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SVD Controllers For H_2-, H_∞-, and μ-Optimal Control

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Abstract

For plants where the directionality is independent of frequency, the singular value decomposition (SVD) is used to decouple the system into nominally independent subsystems of lower dimension. In H_2- and H_∞-optimal control, the controller synthesis can thereafter be performed for each of these subsystems independently, and the resulting overall SVD controller will be optimal. In H_∞-optimal control the resulting controller is also *super-optimal*. For robust control in terms of the structured singular value, μ, the use of the SVD decomposition can lead to significant simplifications in controller synthesis, but the optimality of the SVD controller will depend on the structure of the uncertainty. The results are applied to the ill-conditioned distillation case study of Skogestad et a. (1988), where it is shown that an SVD controller is μ-optimal for the case of unstructured input uncertainty.

1 Introduction

In this paper we study SVD controllers which we define to have the form

$$K(s) = V\Sigma_K(s)U^H \tag{1}$$

Here $\Sigma_K(s)$ is a diagonal matrix with real rational transfer functions on the diagonal, and U and V are real unitary singular vector matrices which are derived from a singular value decomposition (SVD) of the plant $G(s)$. SVD controllers have been studied previously by Hung and MacFarlane [13] and Lau et al. [15]. In both these references the SVD structure is essentially used to counteract interactions at one given frequency, as the problems considered are such that U and V change with frequency. However, in this paper we consider some special problems for which U and V are constant at all frequencies and can be chosen to be real, i.e.

$$G(s) = U\Sigma_G(s)V^H \tag{2}$$

where U and V are constant real unitary (i.e. orthonormal) matrices, and $\Sigma_G(s)$ is a diagonal matrix with real rational transfer functions on the diagonal.

Eq. (2) is the singular value decomposition of the plant $G(s)$ with the slight modification that the diagonal elements of $\Sigma_G(s)$, which we refer to as *singular values*, have *phase*, and without necessarily requiring that the singular values in $\Sigma_G(s)$ are ordered according to their magnitudes. The diagonal elements of $\Sigma_G(s)$ will be denoted "subplants" $\sigma_{Gi}(s)$. To simplify the presentation, we consider only SISO subplants here, but it is straightforward to generalize the results presented here to cases where unitary transformations decompose the plant into MIMO subplants, that is, $\Sigma_G(s)$ is block-diagonal (see [11, 12] for details).

The following two classes of plants are of special interest in applications:

A. Plants with scalar dynamics multiplied by a constant matrix. Let

$$G(s) = k(s)A \tag{3}$$

where A is a constant real matrix. One example is the simplified distillation column model by Skogestad et al. (1988) studied in the example below.

B. Circulant symmetric plants. Plants with symmetric circulant transfer matrices are common in practice, and include a large number of processes with some symmetric spatial arrangement. Examples include paper machines with neglected edge effects [16, 22], dies for plastic films [19], and multizone crystal growth furnaces [1]. In these case we can write $G(s) = R^T\Lambda_G(s)R$ where R is a real matrix which is the same for all symmetric circulant processes of a given dimension.

The objective of the paper is to show how controller design can be simplified for plants which can be diagonalized by constant unitary matrices as shown in Eq. (2). The basis for the simplification is that the H_2- and H_∞-norms

$$\|M(s)\|_2 \equiv \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}(M^H(j\omega)M(j\omega))\,d\omega};$$
$$\|M(s)\|_\infty \equiv \sup_\omega \bar{\sigma}(M(j\omega))$$

are invariant to unitary scalings. To make use of this property, we will need that not only the plant, but the control problem as a whole (including the weights) can be "diagonalized" by unitary matrices. For this special class of problems we find that an SVD controller is optimal, and can be calculated by performing controller synthesis on the n *independent* subsystems defined by $\Sigma_G(s)$ and $\Sigma_K(s)$. We show that in the H_∞ case the resulting controller is *super-optimal*, as the norm is minimized in the worst direction for each of these subsystems.

We also study conditions for which an SVD controller is μ-optimal. We show that this depends on the structure of both the perturbations (uncertainty) and the weights. Again this leads to significant simplifications in controller synthesis, though in this case the subsystems cannot always be considered independently.

2 SVD Control Problem

A general control problem is depicted in Fig. 1 where we have $z = M(s)w$. Here w represents some external input signals (e.g. disturbances, noise, references), and z represents the external output signals (e.g. control error, input signals) which we want to keep small. In this paper we consider control problems where $M(s)$ may be written as a linear fractional transformation (LFT) of the controller $K(s)$ using the interconnection matrix N as shown in Fig. 1. We now define the general class of *SVD problems* which are covered by the results of this paper.

Definition 1 SVD problems. *Consider plants which are diagonalized by constant real unitary matrices U and V, that is, the plant is a $n \times n$ transfer matrix of the form $G(s) = U\Sigma_G(s)V^H$. Consider a control problem where the objective is to minimize some norm of*

$$M(s) = W_O(s)M_0(s)W_I(s),$$

where $M_0(s)$ depends on $G(s)$ and $K(s)$ only and $M_0(s)$ may be written as an LFT in terms of the controller $K(s)$. The matrices $W_O(s) = \mathrm{diag}\{W_{Oi}(s)\}$ and $W_I(s) = \mathrm{diag}\{W_{Ii}(s)\}$ are block-diagonal matrices with weighting matrices along the diagonal with dimensions compatible with the dimensions of the subblocks containing $G(s)$ and $K(s)$ in $M_0(s)$. The weighting matrices $W_{Oi}(s)$ and $W_{Ii}(s)$ should fulfill the following structural requirements:

- $W_{Oi}(s) = U_{Oi}\Sigma_{W_{Oi}}(s)U^H$ *when $W_{Oi}(s)$ premultipies $G(s)$ in $M(s)$;*
- $W_{Oi}(s) = U_{Oi}\Sigma_{W_{Oi}}(s)V^H$ *when $W_{Oi}(s)$ premultiplies $K(s)$ in $M(s)$;*
- $W_{Ii}(s) = V\Sigma_{W_{Ii}}(s)V_{Ii}^H$ *when $W_{Ii}(s)$ postmultiplies $G(s)$ in $M(s)$;*
- $W_{Ii}(s) = U\Sigma_{W_{Ii}}(s)V_{Ii}^H$ *when $W_{Ii}(s)$ postmultiplies $K(s)$ in $M(s)$;*

where in all these cases the matrices U_{Oi} and V_{Ii} are unitary and $\Sigma_{W_{Ii}}(s)$ and $\Sigma_{W_{Oi}}(s)$ are diagonal, but may otherwise be chosen freely.

Remark 1. The main property of an SVD problem is that, after substituting $G(s) = U\Sigma_G(s)V^H$ and $K(s) = V\Sigma_K(s)U^H$ into $M_0(s)$, the unitary matrices U and V are cancelled by the weights when forming $M(s)$.

Remark 2. If a weight $W_i(s)$ is a scalar transfer function times identity matrix, then $W_i(s)$ always has the required structure.

Remark 3. The interconnection matrix $N(s)$, obtained by expressing $M(s)$ as a linear fractional transformation of the controller $K(s)$, will be such that there exists block-diagonal unitary matrices

$$U_W = \mathrm{diag}\{\mathrm{diag}\{U_{Oi}\}, U\}; V_W = \mathrm{diag}\{\mathrm{diag}\{V_{Ii}\}, V\} \tag{4}$$

such that

$$\tilde{N}(s) = U_W^H N(s) V_W \tag{5}$$

is a matrix consisting of diagonal subblocks (as illustrated in the upper part of Fig. 2). The proper rearrangement of the inputs and outputs of $\tilde{N}(s)$ (i.e. permutations) yields a block diagonal matrix as illustrated in the bottom of Fig. 2.

Remark 4. The unitary matrices V_W and U_W used to transform the matrix $N(s)$ into the matrix $\tilde{N}(s)$ are independent of the controller $K(s)$. No assumption about the structure of the controller $K(s)$ is therefore necessary at this point.

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3 H_2- and H_∞-Optimal Control

In this section we consider H_2- and H_∞-optimal control. The results also apply to any other norm which is invariant under unitary transformations.

Motivating example. As an example consider the well-known *mixed sensitivity* problem for which

$$M(s) = \begin{bmatrix} W_1(s)G(s)K(s)(I+G(s)K(s))^{-1} \\ W_2(s)(I+G(s)K(s))^{-1} \end{bmatrix} \tag{6}$$

In terms of the notation in Definition 1,

$$M_0 = \begin{bmatrix} T \\ S \end{bmatrix}; \quad W_O = \mathrm{diag}\{W_1, W_2\}; \quad W_I = I$$

where $S = (I+GK)^{-1}$ is the sensitivity and $T = GK(I+GK)^{-1}$ is the complementary sensitivity. To get an SVD problem, assume that the weights are of the form

$$W_1(s) = U_1 \Sigma_{W_1}(s) U^H; \quad W_2(s) = U_2 \Sigma_{W_2}(s) U^H \tag{7}$$

where U_1 and U_2 are unitary matrices but may otherwise be chosen freely. For this example the interconnection matrix becomes

$$N(s) = \begin{bmatrix} 0 & W_1(s)G(s) \\ W_2(s) & -W_2(s)G(s) \\ I & -G(s) \end{bmatrix} \tag{8}$$

Since the H_∞- and H_2-norms are unitary invariant, we can introduce $U_W = \mathrm{diag}\{U_1, U_2, U\}$ and $V_W = \mathrm{diag}\{U, V\}$[1] to the output and input of $N(s)$, respectively, to give an *equivalent* optimal control problem with

$$\tilde{N}(s) = U_W^H N(s) V_W = \begin{bmatrix} 0 & \Sigma_{W_1}(s)\Sigma_G(s) \\ \Sigma_{W_2}(s) & -\Sigma_{W_2}(s)\Sigma_G(s) \\ I & -\Sigma_G(s) \end{bmatrix} \tag{9}$$

The transformed interconnection matrix $\tilde{N}(s)$ consists of diagonal subblocks (similar to the upper matrix in Fig. 2, with $a_j = 0$ and $e_j = 1$), and we may rearrange (permute) the order of the inputs and outputs such that we get a block-diagonal matrix (similar to the lower matrix in Fig. 2). The optimal controller $\tilde{K}(s)$ must be diagonal. To see this, note that any off-diagonal block of $\tilde{K}(s)$ will only affect the input to a subplant *for whose output it has no measurement.* Therefore any off-diagonal block of the optimal $\tilde{K}(s)$ can be taken to be zero. If we refer to the *optimal* diagonal controller $\tilde{K}$ as $\Sigma_K(s)$, then $\Sigma_K(s) = UK^*(s)V^H$ and the optimal controller $K^*(s)$ for the original system $N(s)$ will have the SVD structure of Eq. (1). To find $\Sigma_K(s)$ we must solve n subproblems of smaller dimension.

Theorem 1 (H_2- and H_∞-Optimality) *For SVD problems, an SVD controller is H_2- or H_∞- optimal. This controller can be obtained by designing n independent SISO H_2- or H_∞-optimal controllers, one for each of the SISO subplants of the plant.*

Proof:

1. Express the matrix whose H_2- or H_∞-norm we want to minimize as a Linear Fractional Transformation (LFT) of the controller $K(s)$ to obtain the interconnection matrix $N(s)$ (see Fig. 1).
2. For an SVD problem, form the matrix $\tilde{N}(s) = U_W^H N(s) V_W$ consisting of diagonal blocks (see Remark 3 to Definition 1).
3. The control problem in terms of $\tilde{N}(s)$ is the same as the original one. This follows since the H_2- and H_∞-norms are invariant to pre- and postmultiplication with unitary matrices.
4. The structure of $\tilde{N}(s)$ means that the controller synthesis problem is decomposed into n independent subproblems, as can be seen by rearranging the order of the inputs and the outputs of $\tilde{N}(s)$ (see Fig. 2). Mathematically, the rearrangement of inputs and outputs corresponds to post- and premultiplication with permutation matrices. All permutation matrices are unitary.
5. Any off-diagonal block of the controller $\tilde{K}(s)$ will only affect the input to a subplant *for whose output it has no measurement.* Therefore any off-diagonal block of the optimal $\tilde{K}(s)$ can be taken to be zero. To recover the corresponding controller $K(s)$ for the original problem, we note that in general $N_{22}(s) = G(s)$ so the unitary transformation which was used above to diagonalize $N_{22}(s)$ is given by $\tilde{N}_{22}(s) = \Sigma_G(s) = U^H G(s) V$. If we refer to the optimal controller $\tilde{K}(s)$ as $\Sigma_K(s)$, then $\Sigma_K(s) = UK(s)V^H$ and the optimal controller for the original system is $K(s) = V\Sigma_K(s)U^H$, which is an SVD controller.
6. For diagonal $\tilde{K}(s)$ the control problem in terms of $\tilde{N}(s)$ consists of n independent synthesis problems of lower dimension, and the controller is obtained by minimizing the appropriate norm for each separate subproblem. This follows since the norm of a block-diagonal matrix is minimized by minimizing each block (for the H_2-norm each block must be minimized, whereas for the H_∞-norm we strictly need only minimize the norm of the block with the largest norm). □

[1] To see the connection with V_W in Eq.(4), note that $W_I = I = UIU^H$ so that $V_I = U$.

Remark 1. In general, the solution to the H_∞ controller synthesis problem is non-unique [7], since many controllers will achieve the optimum H_∞ norm in the worst direction, while doing equally well or better in the other directions. However, using the above approach to obtain the H_∞-optimal controller for each subproblem yields *super-optimality* [14, 10, 21], where the H_∞-norm is optimized not only in the *worst* direction, but in n directions.

Remark 2. In general we solve n independent synthesis subproblems of low dimension. In some cases the problem is even further reduced in size since some of these subproblems are identical. For example, for the case of symmetric circulant systems we need only solve $(n+1)/2$ SISO problems for odd n and $n/2+1$ problem for even n. For the case of parallel processes we need only solve two independent subproblems (since $n-1$ subproblems are identical). For details see [12].

4 μ-Optimal Control

In this section we shall generalize the H_∞-problem studied above to the design of robust optimal controllers. This control problem results when we introduce model uncertainty and want to minimize the H_∞-norm for robust performance, or alternatively want to optimize robust stability.

4.1 The Structured Singular Value

The structured singular value, μ, is used as a means of taking uncertainty in a feedback system explicitly into account. Consider

$$\Delta = diag\{\Delta_i\}, \quad \| \Delta_i \|_\infty < 1 \tag{10}$$

These subblocks Δ_i may represent different *sources* of uncertainty in the system. The most common (and useful) structures for the subblocks Δ_i are:

- Full block uncertainty: Δ_i is a full matrix of the same dimension as the plant $G(s)$.
- Independent diagonal uncertainty: $\Delta_i = diag\{\delta_i\}$ is a diagonal matrix with the same dimension as the plant $G(s)$.
- Repeated diagonal uncertainty: $\Delta_i = \delta I$, i.e., a scalar multiplied with an identity matrix of the same dimension as the plant $G(s)$.

The structured singular value with respect to the uncertainty structure Δ is defined as

$$\mu(M) \equiv \begin{cases} 0 \text{ if there does not exist } \Delta \text{ such that } \det(I+M\Delta)=0 \\ \left[\min_\Delta\{\bar{\sigma}(\Delta)\,|\det(I+M\Delta)=0\}\right]^{-1} \quad \text{otherwise} \end{cases} \tag{11}$$

Currently no simple computational method exists for exactly calculating μ in general, but a tight upper bound is

$$\mu(M) \le \inf_D \bar{\sigma}(DMD^{-1}) \tag{12}$$

where D is an invertible matrix with a structure such that $D^{-1}\Delta D = \Delta$. For example, $D = dI$ if Δ is a full matrix, and D is a full matrix if Δ is repeated scalar ($\Delta = \delta I$). For complex uncertainties the upper bound (12) is equal to μ for three or fewer full blocks [5], and usually within 1-2% for other cases [2].

The standard D-K iteration procedure [6] attempts to find the DMD^{-1}-*optimal* controller. Although convergence to the global optimum is not guaranteed, D-K iteration appears to work well [6]. D-K iteration involves two steps:

D Step: Find $D(s)$ to minimize frequency-by-frequency the upper bound on μ in (12).

K Step: Scale the controller design problem with $D(s)$, and design an H_∞-optimal controller for the scaled design problem DMD^{-1}.

4.2 Robust SVD Problem

In Section 3 we showed that an SVD controller was optimal for SVD problems involving the H_2- or H_∞-norm. Additional conditions on the uncertainty weights have to be imposed to ensure that the structures of Δ and D remain unchanged when M is scaled by unitary matrices.

Definition 2 Robust SVD Problems. *Consider an SVD problem with $M(s) = W_O(s)M_0(s)W_I(s)$ as in Definition 1, and multiple sources of uncertainty $\Delta = diag\{\Delta_i\}$, as illustrated in Fig. 3. In addition to the requirements of Definition 1, the weights $W_{Oi} = U_{Oi}\Sigma_{W_{Oi}}(s)V_{Oi}^H(s)$ and $W_{Ii} = U_{Ii}\Sigma_{W_{Ii}}(s)V_{Ii}^H(s)$ related to each Δ_i should fulfill the following:*

1. *$U_{Oi} = V_{Ii}$ for all repeated scalar Δ_i*
2. *$U_{Oi} = V_{Ii} = I$ for all independent diagonal Δ_i*

For a full Δ_i no additional assumption is necessary.

Now we show that for this class of problems the interconnection matrix N can be pre- and postmultiplied by block-diagonal unitary matrices to arrive at an equivalent interconnection matrix $\tilde{N}$ which consists of diagonal subblocks (as in Fig. 2).

Lemma 1 *Let $\tilde{N}$ be defined as in Eqs. (4) and (5). For μ-optimality and DMD^{-1}-optimality of Robust SVD problems, the "diagonalized" control problem is* equivalent *to the original problem, in the sense that*

$$\min_K \mu(F_l(N,K)) = \min_{\tilde{K}} \mu\left(F_l(\tilde{N},\tilde{K})\right) \quad (13)$$

$$\min_K \inf_D \left(DF_l(N,K)D^{-1}\right) = \min_{\tilde{K}} \inf_D \left(DF_l(\tilde{N},\tilde{K})D^{-1}\right) \quad (14)$$

where both μ problems are with respect to the uncertainty in the original control problem, and the structure of the D matrices in both DMD^{-1}-problems is compatible with this uncertainty.

Proof: In the block diagram for the system, replace G with $U\Sigma_G(s)V^H$, and substitute in the weights $W_{Ii}(s)$ and $W_{Oi}(s)$. Rearranging the block diagram (see Fig. 3) gives $\tilde{N}$ with diagonal subblocks (similar to the top matrix in Fig. 2) with the subblocks of $\tilde{\Delta}$ given by $\tilde{\Delta}_i = V_{Ii}^H\Delta_i U_{Oi}$. Note that under the assumptions on U_{Oi} and V_{Ii} in Definition 2

1. $\tilde{\Delta}_i$ is full if and only if Δ_i is full;
2. $\tilde{\Delta}_i$ is repeated scalar if and only if Δ_i repeated scalar;
3. $\tilde{\Delta}_i$ is independent diagonal if and only if Δ_i independent diagonal.

Thus in Fig. 3 the middle block diagram is equivalent to the rightmost block diagram.

A similar argument holds with regard to the upper bound of μ. Under the assumptions on U_{Ii} and V_{Oi}, for each diagonal or full block Δ_i the corresponding D_i and its inverse commute with U_{Ii} and V_{Oi}. For repeated scalar blocks the U_{Ii} and V_{Oi} can be absorbed into the D_i. □

Remark 1. Requirement 1 in Definition 2 holds regardless of the uncertainty's location when the plant is described by a normal transfer function matrix (e.g., symmetric circulant plants).

Remark 2. Requirement 1 also always holds for multiplicative or inverse multiplicative repeated scalar uncertainty with repeated scalar weights.

Remark 3. Requirement 2 puts strong restrictions on the choice of weights for independent diagonal Δ_i. This choice of weights will usually only make sense for plants which have either U or V equal to identity (that is, the inputs or outputs to the plant are naturally aligned in the direction of the singular values). One example of a plant with $V = I$ is the DV configuration for composition control of distillation columns studied by Skogestad et al. [20].

4.3 Optimality of SVD controller

Here we study the optimality if SVD-controllers for Robust SVD Problems.

Theorem 2 (μ-Optimality) *Consider a Robust SVD problem where the objective is to minimize $\sup_\omega \mu(M)$. Assume that all uncertainty blocks Δ_i are diagonal except possibly one full block. Then an SVD controller is optimal. When all the uncertainty blocks are diagonal, then the μ-optimal control problem decouples into n independent SISO μ-optimal control problems, one for each of the SISO subplants of the plant.*

Proof: If all uncertainty blocks Δ_i are diagonal (including repeated scalar uncertainty), then the system consists of independent subsystems. If one uncertainty block is full, then the diagonal blocks can be absorbed into the interconnection matrix to get a "reduced" $\tilde{N}$ which still consist of diagonal subblocks after absorbing the diagonal uncertainty blocks. Whatever the values of the diagonal blocks, we know from Thm. 1 that an SVD controller is optimal for this "reduced" control problem. Thus an SVD controller is optimal for the original μ problem. □

Theorem 3 (DMD^{-1}-Optimality) *Consider a Robust SVD control problem where the objective is to minimize $\sup_\omega \min_D \| DMD^{-1} \|_\infty$. Assume that all uncertainty blocks Δ_i are full blocks. Then an SVD controller is optimal.*

*Furthermore, for D-K iteration the **K** step (with fixed D) consists of n independent SISO H_∞-optimal control problems, one for each of the SISO subplants of the plant.*

Proof: If all uncertainty blocks Δ_i are full, then all D_i are of the form $D_i = d_i I_i$. Since the structure of the interconnection matrix is maintained for all values of $D(s)$, Theorem 1 can be applied to give the result. □

The above theorems complement each other in that Theorem 2 handles one form of uncertainty (diagonal) and Theorem 3 handles another (full). By assuming μ is equal to its upper bound we can handle both types of uncertainty.

Theorem 4 (μ- and DMD^{-1}-Optimality) *Consider a Robust SVD control problem, and assume that μ is equal to its upper bound (12). Then*

1. *An SVD controller is μ-optimal.*
2. *For the D-K-iteration procedure the **K** step consists of n independent SISO H_∞-optimal control problems, one for each of the SISO subplants of the plant.*
3. *For repeated diagonal uncertainty: D_i can be taken to be* diagonal *rather than full in the **D** step.*

Proof:

1. All diagonal blocks (repeated or independent) can be absorbed into the interconnection matrix $\tilde{N}$ without changing its structure. By Thm. 3 an SVD controller is optimal for this "reduced" control problem for all values of the diagonal blocks. Thus an SVD controller is optimal for the original μ problem.
2. For independent diagonal and full block Δ_i, D_i is diagonal and cannot induce interaction between individual subproblems. This also holds for D_i corresponding to repeated scalar Δ_i. To see this, again consider the "reduced" control problem. If the D_i corresponding to the repeated scalar blocks introduced interaction between subproblems, they would effectively allow for a larger class of uncertainty than the original uncertainty description.
3. Scalings D_i which do not cause interactions between subproblems are parametrized by unitary times diagonal matrices. The unitary matrices do not affect the value of the H_∞-norm, so can be ignored. □

Remark 1. The assumption that μ is equal to its upper bound is not restrictive. This equality always holds when all uncertainty subblocks Δ_i are full and three or less, or when one block is full and one is repeated scalar, and has been found to approximately hold (within 1-2%) for all problems of practical interest [2].

Remark 2. Theorem 4 includes another interesting structural condition: for interconnection matrices with diagonal subblocks all repeated diagonal uncertainty can be treated as independent diagonal uncertainty (even in the presence of full blocks).

D-K Iteration: Reduction of Computational Effort

The above results can be used to reduce the computational effort involved in the K step of the D-K iteration procedure in two ways. First, instead of solving one large H_∞-synthesis problem, one may solve n smaller H_∞-synthesis subproblems. Second, some of these n subproblems may be repeated (identical), for example, this occurs for the important case when both the plant and weights are symmetric circulant (or parallel).

In general, the computational effort is *not* reduced in the D step where the upper bound to μ is computed, since for the case of full block uncertainty we have $D = dI$ and we would have to restrict d to be the same for all subproblems. However, all *repeated* subproblems need only be considered once in finding the D_i (see item 3 in Theorem 4). Thus repeated subproblems can be deleted before starting the D-K iteration design procedure, and for a large number of subsystems the size of the D-K iteration and μ-analysis problems can be reduced dramatically.

When all uncertainty blocks are diagonal except possibly one full block, and the weights for the diagonal blocks satisfy Definition 2 (which is quite restrictive, recall Remark 3 following Definition 2), the subproblems can be considered independently for the D step, since the D_i corresponding to the full block can be normalized to be the identity matrix.

Performing D-K iteration on the transformed system will converge faster and is numerically better conditioned than on the original system. This is both because the H_∞ subproblems are smaller than the original problem, and because the algorithm will be initialized with a controller which has the correct (optimal) directionality. This will be illustrated in the examples in Section 5.

5 Example: Distillation Column

Consider the robust controller design problem for the simplified distillation column example studied by Skogestad et al. [20], which under certain assumptions regarding the structure of the uncertainty can be shown to be a Robust SVD problem according to Definition 2. The nominal plant for this problem is of the form $G(s) = k(s)A$ given in Eq. (3):

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{bmatrix} \tag{15}$$

The plant has a condition number of 141.7 at all frequencies [20]. Although not a good model of a real distillation column, this model is an excellent example for demonstrating the problems with ill-conditioned plants and has been studied by many other researchers. For example, in a somewhat altered form this robust controller design problem has been considered by Yaniv and Barlev [23], and was used as a benchmark for the 1991 CDC [3].

For this problem, the relative magnitude of the uncertainty in each of the manipulated variables is given by $w_1(s) = 0.2(5s+1)/(0.5s+1)$. The robust performance specification is that $\|w_2 S_p\|_\infty < 1$ where $w_2(s) = 0.5(10s+1)/10s$ and S_p is the worst sensitivity function possible with the given bounds on the uncertainty in the manipulated variables. The resulting μ condition for Robust Performance (RP) becomes:

$$\text{RP} \iff \mu(M) < 1 \quad \forall\omega \tag{16}$$

$$M = \begin{bmatrix} -W_1KSG & W_1KS \\ W_2SG & -W_2S \end{bmatrix}; \quad \Delta = \text{diag}\{\Delta_1, \Delta_2\} \tag{17}$$

where Δ_1 is a *diagonal* 2×2 perturbation block, Δ_2 is a full 2×2 perturbation block, and $W_1 = w_1 I_2$ and $W_2 = w_2 I_2$. Note that in this case with only three perturbation blocks the upper bound in terms of the scaled singular value is equal to the structured singular value. As stated this is not a Robust SVD Problem according to Definition 2. However, if we allow unstructured (full block) input uncertainty, i.w. Δ_I is a full rather than diagonal matrix, then this is a Robust SVD Problem, and we know from Theorem 4 that an SVD controller $K(s) = V\Sigma_K(s)U^H$ will be μ-optimal.

Controller Design

Skogestad et al. [20] used DK-iteration with some early H_∞-software to design a controller with 6 states giving a value of $\mu = 1.067$. Lundström et al. [18] assumed full block input uncertainty (for numerical convenience) and used the latest state-space H_∞ software [2] to design a μ-optimal controller with 22 states and with $\mu = 0.978$. As just noted we know that the μ-optimal controller for this case with full block input uncertainty should be an SVD controller. Indeed, Engstad [8] found for Lundström's [18] controller that the diagonal elements in $\tilde{K}(s) = V^H K(s) U$ were more than 10^7 times larger than the off-diagonal elements, and removing these off-diagonal elements did not affect the value of μ, which suggests that Lundström's controller is nearly μ-optimal. We have made attempts to improve on the design which gave $\mu = 0.978$ by considering diagonal rather than full block input uncertainty. Somewhat surprisingly, this has not proved successful. Actually, the value of μ with Lundström's [18] controller is not reduced by restricting the input uncertainty to be diagonal. Thus, it seems that in this special case the worst-case uncertainty Δ_i occurs when Δ_i is diagonal. Though we have no proof of this, it does seem reasonable since the input singular vector matrix V corresponding to G in Eq. (15) has large off-diagonal terms, which allows independent input uncertainty to cause strong interactions between the nominal subplants $\Sigma_{Gi}(s)$.

Design of SVD controller. The optimal SVD controller may be obtained by designing two SISO-controllers, $\sigma_{K1}(s)$ and $\sigma_{K2}(s)$, using DK-iteration which involves solving two independent 2×2 H_∞-problems in the K-step and considering the full 4×4 μ-problem in the D-step to obtain the scaling $D = \text{diag}\{d(s)I_2, I_2\}$.

Alternatively, one may design directly a low-order SVD-controller using "μ-K iteration", that is, by optimizing the parameters in a given controller to minimize μ. This approach only requires software to *compute* the structured singular value, as the D-K iteration involving H_∞-norm minimization is not used. Freudenberg [9] used this approach. He *assumed* the controller to be on the SVD form and obtained two SISO controller with 2+3=5 states giving $\mu = 1.054$, and he also used this problem as an example in [4]. Lin [17] used the same approach and obtained two SISO controllers with 7+4=11 states giving $\mu = 1.038$ (observed from plot).

Engstad [8] also used the same approach, but he restricted the input uncertainty to be diagonal rather than full, and used PID controllers of the form

$$\sigma_{Kj} = K_j \frac{1+\tau_{Ij}s}{\tau_{Ij}s}\frac{1+\tau_{Dj}s}{1+0.1\tau_{Dj}s} \tag{18}$$

Each controller has two states and three adjustable parameters. By numerical optimization[2] he obtained a value of $\mu = 1.036$ which is only slightly higher than the optimal value of 0.978, in spite of the fact that the overall controller only has 4 states. The optimal PID parameters for the SVD controller were: $K_1 = 38.3$, $\tau_{I1} = 3.21$, $\tau_{D1} = 0.50$, $K_2 = 5.65$, $\tau_{I2} = 1.24$, $\tau_{D2} = 79.2$.

6 Discussion

The results of this paper are easily generalized to cases with multivariable, possibly nonsquare subplants.

The structure of an SVD controller may be useful also for problems that do not fit into the problem definition in this paper. The reason is that we convert a multivariable design problem into designing n single-loop controllers. The results of this paper (see above) imply that at a fixed frequency the SVD structure is optimal (with some restrictions on the structures of the perturbation blocks given in Definition 2). This provides a theoretical justification for a design method based on obtaining an SVD of the plant at some important frequency, for example, the closed loop bandwidth, and use this as a basis for design a realizable controller. Indeed this has been suggested by several authors [15, 13]. One problem is that we need to obtain real approximations of the singular vector matrices U and V. The ALIGN algorithm of MacFarlane [13] deals with this particular issue.

D-K iteration is known for resulting in controllers with many states. We have shown that the SVD controller is the optimal structure for a certain class of problems, and this may be used for designing controllers with a low number of states. Using V as a pre-compensator and U^H as a post-compensator, we are left with n SISO controllers to design for a plant of dimension $n\times n$. This design problem is similar to the conventional decentralized control problem.

The SVD controller structure can be used for obtaining a simple lower bound on the achievable value for μ. The frequency response of the μ interconnection matrix can be decomposed frequency-by-frequency. At each frequency the plant can be decomposed into its singular value decomposition $G = U\Sigma_G V^H$ where in this case the matrices may be complex. This frequency-by-frequency approach will not yield a realizable controller, since issues such as causality and phase-gain relationships are ignored. Instead, the resulting value for μ will be a lower bound on the structured singular value obtainable by any realizable controller, and may therefore be regarded as a controllability measure.

7 Conclusions

For SVD Problems a plant of dimension $n\times n$ can be decomposed into n SISO subplants which may be considered independently when designing H_2- or H_∞-optimal controller and when performing the K step (controller synthesis) in D-K iteration. When minimizing the H_∞-norm the resulting controller is super optimal, as the norm is minimized in the worst direction for each of the n subproblems.

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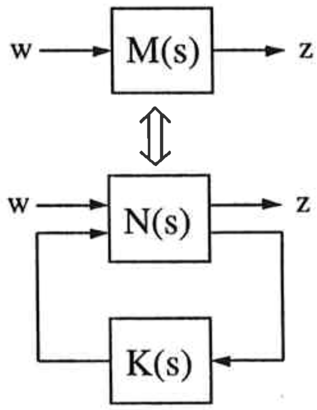

Figure 1: Expressing $M(s)$ as a linear fractional transformation of the controller $K(s)$.

$$\begin{bmatrix}
a_1 & & & b_1 & & \\
& \ddots & & & \ddots & \\
& & a_n & & & b_n \\
c_1 & & & d_1 & & \\
& \ddots & & & \ddots & \\
& & c_n & & & d_n \\
e_1 & & & f_1 & & \\
& \ddots & & & \ddots & \\
& & e_n & & & f_n
\end{bmatrix}$$

$\Updownarrow$ Permutations

$$\begin{bmatrix}
a_1 & b_1 & & & & & \\
c_1 & d_1 & & & & & \\
e_1 & f_1 & & & & & \\
& & a_2 & b_2 & & & \\
& & c_2 & d_2 & & & \\
& & e_2 & f_2 & & & \\
& & & & \ddots & & \\
& & & & & a_n & b_n \\
& & & & & c_n & d_n \\
& & & & & e_n & f_n
\end{bmatrix}$$

Figure 2: Top: $\tilde{N}$ for a case with 3×2 main blocks. Bottom: $\tilde{N}$ permuted to have the n independent synthesis subproblems along the main diagonal. From the bottom matrix it is apparent that the controller design problem consists of n independent subproblems.

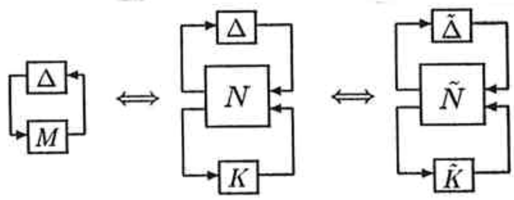

Figure 3: Equivalent representations of system M with perturbation Δ.