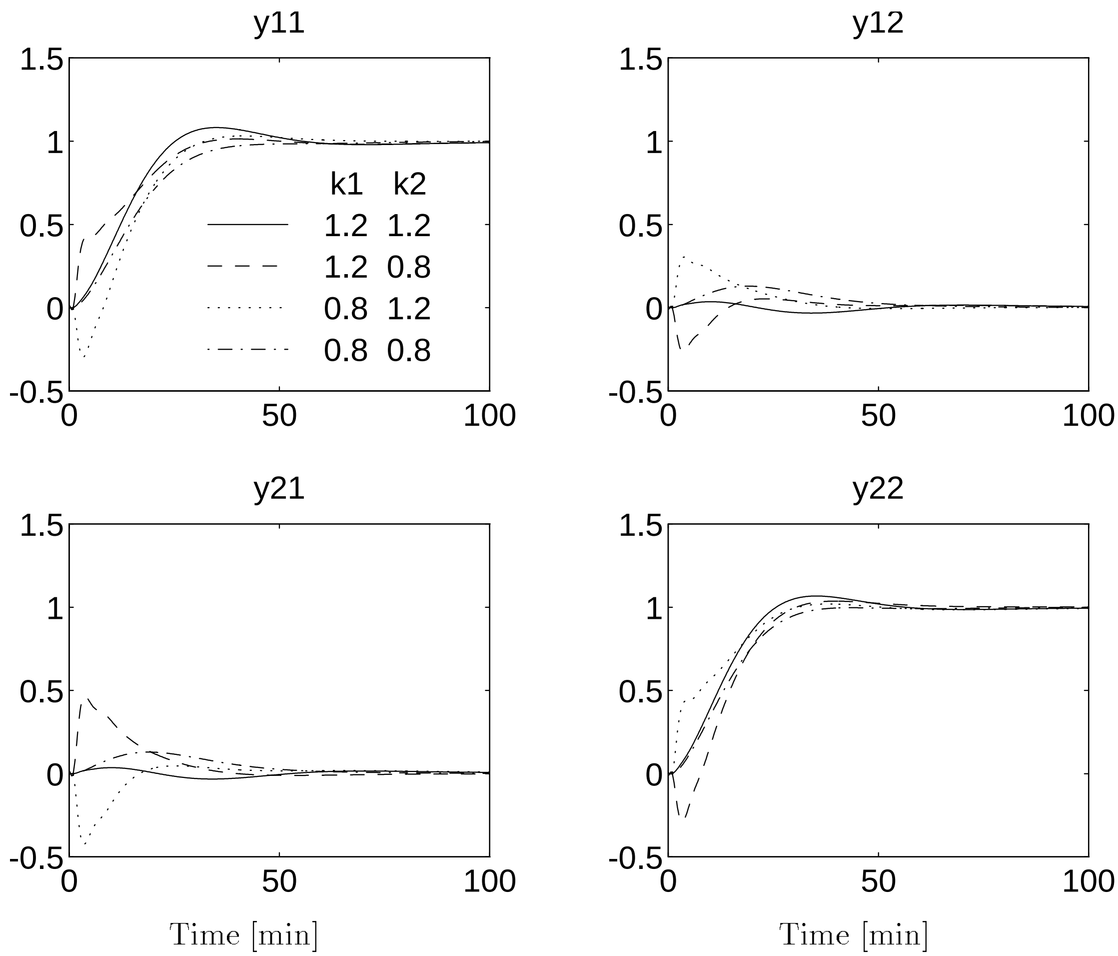

Figure 14: Output responses for TDF-Alt.2 controller with plant-model mismatch. y_{ij} shows response in output i for step change of set-point j at $t = 0$. All responses with 1 min delay (2nd order Padé).

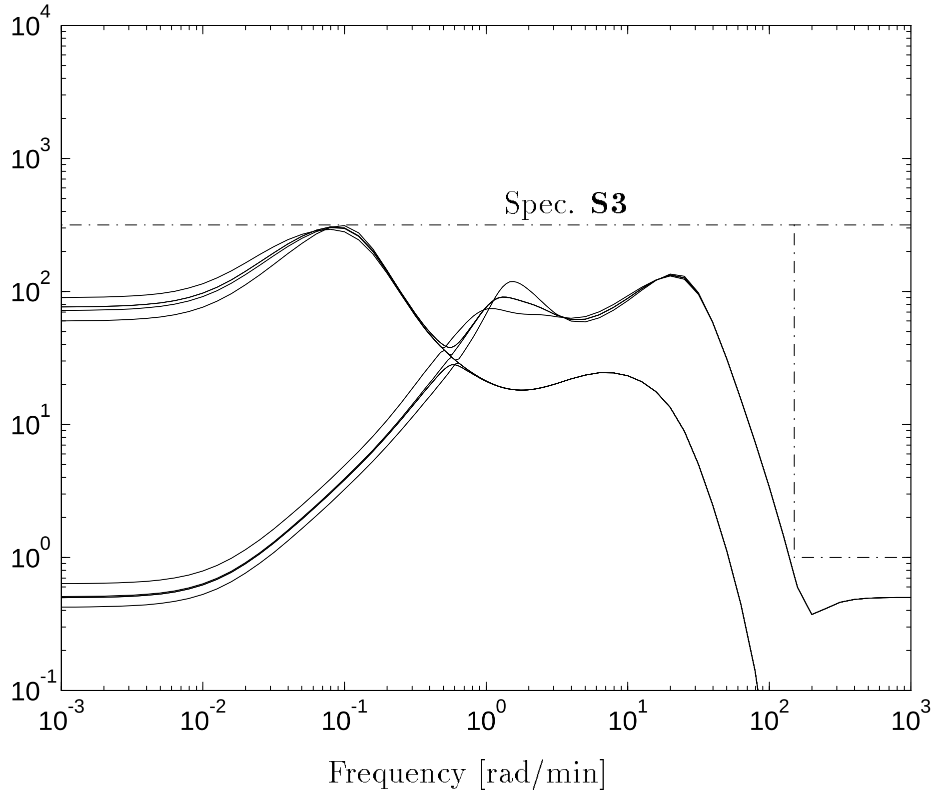

Figure 15: Maximum and minimum singular values of $K_y\hat{S}$ for TDF-Alt.2 controller for four different plants.