EFFECT OF RHP ZEROS AND POLES ON PERFORMANCE IN MULTIVARIABLE SYSTEMS

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Abstract. This paper examines the implications of RHP-zeros and poles on performance of multivariable feedback systems. The results quantify the fundamental limitations imposed by RHP-zeros and poles in terms of lower bounds on the peak of the weighted sensitivity and complementary sensitivity functions.

1. INTRODUCTION

This paper considers linear time invariant systems on state-space form

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx + Du \tag{2}$$

where A, B, C and D are real matrices. These equations may be rewritten as

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

This gives rise to the short-hand notation

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{3}$$

which is frequently used to describe a state-space model of a system G. The transfer function of G (of size $l \times m$) defined by (3) can be evaluated as a function of the complex variable s, $G(s) = C(sI - A)^{-1}B + D$. We often omit to show the dependence on the complex variable s for transfer functions. The feedback controller is denoted K. The loop transfer function is defined by $L \triangleq GK$. The sensitivity and complementary sensitivity functions are defined by $S \triangleq (I + L)^{-1}$ and $T \triangleq L(I + L)^{-1} = LS = I - S$.

The results in this paper quantify the fundamental limitations imposed by RHP-zeros and poles in terms of lower bounds on the peak of the weighted sensitivity and complementary sensitivity functions. To derive the results we have made use of output factorizations of RHP-zeros and poles in all-pass filters $B(s)$. Further details on how to do this factorizations can be found in (Havre and Skogestad, 1996).

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The outline of the paper is as follows: fist we discuss zeros and poles of multivariable systems and their directions. Then we derive constraints on the sensitivity and the complementary sensitivity functions imposed by RHP-zeros and poles. Next we consider the lower bounds on the peak of the weighted sensitivity and complementary sensitivity functions. At the end we give two examples and a conclusion. All proofs are given in appendix A.

2. ZEROS AND POLES OF MULTIVARIABLE SYSTEMS

2.1 Zeros

Zeros of a system may arise when competing effects internal to the system are such that the output is zero even when the inputs (and the states) are not themselves identically zero. For Single Input Single Output (SISO) systems, the zeros are the solutions $s = z_i$ to $G(s) = 0$, and thus it could be argued that they are values of s at which $G(s)$ looses rank (from rank 1 to rank 0). This is the basis for the following definition for zeros for the multivariable system (MacFarlane and Karcanias, 1976).

DEFINITION 1. (ZEROS). *$z_i \in \mathbb{C}$ is a zero of $G(s)$ if the rank of $G(z_i)$ is less than the normal rank of $G(s)$. The zero polynomial is defined as $z(s) = \prod_{i=1}^{N_z}(s - z_i)$ where N_z is the number of finite zeros of $G(s)$.*

The normal rank of $G(s)$ is defined as the rank of $G(s)$ at all s except a finite number of singularities (which are the zeros). This definition of zeros is based on the transfer function matrix, corresponding to a minimal realization of a system. These zeros are sometimes called "transmission zeros", but we shall simply call them "zeros". We continue with the definitions of input and output zero directions.

DEFINITION 2. (ZERO DIRECTIONS). *If $G(s)$ has a zero for $s = z \in \mathbb{C}$ then there exist non-zero vectors labeled the output zero direction $y_z \in \mathbb{C}^l$ and the input zero direction $u_z \in \mathbb{C}^m$, such that $y_z^H y_z = 1$, $u_z^H u_z = 1$ and*

$$y_z^H G(z) = 0; \quad G(z)u_z = 0 \tag{4}$$

It follows that the input and output zero directions lie in the null-space and the left null-space of $G(z)$. The dimension of these spaces may be larger than the multiplicity of the zeros z. In particular, this is the case for non-square G. The definitions of input and output zero directions can further be extended with the state input and output zero directions through the use of generalized eigenvalues. For a system $G(s)$, the zeros z of the system, the zero input directions u_z and the zero input state directions $x_{z,\mathrm{I}} \in \mathbb{C}^{n_x}$ (n_x is the number of states) can all be computed from the generalized eigenvalue problem

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} x_{z,\mathrm{I}} \\ u_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

In this setup we normalize the length of u_z, i.e. $u_z^H u_z = 1$. This imply that the length of $x_{z,\mathrm{I}}$ is different from one. Similarly one can compute the zeros z, the output zero direction y_z and the output zero state direction $x_{z,\mathrm{O}} \in \mathbb{C}^{n_x}$ through the generalized eigenvalue problem

$$\begin{bmatrix} x_{z,\mathrm{O}}^H & y_z^H \end{bmatrix} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{6}$$

Where the length of y_z is normalized, so that $y_z^H y_z = 1$. By taking the transpose of (6) one obtains

$$\begin{bmatrix} A^T - sI & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} \bar{x}_{z,\mathrm{O}} \\ \bar{y}_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

From this we see that the input directions of the transposed system G^T is equal to the conjugate of the output directions of G. In MATLAB the generalized eigenvalue problem (6) can be solved via the transposed problem.

2.2 Poles

DEFINITION 3. (POLES). *The poles $p_i \in \mathbb{C}$ of a system with state-space description (3) are the eigenvalues $\lambda_i(A)$, $i = 1, \ldots, n_x$ of the matrix A. The pole or characteristic polynomial $\phi(s)$ is defined as*

$$\phi(s) = \det(sI - A) = \prod_{i=1}^{n_x} (s - p_i) \tag{8}$$

Thus, the poles are the roots of the characteristic equation

$$\phi(s) = \det(sI - A) = 0 \tag{9}$$

The gain of the system G evaluated at $s = p$, $G(p)$ is infinite in some directions at the input and the output. This is the basis for the following definition of input and output pole directions.

DEFINITION 4. (POLE DIRECTIONS). *If $s = p \in \mathbb{C}$ is a pole of $G(s)$ then there exist an output direction $y_p \in \mathbb{C}^l$ and an input direction $u_p \in \mathbb{C}^m$ with infinite gain for $s = p$.*

The following result shows how to compute the pole directions for a general system with state space realization (3).

LEMMA 1. (POLE DIRECTIONS). *For a system G with state space description (3) the pole directions associated with the pole $p \in \mathbb{C}$ can be computed from*

$$y_p = C x_R; \quad u_p = B^H x_L \tag{10}$$

where $x_R \in \mathbb{C}^{n_x}$ and $x_L \in \mathbb{C}^{n_x}$ are the eigenvectors corresponding to the two eigenvalue problems

$$A x_R = p x_R; \quad x_L^H A = x_L^H p$$

3. CONSTRAINTS ON T AND S

The condition $S + T = I$ holds for MIMO-systems, and it follows from Fan's theorem in (Horn and Johnson, 1985, p. 140 and p. 178) $\sigma_i(A) - \bar{\sigma}(B) \leq \sigma_i(A+B) \leq \sigma_i(A) + \bar{\sigma}(B)$ with $i = 1$ that

$$|1 - \bar{\sigma}(S)| \leq \bar{\sigma}(T) \leq 1 + \bar{\sigma}(S) \tag{11}$$

$$|1 - \bar{\sigma}(T)| \leq \bar{\sigma}(S) \leq 1 + \bar{\sigma}(T) \tag{12}$$

This shows that we cannot have both S and T small simultaneously and that $\bar{\sigma}(S)$ is large if and only if $\bar{\sigma}(T)$ is large. For MIMO-systems the interpolation constraints on T and S caused by RHP-poles and RHP-zeros have directions. This follows since RHP-zeros and RHP-poles themselves have directions as discussed in section 2.

CONSTRAINT 1. (RHP-ZERO). *If $G(s)$ has a RHP-zero at z with output zero direction y_z, then for internal stability of the feedback system the following interpolation constraints must apply*

$$y_z^H T(z) = 0; \quad y_z^H S(z) = y_z^H \tag{13}$$

In words, (13) says that T must have a RHP-zero in the same direction as G and that $S(z)$ has an eigenvalue of 1 corresponding to the left eigenvector y_z.

CONSTRAINT 2. (RHP-POLE). *If $G(s)$ has a RHP-pole at p with output direction y_p, then for internal stability of the feedback system the following interpolation constraints must apply*

$$S(p) y_p = 0; \quad T(p) y_p = y_p \tag{14}$$

Similar constraints apply to L_I, S_I and T_I, but these are in terms of the input zero and pole directions, u_z and u_p.

For more detailed information on integral relations on sensitivity function with RHP-poles and zeros refer to (Zhou *et al.*, 1996; Chen, 1995; Freudenberg and Looze, 1988; Boyd and Desoer, 1985).

4. LOWER BOUNDS ON $\|w_P S(s)\|_\infty$ AND $\|w_T T(s)\|_\infty$

In this section we deduce lower bounds on the weighted sensitivity functions $\|w_P S(s)\|_\infty$ and $\|w_T T(s)\|_\infty$ when RHP-zeros and poles are present. We use the $\mathcal{H}_\infty$-norm in terms of the maximum singular value.

4.3 Limitations imposed by RHP-zeros

The following result is originally from Zames (1981).

THEOREM 1. (RHP-ZERO AND $\|w_P S(s)\|_\infty$). *Suppose the plant $G(s)$ has a RHP-zero at $s = z$. Let $w_P(s)$ be a scalar transfer function. Then for closed-loop stability the weighted sensitivity function must satisfy*

$$\|w_P S(s)\|_\infty = \sup_\omega \bar{\sigma}(w_P S(j\omega)) \geq |w_P(z)| \quad (15)$$

REMARK 1. Condition (15) shows that there are inherent performance limitations imposed by RHP-zeros.
REMARK 2. Note that (15) involves the maximum singular value (the "worst" direction), and therefore the RHP-zero may not be a limitation in other directions.
REMARK 3. The assumption of a scalar weight is somewhat restrictive. However, the assumption is less restrictive if one follows a scaling procedure and scales all outputs by their allowed variations such that their magnitudes are of approximately equal importance.

4.4 Limitations imposed by RHP-poles

The following result is based on the interpolation constraints $S(p)y_p = 0$ and $T(p)y_p = y_p$ which apply when $G(s)$ has a RHP-pole at $s = p$.

THEOREM 2. (RHP-POLE AND $\|w_T T(s)\|_\infty$). *Suppose the plant $G(s)$ has a RHP-pole at $s = p$. Let $w_T(s)$ be a scalar transfer function. Then for closed-loop stability the weighted complementary sensitivity function must satisfy*

$$\|w_T T(s)\|_\infty = \sup_\omega \bar{\sigma}(w_T T(j\omega)) \geq |w_T(p)| \quad (16)$$

4.5 RHP-poles combined with RHP-zeros

By considering the effect of one RHP-zero and one RHP-pole separately we derived in (15) and (16) the conditions

$$\|w_P S(s)\|_\infty \geq c_1 |w_P(z)| \quad (17)$$
$$\|w_T T(s)\|_\infty \geq c_2 |w_T(p)| \quad (18)$$

with $c_1 = c_2 = 1$. These conditions may be optimistic in that the lower bound may be to small, and indeed we may have $c_1 \geq 1$ and $c_2 \geq 1$ for the case when we have both a RHP-pole and a RHP-zero with some alignment in the same direction.

Consider a plant $G(s)$ with RHP-poles p_i and RHP-zeros z_j, and factorize $G(s)$ in terms of *Blaschke products* as follows

$$G(s) = B_p(s)G_p(s), \quad G(s) = B_z^{-1}(s)G_z(s)$$

where $B_p(s)$ and $B_z(s)$ are stable all-pass transfer matrices (all singular values are equal to 1 for $s = j\omega$) containing the RHP-poles and RHP-zeros, respectively. $B_p(s)$ is obtained by factorizing at the output one RHP-pole at a time, starting with $G(s) = B_{p1}(s)G_{p1}(s)$ where

$$B_{p1}(s) = I + \frac{2\text{Re}p_1}{s - p_1}\hat{y}_{p1}\hat{y}_{p1}^H$$

and $\hat{y}_{p1} = y_{p1}$ is the output pole direction for p_1. This procedure may be continued to factor out p_2 from $G_{p1}(s)$ where $\hat{y}_{p2}$ is the output pole direction of G_{p1} (which need not coincide with y_{p2}, the pole direction of G), and so on. A similar procedure may be used for the RHP-zeros. We get

$$B_p(s) = \prod_{i=1}^{N_p}(I + \frac{2\text{Re}(p_i)}{s - p_i}\hat{y}_{pi}\hat{y}_{pi}^H) \quad (19)$$

$$B_z(s) = \prod_{j=1}^{N_z}(I + \frac{2\text{Re}(z_j)}{s - z_j}\hat{y}_{zj}\hat{y}_{zj}^H) \quad (20)$$

REMARK. For further details regarding state-space realizations of the factorizations and properties of the all pass filters. see (Havre and Skogestad, 1996). the output factorization of RHP-zeros are also given in (Zhou *et al.*, 1996, p.145).

With those two factorization we have the following theorem.

THEOREM 3. (MIMO SENSITIVITY PEAK). *Suppose the plant $G(s)$ has N_z RHP-zeros z_j with output directions y_{zj} and N_p RHP-poles p_i with output directions y_{pi}. Define the all-pass transfer matrices in (19) and (20) and compute the real constants*

$$c_{1,j} = \|y_{zj}^H B_p(z_j)\|_2 \geq 1 \quad (21)$$
$$c_{2,i} = \|B_z(p_i)y_{pi}\|_2 \geq 1 \quad (22)$$

Then for closed-loop stability the weighted sensitivity function must satisfy for each z_j

$$\boxed{\|w_P S(s)\|_\infty \geq c_{1,j}|w_p(z_j)|} \quad (23)$$

and the weighted complementary sensitivity function must satisfy for each p_i

$$\boxed{\|w_T T(s)\|_\infty \geq c_{2,i}|w_T(p_i)|} \quad (24)$$

For the case with one RHP-zero and one RHP-pole we have the result given in Corollary 1. A similar result was first proved by Boyd and Desoer (1985) and an alternative proof is given in Chen (1995) who presents a slightly improved bound. One disadvantage with the lower bound in (Boyd and Desoer, 1985, (3.15) on p. 164) is that it is zero when the angle between the pole and the zero direction is 90° whereas the bound presented next is unity.

COROLLARY 1. (ONE RHP-ZERO AND ONE RHP-POLE). *Given the system $G(s)$ with one* RHP*-pole and one* RHP*-zero. The constants c_1 and c_2 in (21) and (22) are given by the equation*

$$c = c_1 = c_2 = \sqrt{\sin^2(\phi) + \frac{|z+p|^2}{|z-p|^2}\cos^2(\phi)} \quad (25)$$

where $\phi = cos^{-1}(|y_z^H y_p|)$.

For SISO-systems, Theorem 3 simplifies to:

COROLLARY 2. (SISO SENSITIVITY PEAK). *Suppose the* SISO*-system $G(s)$ has N_z* RHP*-zeros z_j, and N_p* RHP*-poles p_i. Then for closed-loop stability the weighted sensitivity function must satisfy for* each RHP*-zero z_j*

$$\|w_P S(s)\|_\infty \geq c_{1,j} |w_P(z_j)| \quad (26)$$

$$c_{1,j} = \prod_{i=1}^{N_p} \frac{|z_j + \bar{p}_i|}{|z_j - p_i|} \geq 1 \quad (27)$$

and the weighted complementary sensitivity function must satisfy for each RHP*-pole p_i*

$$\|w_T T(s)\|_\infty \geq c_{2i} |w_T(p_i)| \quad (28)$$

$$c_{2,i} = \prod_{j=1}^{N_z} \frac{|\bar{z}_j + p_i|}{|z_j - p_i|} \geq 1 \quad (29)$$

This result is stated in (Skogestad and Postlethwaite, 1996, Theorem 5.5, p. 171), it follows easily from Theorem 3 by setting the pole and zero directions equal to 1.

PEAK IN S AND T. From Theorem 3 we get by selecting $w_P(s) = 1$ and $w_T(s) = 1$

$$\boxed{\|S(s)\|_\infty \geq \max_{\text{zeros},j} c_{1,j}} \quad (30)$$

$$\boxed{\|T(s)\|_\infty \geq \max_{\text{poles},i} c_{2,i}} \quad (31)$$

Thus, a peak for $\bar{\sigma}(S(j\omega))$ and $\bar{\sigma}(T(j\omega))$ larger than 1 is unavoidable if the plant has both a RHP-pole and a RHP-zero (unless their relative angle ϕ is 90°).

5. EXAMPLES

EXAMPLE 1. BALANCING A ROD. This example is taken from Doyle *et al.* (1992). Consider the problem of balancing a rod in the palm of one's hand. The objective is to keep the rod upright, by small hand movements, based on observing the rod either at its far end (output y_1) or the end in one's hand (output y_2). The linearized transfer functions for the two cases are

$$G_1(s) = \frac{-g}{s^2\,(Mls^2 - (M+m)g)}$$

$$G_2(s) = \frac{ls^2 - g}{s^2\,(Mls^2 - (M+m)g)}$$

Here l [m] is the length of the rod and m [kg] its mass. M [kg] is the mass of your hand and g [≈ 10 m/s^2] is the acceleration due to gravity. In both cases, the plant has three unstable poles: two at the origin and one at $p = \sqrt{\frac{(M+m)g}{Ml}}$. A short rod with a large mass gives a large value of p, (the pole is far from the imaginary axis in the RHP) and this in turn means that the system is more difficult to stabilize. For example, with $M = m$ and $l = 1$ [m] we get $p = 4.5$ [rad/s] and we desire a bandwidth of about 9 [rad/s] (corresponding to a response time of about 0.1 [s]).

If one is measuring y_1 (looking at the far end of the rod) then achieving this bandwidth is the main requirement. However, if one tries to balance the rod by looking at one's hand (y_2) there is also a RHP-zero at $z = \sqrt{\frac{g}{l}}$. If the mass of the rod is small (m/M is small), then p is close to z and stabilization is in practice impossible with any controller. However, even with a large mass, stabilization is very difficult because $p > z$ whereas we would normally prefer to have the RHP-zero far from the origin and the RHP-pole close to the origin ($z > p$). So although in theory the rod may be stabilized by looking at one's hand (G_2), it seems doubtful that this is possible for a human. To quantify these problems use (26) with c_1 from (27), or use (28) with c_2 from (29). We get

$$c_1 = c_2 = \frac{|z+p|}{|z-p|} = \frac{|1+\gamma|}{|1-\gamma|},\ \gamma = \sqrt{\frac{M+m}{M}}$$

Consider a light weight rod with $m/M = 0.1$, for which we expect stabilization to be difficult. We obtain $c_1 = c_2 = 42$, and we must have

$$\|S(s)\|_\infty \geq 42 \quad \text{and} \quad \|T(s)\|_\infty \geq 42$$

so poor control performance is inevitable if we try to balance the rod by looking at our hand (y_2).

The difference between the two cases, measuring y_1 and measuring y_2, highlights the importance of sensor location on the achievable performance of control.

EXAMPLE 2. RHP-POLE AND RHP-ZERO WITH ALIGNMENT. We consider the following plant

$$G(s) = \begin{bmatrix} \frac{1}{s-p} & 0 \\ 0 & \frac{1}{s+p} \end{bmatrix} \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}_{U_\alpha} \begin{bmatrix} \frac{s-z}{0.1s+1} & 0 \\ 0 & \frac{s+z}{0.1s+1} \end{bmatrix};$$

$$z = 2,\ p = 3 \quad (32)$$

which has a RHP-zero at $z = 2$ and a RHP-pole at $p = 3$. For $\alpha = 0°$ the rotation matrix $U_\alpha = I$, and the plant consists of two decoupled subsystems

$$G_0(s) = \begin{bmatrix} \frac{s-z}{(0.1s+1)(s-p)} & 0 \\ 0 & \frac{s+z}{(0.1s+1)(s+p)} \end{bmatrix}$$

The subsystem g_{11} has both a RHP-pole and a RHP-zero, and closed-loop performance is expected to be poor. On the other hand, there are no particular control problems related to subsystem g_{22}. With $\alpha = 90°$, $U_\alpha = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, which gives

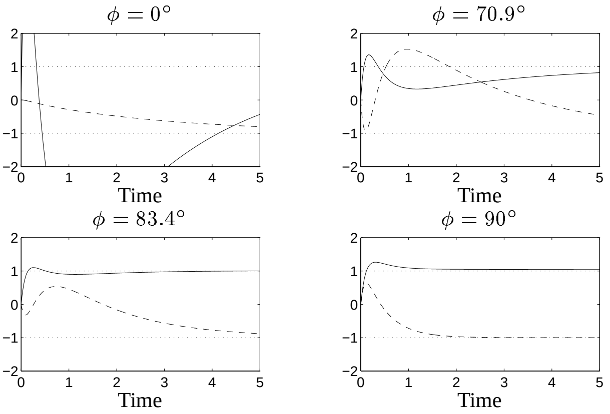

Figure 1. MIMO-plant with angle ϕ between RHP-pole and RHP-zero. Response to step in reference with $\mathcal{H}_\infty$-controller for four different values of ϕ. Solid line: y_1; Dashed line: y_2.

$$G_{90}(s) = \begin{bmatrix} 0 & -\frac{s+z}{(0.1s+1)(s-p)} \\ \frac{s-z}{(0.1s+1)(s+p)} & 0 \end{bmatrix}$$

and we again have two decoupled subsystems, but this time in the off-diagonal elements. The main difference, is that there is no interaction between the RHP-pole and RHP-zero in this case, so we expect this plant to be easier to control. For other values of α we do not have decoupled subsystems, and there will be some interaction between the RHP-pole and RHP-zero. Since the pole is located at the output of the plant, its output direction is fixed, we find $y_p = [1 \quad 0]^T$ for all values of α. On the other hand the zero direction changes from $[1 \quad 0]^T$ for $\alpha = 0°$ to $[0 \quad 1]^T$ for $\alpha = 90°$. Thus, the angle between the pole and zero direction, ϕ, will also vary between 0° and 90° as α varies from 0° to 90°, as seen from Table 1, where we also give c_1 and c_2 for four rotation angles, $\alpha = 0°, 30°, 60°$ and $90°$. The table also shows the values of

TABLE 1: Results from Example 2.

α	$0°$	$30°$	$60°$	$90°$
y_z	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}0.33\\-0.94\end{bmatrix}$	$\begin{bmatrix}0.11\\-0.99\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$
$\phi = \cos^{-1}\lvert y_z^H y_p \rvert$	$0°$	$70.9°$	$83.4°$	$90°$
$c = c_1 = c_2$	5.0	1.89	1.15	1.0
$\lVert S(s)\rVert_\infty$	7.00	2.60	1.59	1.98
$\lVert T(s)\rVert_\infty$	7.40	2.76	1.60	1.31
$\gamma(S/KS)$	9.55	3.53	2.01	1.59

$\|S(s)\|_\infty$ and $\|T(s)\|_\infty$ using controllers obtained by an $\mathcal{H}_\infty$-optimal S/KS-design using the following weights

$$W_u = I; \quad W_p = (\frac{s/M + \omega_B^*}{s})I \tag{33}$$

with $M = 2$ and $\omega_B^* = 0.5$. The weight w_P for the weighted sensitivity means that we require $\|S(j\omega)\|_\infty$ less than 2, and require tight control up to a frequency of about $\omega_B^* = 0.5\ rad/s$. The minimum $\mathcal{H}_\infty$-norm for the S/KS problem is given by the value of γ in the Table 1. The corresponding responses to a step change in the reference $r = [1 \quad -1]^T$ are shown in Figure 1.

Several things are worth noting:

1. We see from the simulation for $\phi = 0°$ in Figure 1 that the response for y_1 is very poor. This is as expected because of the closeness of the RHP-pole and zero ($z = 2, p = 3$).
2. The bound c_1 on $\|S(s)\|_\infty$ in (30) is tight in this case. This can be shown numerically by selecting $W_u = 0.01I$, $\omega_B = 0.01$ and $M_s = 1$ (W_u and ω_B are small so the main objective is to minimize the peak of S). We find that the $\mathcal{H}_\infty$-designs for the four angles yield $\|S(s)\|_\infty = 5.04, 1.905, 1.155, 1.005$, which is very close to c_1.
3. The angle ϕ between the pole and zero, is quite different from the rotation angle α at intermediate values between 0° and 90°. The reason for this is the influence of the RHP-pole in output 1, which yields a strong gain in this direction, and thus tends to push the zero direction toward output 2.
4. For $\alpha = 0°$ we have $c_1 = c_2 = 5$ so $\|S(s)\|_\infty \geq 5$ and $\|T(s)\|_\infty \geq 5$, so it is clearly impossible to get $\|S(s)\|_\infty$ less than 2, as required by the performance weight w_P.
5. The $\mathcal{H}_\infty$-optimal controller is unstable for $\alpha = 0°$ and 30°. This is not surprisingly, because for $\alpha = 0°$ the plant is two SISO systems one of which needs an unstable controller to stabilize it, since $p > z$.

6. CONCLUSION

We have presented lower bounds on the peak in weighted sensitivity and complementary sensitivity functions for systems with RHP-zeros and poles. Peaks in the sensitivity and complementary sensitivity functions are unavoidable if the plant has both a RHP-zero and a RHP-pole with some alignment.

We expect the bounds derived to have importance for the selection of performance weights for controller design and analysis.

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A. PROOFS OF THE RESULTS

Proof of Lemma 1. We have for $s = p$, $G(p) = C(pI - A)^{-1}B + D$. Since p is an eigenvalue of A and x_R is the eigenvector corresponding to the pole p, $(pI - A)x_R = 0$. Therefore x_R is the output state direction with infinite gain for $(pI - A)^{-1}$. The output pole direction becomes $y_p = Cx_R$ as long as $\|D\|$ is finite. The input pole direction u_p follows similarly as the conjugate of the output direction of the transposed system G^T. □

Proof of (13). The output direction is given by $y_z^H G(z) = 0$. For internal stability the controller cannot cancel the RHP-zero and it follows that $L = GK$ has a RHP-zero in the same direction, i.e. $y_z^H L(z) = 0$. Now $S = (I + L)^{-1}$ is stable and thus has no RHP-pole at $s = z$. It then follows from $T = LS$ that $y_z^H T(z) = 0$ and $y_z^H(I - S) = 0$. □

Proof of (14). The square matrix $L(s) = GK(s)$ has a RHP-pole at $s = p$, and if we assume that $L(s)$ has no RHP-zero at $s = p$, then $L^{-1}(p)$ exists and the output pole direction y_p is given by $L^{-1}(p)y_p = 0$. Since T is stable, it has no RHP-pole at $s = p$, so $T(p)$ is finite. It then follows from $S = TL^{-1}$ that $S(p)y_p = T(p)L^{-1}(p)y_p = 0$. □

Proof of Theorem 1. Introduce the scalar function

$$f(s) = y_z^H w_P(s)S(s)y_z$$

which is analytic in the RHP. We then have

$$\|w_P S(s)\|_\infty \ge \|f(s)\|_\infty \ge |f(z)| = |w_P(z)| \qquad \text{(A.1)}$$

The first inequality follows since the singular value measures the maximum gain of a matrix independent of direction, so $\bar\sigma(A) \ge \|Aw\|_2$ and $\bar\sigma(A) \ge \|wA\|_2$ for any vector w with $\|w\|_2 = 1$. The second inequality follows from the maximum modulus theorem. The final equality follows since $w_P(s)$ is a scalar and from the interpolation constraint $y_z^H S(z) = y_z^H$ we get $y_z^H S(z)y_z = y_z^H y_z = 1$. □

Proof of Theorem 2. Introduce the scalar function

$$f(s) = y_p^H w_T(s)T(s)y_p$$

which is analytic in the RHP since $w_T T(s)$ is stable. We then have

$$\|w_T T(s)\|_\infty \ge \|f(s)\|_\infty \ge |f(p)| = |w_T(p)| \qquad \text{(A.2)}$$

The first inequality follows since the singular value measures the maximum gain of a matrix independent of direction and $\|y_p\|_2 = 1$. The second inequality follows from the maximum modulus theorem. The final equality follows since $w_T(s)$ is a scalar and from the interpolation constraint $T(p)y_p = y_p$ we get $y_p^H T(p)y_p = y_p^H y_p = 1$. □

Proof of $c_{1,j}$ in Theorem 3. We consider one RHP-zero z with direction y_z at a time (the subscript j is omitted). Factorize the N_p RHP-poles p_i in $G(s) = B_p G_m(s)$, where $B_p(s)$ is given by (19). It follows that $G_m(s)$ is stable, $B_p(s)$ has all singular values and absolute value of all eigenvalues equal to one for $s = j\omega$ and $\bar\sigma(B_p(s)) \ge 1$ whenever $\mathrm{Re}(s) \ge 0$, see (Havre and Skogestad, 1996, Lemma 2). The loop transfer function can then be written

$$L(s) = GK(s) = B_p G_m K(s) \triangleq B_p(s)L_m(s)$$

then

$$S = TL^{-1} = TL_m^{-1}(s)B_p^{-1}(s) \triangleq S_m B_p^{-1}(s)$$

Introduce the scalar function $f(s) = y_z^H w_P(s)S_m(s)y$ which is analytic (stable) in RHP. We want to choose y so that $|f(s)|$ obtains maximum

$$J(s) = \max_{\|y\|=1} |f(s)| = \max_{\|y\|=1} |y_z^H w_P(s)S_m(s)y|$$

We then get

$$\|w_P(s)S(s)\|_\infty = \|w_P(s)S_m(s)\|_\infty \ge \|J(s)\|_\infty \ge |J(z)| = \max_{\|y\|=1} |w_P(z)|\,|y_z^H B_p(z)y| = |w_P(z)|\,\|y_z^H B_p(z)\|_2 \qquad \text{(A.3)}$$

The first equality follows since $B_p(s)$ is all pass for $s = j\omega$. The first inequality follows since the singular value measures the maximum gain of a matrix independent of direction, so $\bar\sigma(A) \ge \|Aw\|_2$ and $\bar\sigma(A) \ge \|wA\|_2$ for any vector w with $\|w\|_2 = 1$. The second inequality follows from the maximum modulus theorem. The second equality follows from $y_z^H S_m(z) = y_z^H S(z)B_p(z) = y_z^H B_p(z)$ and the fact that $w_P(s)$ is a scalar. The last equality follows from the fact that the largest singular value measure the strongest gain direction and is equivalent to the second norm $\|\cdot\|_2$. The fact that $c_{1,j} \ge 1$ follows from $\sigma_i(B_p(s)) \ge 1\, \forall i$ when $\mathrm{Re}(s) \ge 0$. □

Proof of $c_{2,i}$ in Theorem 3. We consider one RHP-pole p with direction y_p at a time (the subscript i is omitted). Factorize the N_z RHP-zeros z_i in $G(s) = B_z^{-1}(s)G_m(s)$, where $B_z(s)$ is given by the (20). It follows that that $G_m(s)$ is minimum phase, $B_z(s)$ has all singular values and absolute value of all eigenvalues equal to one for $s = j\omega$, see (Havre and Skogestad, 1996, Lemma 2). The loop transfer function becomes

$$L(s) = GK(s) = B_z^{-1}G_m K(s) \triangleq B_z^{-1}L_m(s)$$

Factorize $T = LS = B_z^{-1}L_m S \triangleq B_z^{-1}T_m$ and introduce the scalar function $f(s) = y^H w_T T_m(s)y_p$ which is analytic in RHP. We want to choose y so that $|f(s)|$ obtains maximum

$$J(s) = \max_{|y|=1} |f(s)| = \max_{|y|=1} |y^H w_T(s)T_m(s)y_p|$$

We then get

$$\|w_T(s)T(s)\|_\infty = \|w_T(s)T_m(s)\|_\infty \ge \|J(s)\|_\infty \ge |J(p)| = \max_{|y|=1} |w_T(p)|\,|y^H B_z(p)y_p| = |w_T(p)|\,\|B_z(p)y_p\|_2 \qquad \text{(A.4)}$$

The first equality follows since $B_z(s)$ is all pass for $s = j\omega$. The first inequality follows since the singular value measures the maximum gain of a matrix independent of direction, so $\bar\sigma(A) \ge \|Aw\|_2$ and $\bar\sigma(A) \ge \|wA\|_2$ for any vector w with $\|w\|_2 = 1$. The second inequality follows from the maximum modulus theorem. The second equality follows from $T_m(p)y_p = B_z(p)T(p)y_p = B_z(p)y_p$. The last equality follows from the fact that the largest singular value measure the strongest gain direction and is equivalent to the second norm $\|\cdot\|_2$. The fact that $c_{2,i} \ge 1$ follows from $\sigma_i j(B_z(s)) \ge 1\, \forall j$ when $\mathrm{Re}(s) \ge 0$. □

Proof of $c = c_1 = c_2$ in Corollary 1. Note that when $N_z = N_p = 1$ both z and p are real and positive, so $\bar z = z$ and $\bar p = p$. Consider c_2

$$\begin{aligned} c_2 &= \left\| (I + \frac{2\mathrm{Re}(z)}{p - z} y_z y_z^H) y_p \right\|_2 \\ &= \left\| [\,U \quad y_z\,] \begin{bmatrix} I & 0 \\ 0 & \frac{p+\bar z}{p-z} \end{bmatrix} \begin{bmatrix} U^H \\ y_z^H \end{bmatrix} y_p \right\|_2 \\ &= \left\| UU^H y_p + \frac{p + \bar z}{p - z} y_z y_z^H y_p \right\|_2 \\ &= \sqrt{\sin^2(\phi) + \frac{|\bar z + p|^2}{|z - p|^2}\cos^2(\phi)} \end{aligned}$$

The matrix U contains a basis for the orthogonal subspace to y_z, $y_z^\perp$. The angle between y_p and $y_z^\perp$ is $90 - \phi$, $\cos(90 - \phi) = \sin(\phi)$ and (A.5) follows. We can interpret (A.5) as a weighted projection of y_p on the subspaces $y_z^\perp$, with weight 1, and y_z, with weight $\frac{|\bar z + p|^2}{|z - p|^2}$. In (Boyd and Desoer, 1985, (3.15) on p. 164) it is the projection on the orthogonal subspace $y_z^\perp$ which lacks. By interchanging the roles of the pole and zero directions the bound the bound c_1 follows similarly. □