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Pairing Criteria for Decentralized Control of Unstable Plants

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The widely used pairing criteria involving the Niederlinski index and the steady-state relative gain array (RGA) used for evaluating integrity under decentralized control are extended to open-loop unstable plants it will in some cases be preferable to choose pairings giving negative Niederlinski index and/or RGA. The relationship between the two measures is also clarified.

1. Introduction

Decentralized control remains popular in the industry, despite developments of advanced controller synthesis procedures leading to full multivariable controllers. Some reasons for the continued popularity are the following:

1. Decentralized controllers are easy to implement.
2. They are easy for operators to understand.
3. The operators can easily retune the controllers to take into account changing process conditions (as a result of 1 above).
4. Some measurements or manipulated variables may fail. Tolerance of such failures is more easily incorporated into the design of decentralized controllers than full controllers.
5. The control system can be brought gradually into service during process startup and taken gradually out of service during shutdown.

The design of a decentralized control system consists of two main steps:

Step 1 is the control structure selection, that is, pairing inputs and outputs.

Step 2 is the design of a single-input single-output (SISO) controller for each loop.

In this paper we consider the pairing problem (step 1).

Selected pairings based on intuition and physical insight usually results in pairings for which the controlled and manipulated variables are close; i.e., the manipulated variable has a fast and direct effect on the controlled variable in open loop. Whereas such pairings may give good performance, interactions between the loops at low frequencies may give rise to problems with *integrity*. Integrity means that the system remains stable as subsystem controllers are arbitrarily brought in and out of service and is defined more precisely next.

Definition of integrity. *Consider a plant $G(s)$ and a controller $C(s)$. Then the closed-loop system possesses integrity if the controller $C'(s) = \Delta C(s)$ stabilizes the system for all*

$$\Delta = \{\mathrm{diag}(\delta_i) | \delta_i \in \{0,1\}, i = 1, ..., n\} \qquad (1)$$

Normally, one requires the plant $G(s)$ to be stable as a prerequisite for integrity because the uncontrolled systems (corresponding to $\delta_i = 0$ in eq 1) may otherwise be unstable, for example, with $C'(s) = 0$. However, in this paper we will not consider the stability of the uncontrolled subsystems, and we can then consider integrity also for unstable plants.

A significant amount of work has been done on the choice of pairings for decentralized control of stable plants, e.g., Bristol (1966), Niederlinski (1971), McAvoy (1983), Grosdidier and Morari (1986), and Skogestad and Morari (1988), to reference a few. Most of the proposed pairing criteria are steady-state conditions for achieving integrity assuming integral action. This is the case for the widely used conditions based on the relative gain array (RGA) (Bristol, 1966) and Niederlinski index (N_I) (Niederlinski, 1971), where in order to achieve integrity one must avoid pairings corresponding to negative steady-state values.

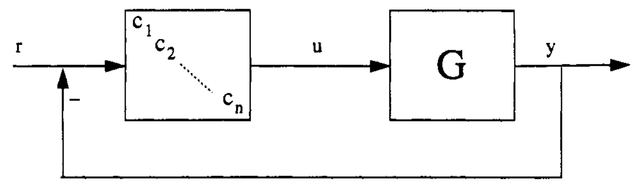

Figure 1. Block diagram of control system with a decentralized controller.

However, these pairing criteria were derived for open-loop stable plants, and their application or relevance for open-loop unstable plants have not been clarified previously. The objective of this paper is to generalize these "steady-state" pairing criteria to unstable plants.

Remark: We will in this paper avoid the term "steady state" since a steady state does not exist for an unstable system in open loop. Instead we refer to conditions at zero frequency ($\omega = 0$ or $s = 0$), and use, for example, the term "RGA(0)" instead of "steady-state RGA".

2. Tools for Pairing Selection

In the paper $G(s)$ denotes a square $n \times n$ plant and $C(s)$ the decentralized controller; see Figure 1. Without loss of generality we assume that $C(s)$ is diagonal (after possibly rearranging the order of columns and rows in $G(s)$). We also assume that $GC(s)$ contains integrators in all channels, and that $C(s)$ otherwise is stable. G may be unstable and we let P denote the number of unstable poles in the open right half plane (RHP), not counting poles at $s = 0$ (integrators). Negative feedback is assumed throughout the paper, and we assume that $G(s)\ C(s)$ has no internal pole-zero cancellations in the right half plane. We then have that closed-loop stability is guaranteed if and only if $(I + G(s)\ C(s))^{-1}$ has no RHP poles. The Laplace variable s will be dropped where it is not needed for clarity.

We next define the Niederlinski index and the relative gain array where the rule for stable systems is to avoid pairings corresponding to negative steady-state values.

Niederlinski Index. The Niederlinsk index, N_I, is defined as

$$N_I(s) = \det(G)/\prod_{i=1}^{n} g_{ii} = \det G/\det \tilde{G} \qquad (2)$$

where

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$$\tilde{G} = \mathrm{diag}\{g_{ii}\} = \begin{pmatrix} g_{11} & & & \\ & g_{22} & & \\ & & \ddots & \\ & & & g_{nn} \end{pmatrix} \quad (3)$$

For later use let $\tilde{P}$ denote the number of unstable poles in $\tilde{G}$. Note that $\tilde{P}$ is equal to the sum of unstable poles in the diagonal elements of $G(s)$. Thus in most cases $\tilde{P} > P$.

Relative Gain Array. The RGA elements, λ_{ij}, are defined as follows

$$\lambda_{ij}(s) = g_{ij}[G^{-1}]_{ji} \quad (4)$$

where $[G^{-1}]_{ji}$ represents the jith element of G^{-1}. We have $[G^{-1}]_{ji} = c_{ij}/\det(G)$ where $c_{ij} = (-1)^{i+j}\det(G^{ij})$ denotes the ijth cofactor of G, and G^{ij} is obtained from G by removing row i and column j. In general, the RGA is computed as a function of frequency, but we shall mainly consider its value at $s = 0$.

We will here consider the diagonal RGA elements for which we derive

$$\lambda_{ii}(s) = g_{ii}[G^{-1}]_{ii} = \frac{g_{ii}\det G^{ii}}{\det G} = \det G'_{ii}/\det G \quad (5)$$

where

$$G'_{ii} = \begin{pmatrix} g_{ii} & 0 \\ 0 & G^{ii} \end{pmatrix} \quad (6)$$

For later use let P'_{ii} denote the number of unstable poles in G'_{ii}. For plants with many inputs and outputs we usually have $\tilde{P} > P'_{ii} > P$. For example, consider a $n \times n$ plant G with a single unstable pole which appears in all the elements of G. In this case $\tilde{P} = n$, $P'_{ii} = 2$, $\forall i$ and $P = 1$. However, note that P'_{ii} in general may be different for different loops i.

For the special case of 2×2 plants we get $\tilde{G} = G'_{ii}$, so $P_{ii} = \tilde{P}$, and we also see from (2) and (5) that $N_I = 1/\lambda_{ii}$. Consequently, for 2×2 plants the RGA and N_I contain the same information. At the end of the paper we shall discuss how they differ.

3. Nyquist Stability Condition

We showed above that $N_I(s) = \det G/\det \tilde{G}$ and $\lambda_{ii}(s) = \det G'_{ii}/\det G$, that is, both the N_I and the RGA elements are given by the ratio of two determinants, one involving the plant model (G) and one involving some "alternative" model ($\tilde{G}$ or G'_{ii}). The following derivation makes it possible to derive the results for these two measures in a unified manner.

Let $\hat{G}$ denote some alternative model of the plant G. Throughout this paper we assume that both that both of the square matrices $G(0)$ and $\hat{G}(0)$ are nonsingular. Introduce the complementary sensitivity function in terms of $\hat{G}$

$$\hat{H} = \hat{G}C(I + GC)^{-1} \quad (7)$$

and the relative "error"

$$E = (G - \hat{G})\hat{G}^{-1} \quad (8)$$

Then the following factorization applies (e.g., Grosdidier and Morari, 1986)

$$(I + GC) = (I + E\hat{H})(I + \hat{G}C) \quad (9)$$

Proof: Trivial by noting that $\hat{H} = \hat{G}C(I + \hat{G}C)^{-1} = (I + \hat{G}C)^{-1}\hat{G}C$.

The following lemma then follows directly from the multivariable Nyquist theorem. For completeness a proof is given in Appendix 1, but similar results, at least for stable plants, have been presented before (e.g., Grosdidier and Morari, 1986; Nwokah and Perez, 1991).

Lemma 1. *Let the number of open-loop unstable poles of $G(s)$ $C(s)$ and $\hat{G}(s)$ $C(s)$ be P and $\hat{P}$, respectively. Assume that the closed-loop system consisting of $\hat{G}(s)$ $C(s)$ is stable. Then the closed-loop system consisting of $G(s)$ $C(s)$ is stable if and only if*

$$\mathcal{N}(\det(I + E\hat{H})) = \hat{P} - P \quad (10)$$

where $\mathcal{N}$ denotes the number clockwise encirclements of the origin as s traverses the Nyquist D contour.

In other words, (10) says that for stability $\det(I + E\hat{H})$ must provide for the difference in the number of required encirclements between $\det(I + GC)$ and $\det(I + \hat{G}C)$. If this is not the case, then at least one of the systems consisting of GC or $\hat{G}C$ must be unstable.

We will now see that information about what happens at $s = 0$ may provide useful information about the number of encirclements.

Lemma 2. *Let the number of open-loop unstable poles (excluding poles at $s = 0$) of $G(s)$ $C(s)$ and $\hat{G}(s)$ $C(s)$ be P and $\hat{P}$, respectively. Assume that the controller C is such that $\hat{G}C$ has integral action in all channels, and that the transfer functions GC and $\hat{G}C$ are strictly proper. Then if*

$$\det G(0)/\det \hat{G}(0) \begin{cases} <0 & \text{for } \hat{P} - P \text{ even} \\ >0 & \text{for } \hat{P} - P \text{ odd} \end{cases} \quad (11)$$

at least one of the following instabilities will occur: (a) the closed-loop system consisting of GC is unstable; (b) the closed-loop system consisting of $\hat{G}C$ is unstable.

Proof: For stability of both $(I + GC)^{-1}$ and $(I + \hat{G}C)^{-1}$ we know from lemma 1 that $\det(I + E(s)\hat{H}(s))$ needs to encircle the origin $\hat{P} - P$ times as s traverses the Nyquist D contour. $\hat{H}(0) = I$ because of the requirement for integral action in all channels of $\hat{G}C$. Also, since GC and $\hat{G}C$ are strictly proper, $E\hat{H}$ is strictly proper, and hence $E(s)\hat{H}(s) \rightarrow 0$ as $s \rightarrow \infty$. We therefore have

$$\lim_{s\to\infty} \det(I + E(s)\hat{H}(s)) = 1 \quad (12)$$

$$\lim_{s\to 0} \det(I + E(s)\hat{H}(s)) = \lim_{s\to 0} \det(G(s)\hat{G}^{-1}(s)) \quad (13)$$

Thus, the map of $\det(I + E\hat{H})$ starts at $\det G(0)/\det \hat{G}(0)$ (for $s = 0$) and ends at 1 (for $s = \infty$). A more careful analysis of the Nyquist plot reveals that the number of encirclements will be even for $\det G(0)/\det \hat{G}(0) > 0$, and off for $\det G(0)/\det \hat{G}(0) < 0$. Thus, if this parity (odd or even) does not match that of $\hat{P} - P$ we will get instability, and the theorem follows.

Remarks:

1. Lemmas 1 and 2 are not restricted to decentralized control, but their interpretation in terms of RGA and Niederlinski index given in the next section are.

2. The results in the next section follow directly from lemma 2 since $\det G(0)/\det \tilde{G}(0) = N_I(0)$ and $\det G(0)/\det G'_{ii}(0) = 1/\lambda_{ii}(0)$.

4. Pairing Criteria for Unstable Plants

Theorem 1: Niederlinski Index. *Let the number of open-loop unstable poles (excluding poles at $s = 0$) of*

$G(s)$ and $\tilde{G}(s) = diag\{g_{11}(s), g_{22}(s), ..., g_{nn}(s)\}$ be P and $\tilde{P}$, respectively. Assume that the controller C is such that $\tilde{G}C$ has integral action in all channels and is otherwise stable, *and assume that GC is strictly proper. Then if* $N_I(0)$ *has the "wrong" sign, that is, if*

$$N_I(0) \begin{cases} <0 & \text{for } \tilde{P} - P \text{ even} \\ >0 & \text{for } \tilde{P} - P \text{ odd} \end{cases} \tag{14}$$

at least one of the following instabilities will occur:

(a) $(I + GC)^{-1}$ is unstable, that is, the overall system is unstable.

(b) $(I + \tilde{G}C)^{-1}$ is unstable, that is, for decentralized control at least one of the loops is unstable by itself.

Proof: Let $\hat{G} = \tilde{G}$ in lemma 2 and use the definition of N_I in (2).

Remarks:

1. Theorem 2 is a direct generalization of the Niederlinski pairing criterion (see theorem 3 in Grosdidier et al. (1985)) to unstable plants.

2. The matrix $G\tilde{G}^{-1}$ which appears in the Niederlinski index is the inverse of the PRGA matrix introduced by Hovd and Skogestad (1992). The diagonal elements of the PRGA are identical to the diagonal elements of the RGA.

Theorem 2: Relative Gain Array. *Let the number of open-loop unstable poles (excluding poles at $s = 0$) of $G(s)$ and $G'_{ii}(s) = diag\{g_{ii}(s), G^{ii}(s)\}$ be P and P'_{ii}, respectively. Assume that the decentralized controller C is such that $G'_{ii}C$ has integral action in all channels and is otherwise stable, and assume that GC is strictly proper. Then if $\lambda_{ii}(0)$ has the "wrong" sign, that is, if*

$$\lambda_{ii}(0) \begin{cases} <0 & \text{for } P'_{ii} - P \text{ even} \\ >0 & \text{for } P'_{ii} - P \text{ odd} \end{cases} \tag{15}$$

at least one of the following instabilities will occur:

(a) The overall system is unstable; i.e., $(I + GC)^{-1}$ is unstable.

(b) Loop i is unstable by itself; i.e., $(1 + g_{ii}c_i)^{-1}$ is unstable.

(c) The system is unstable as loop i is removed; i.e., $(I + G^{ii}C^{ii})^{-1}$ is unstable.

Proof: Let $\hat{G} = G'_{ii}$ in lemma 2 and use the definition of RGA in (5).

Remarks:

1. All of these instabilities are undesirable. Of course, the worst is that the overall system is unstable. The last possibility for instability is also most undesirable. It will imply instability of the system should loop i become inactive, for example, due to input saturation (in which case u_i is constant).

2. Theorem 2 is a direct generalization of the widely used RGA pairing criterion (see theorem 6 in Grosdidier et al. (1985)) to unstable plants.

3. P'_{ii} is equal to the sum of unstable poles in g_{ii} and G^{ii}.

4. For 2×2 plants $N_I(0) = 1/\lambda_{ii}(0)$, but for larger systems the measures contain different information. Specifically, consider a plant of size 3×3 or larger where the overall closed-loop system (consisting of GC) is stable. Then if $\lambda_{ii}(0)$ has the wrong sign, we know that loop i by itself is unstable or that the system with loop i removed is unstable, both of which are undesirable. Thus, pairing such that $\lambda_{ii}(0)$ has the wrong sign does not necessarily mean that any of the individual loops are unstable. On the other hand, if $N_I(0)$ has the wrong sign, we know that (at least) one of the individual loops are unstable.

5. Since $\lambda_{ii}(0)$ and $N_I(0)$ contain different information for plants larger than 2×2, we may have cases where all $\lambda_{ii}(0)$'s ($i = 1, ..., n$) have the right sign, whereas $N_I(0)$ has the wrong sign, and *vice versa*.

Summary of Pairing Rules for Integrity. For *stable* plants one should select pairings corresponding to *positive* values of the Niederlinski index and RGA(0).

For the special case of a $n \times n$ plant with *one* unstable pole which appears in *all* the elements of $G(s)$, we have $\tilde{P} = n$ and $P'_{ii} = 2$. In this case theorems 1 and 2 yield (1) selection of a set of pairings such that $N_I(0)$ is positive if n is odd and negative if n is even and (2) pair on *negative* RGA elements (i.e., $\lambda_{ii}(0) < 0$).

For the special case of a $n \times n$ plant with P unstable poles which appear in *all* the elements of $G(s)$, we have $\tilde{P} = nP$ and $P'_{ii} = 2P$. In this case theorems 1 and 2 yield (1) selection of a set of pairings such that $N_I(0)$ is positive if $(n - 1)P$ is even and negative if $(n - 1)P$ is odd and (2) pair on positive RGA elements (i.e., $\lambda_{ii}(0) > 0$) if P is even and on negative RGA elements if P is odd.

5. Examples

Consider a 2×2 plant with one unstable pole

$$G(s) = \begin{bmatrix} \dfrac{9s+1}{(-s+1)(s+1)} & \dfrac{2s-18}{(-s+1)(s+1)} \\ \dfrac{-1.5s-6}{(-s+1)(0.5s+1)} & \dfrac{12}{(-s+1)(0.5s+1)} \end{bmatrix} \tag{16}$$

It may not be obvious the there is only one unstable pole, but a minimal state space realization shows that this is the case, i.e., $P = 1$, For $\tilde{G}$ we get $\tilde{P} = 2$ because the RHP pole appears in both diagonal elements of G.

At $s = 0$ we get

$$G(0) = \begin{pmatrix} 1 & -18 \\ -6 & 12 \end{pmatrix} \tag{17}$$

The pairing ($y_1 - u_1$, $y_2 - u_2$) indicated by (17) we term "pairing 1", and the opposite pairing we term "pairing 2". We get for pairing 1 the RGA matrix and the Niederlinski index

$$\Lambda(0) = \begin{pmatrix} -0.13 & 1.13 \\ 1.13 & -0.13 \end{pmatrix}; \quad N_I(0) = -8$$

Since both the diagonal RGA elements and the Niederlinski index are negative for pairing 1, from the conventional pairing rules we would prefer pairing 2. However, from both theorem 1 and theorem 2 we know that some kind of undesirable instability will occur for pairing 2. On this basis pairing 1, which corresponds to pairing on negative values of $N_I(0)$ and RGA(0), is preferable.

Nyquist plots. For pairing 1 we use the controllers

$$c_1(s) = -\frac{s+1}{s}, \quad c_2(s) = -\frac{(s+1)(0.1s+1)}{s(0.01s+1)}$$

Figure 2 shows Nyquist plots of $1 + g_{11}c_1$, $1 + g_{22}c_2$, and $I + E_H\tilde{H}$ for this choice. We find that $1 + g_{11}c_1$ and $1 + g_{22}c_2$ each make one counterclockwise encirclement of the origin (there seem to be two when negative frequencies are included, but one of them comes from the integrator) and we get from the Nyquist stability criterion that the individual loops are stable. Furthermore, $\det(I + E_H\tilde{H})$ makes one clockwise encirclement of the origin, which is equal to $\tilde{P} - P$, and the system is closed-loop stable according to lemma 1. This confirms the predictions based on the Niederlinski index and RGA(0).

Comment. It may be of interest to check the result using the Nyquist stability condition. From (9) we have

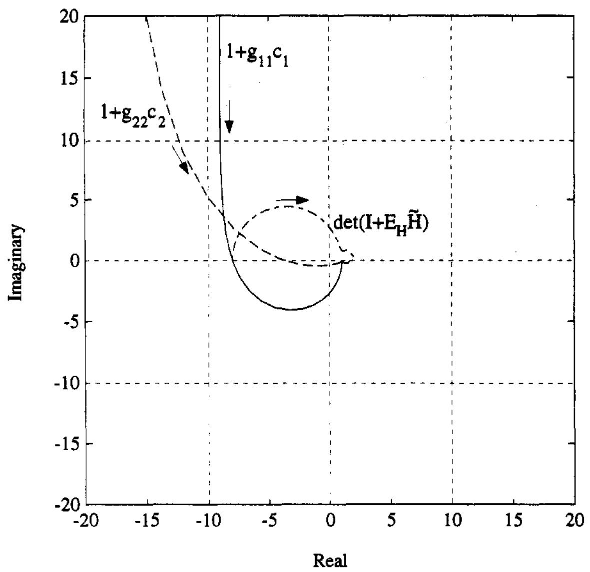

Figure 2. Example 1. Nyquist plots for $1 + g_{11}c_1$, $1 + g_{22}c_2$, and $\det(I + E_H\tilde{H})$. The arrows show the direction of increasing frequency. Only positive frequencies are shown.

$$\det(I + GC) = \det(I + E\tilde{H})\cdot\prod_{i=1}^{n}(1 + g_{ii}c_i)$$

and the number of clockwise encirclements of the origin for $\det(I + GC)$ is therefore $\mathcal{N}(\det(I + GC)) = +1 - 1 - 1 = -1 = -P$, and the system is stable according to the generalized Nyquist stability theorem.

Example 2. Consider a CSTR in which the exothermic reaction A → B takes place. Heat is removed from the CSTR by external cooling. It is desired to control the concentration of component A, c_A, the concentration of component B, c_B, and the temperature T. Available manipulated variables are the feed flow rate, F, the concentration of component A in the feed, c_{AF}, and the cooling water temperature, T_C. The nonlinear model is given in Appendix 2. After linearization, the following model is obtained:

$$\dot{x} = \begin{bmatrix} -0.1562 & 0 & -0.01553 \\ 0.0562 & -0.1000 & 0.01553 \\ 0.7803 & 0 & 0.07958 \end{bmatrix} x + \begin{bmatrix} 0 & 0.1000 & 0.1122 \\ 0 & 0 & -0.1124 \\ 0.0361 & 0 & -0.2000 \end{bmatrix} u \quad (18)$$

where $x = [c_A\, c_B\, T]^T$ and $u = [T_C\, c_{AF}\, F]^T$. The eigenvalues of A are −0.100, −0.081, and 0.004, and we find that there is one unstable pole, i.e., $P = 1$. All three states are measured, i.e., $y = x$, and we get

$$G(0) = \begin{bmatrix} 1.80 & 25.57 & 18.71 \\ -1.80 & -24.57 & -18.71 \\ -18.12 & -250.74 & -180.96 \end{bmatrix};\quad \Lambda(0) = \begin{bmatrix} -34.0 & 25.6 & 9.4 \\ 8.9 & -24.5 & 16.6 \\ 26.1 & -0.1 & -25.0 \end{bmatrix} \quad (19)$$

Conventional pairing. Conventional pairing rules for stable plants would indicate the pairing $T \leftrightarrow T_C$, $c_B \leftrightarrow F$, $c_A \leftrightarrow c_{AF}$, to be preferable, since this is the only pairing corresponding to all positive RGA's, and the value of the Niederlinski index for this pairing is 0.015. This choice of pairings also happens to makes sense from a physical point of view and corresponds to RGA close to the identity matrix at high frequency.

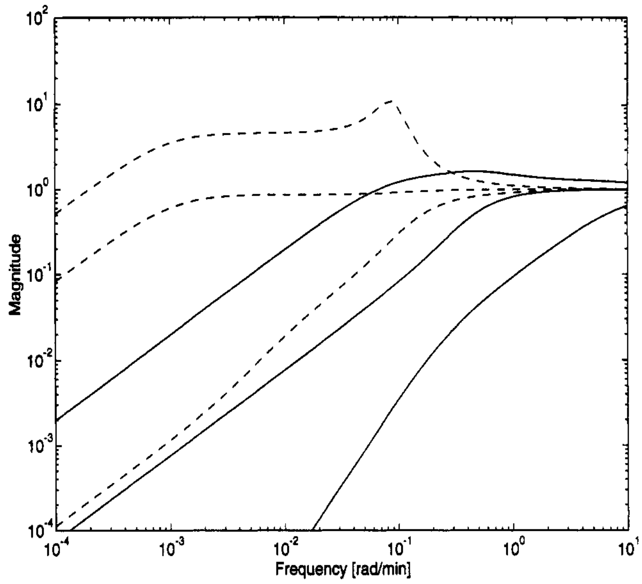

Figure 3. Example 2. Singular values of the sensitivity function, $\sigma_i(S)$, with the "conventional pairing" (solid) and the "pairing for integrity" (dashed).

However, the plant has one unstable pole, which appears in all elements of the transfer function matrix, and therefore $\tilde{P} = 3$ for all pairings, $P'_{ii} = 2$ for all loops for all pairings, and integrity considerations require pairing on a positive Niederlinski index, but *negative RGA's*. In particular, theorem 2 tells us that with the "conventional" pairing, we will have an integrity problem either where the overall system is unstable, or where one of the loops is unstable by itself, or where the system becomes unstable as one of the loops is removed. Nevertheless, in a practical situation, this pairing may be preferable, because it may yield better closed-loop performance with all loops closed. Thus, a choice may have to be made between performance and integrity. We shall return to this point in the discussion.

Pairing for integrity. The only pairing fulfilling the requirement of pairing on negative RGA elements is $c_A \leftrightarrow T_C$ (loop 1), $c_B \leftrightarrow c_{AF}$ (loop 2), $T \leftrightarrow F$ (loop 3), which in addition to having negative RGA's has a Niederlinski index of 0.0016.

For this pairing we may use the controllers

$$c_1 = -5\frac{125s + 1}{125s};\quad c_2 = 0.3\frac{150s + 1}{150s};\quad c_3 = 0.04\frac{200s + 1}{200s} \quad (20)$$

With these controllers we find (1) the overall system is stable, (2) each loop by itself is stable (i.e., each loop stabilizes the plant), and (3) the system remains stable as one of the loops is taken out of service (i.e., due to input saturation).

Thus we have stability in all seven cases, which may be confirmed by computing the closed-loop poles for the overall 3 × 3 system, the three 2 × 2 subsystems, and the three single loops. Note that in this case closing any one of the three single loops will stabilize the overall system. Thus, the only situation for which we do not have stability is the $C(s) = 0$.

However, the resulting closed-loop performance with this controller is poor as is illustrated by the singular values of the sensitivity function in Figure 3 (dashed lines). We note that the speed of response in the worst direction is very slow (time constant about 10^4 min), in spite of the fact that the sensitivity function has a peak value larger than 10. No attempt has been made at optimizing closed-loop performance, but it appears difficult to improve close-loop performance while maintaining stability of the individual loops.

6. Discussion

RGA as a Special Case of the Niederlinski Index. As discussed above, the Niederlinski index, $N_I(0)$, and the RGA elements, $\lambda_{ii}(0)$, contain different information for $n \times n$ plants with $n \geq 3$. Nevertheless, it is clear from (2) and (5) that the two quantities are closely related, and we will show that if we consider the Niederlinski index also of the subsystems then the RGA is strictly not needed.

Define the Niederlinski index for subsystem ii (that is, with loop i removed) as

$$N_I^{ii}(s) = \deg(G^{ii})/\prod_{j\neq i} g_{jj} \quad (21)$$

Then we get from (2) and (5)

$$\lambda_{ii} = N_I^{ii}/N_I \quad (22)$$

(this expression applies to the diagonal RGA elements, and not to the off-diagonal ones). It is then clear that if we evaluate the Niederlinski index also of the subsystems then we will have all the information given by the diagonal RGA elements (and more).

Still, the RGA matrix is very useful because it has nice algebraic properties, since it can be evaluated once for all possible choices of pairings (if one computes the entire RGA matrix and not only the diagonal elements), and because one can often with one "glance" decide on an appropriate set of pairings.

Integrity versus Performance. Above we derived conditions for integrity in terms of the Niederlinski index and RGA at $s = 0$ (theorems 1 and 2). However, the feedback control properties are mainly determined by the model behavior at frequencies corresponding to the closed-loop bandwidth, and in most cases one prefers to pair on variables "close to each other", which normally corresponds to RGA close to the identity matrix at high frequency. If the pairing criterions at low and high frequency agree, then there is no problem. However, as illustrated in example 2 there may be cases where a trade-off has to be made between integrity and performance.

For the specific case in example 2 it is likely that performance considerations may be more important, such that one would sacrifice integrity and select the "conventional pairing" which corresponds to pairing on RGA elements with the wrong sign. However, also in this case the results presented in this paper may provide useful information. Assume we close the three loops sequentially. In generally, it is reasonable to first stabilize the plant and close a loop where it is unlikely that the measurement will fail or the actuator will saturate. Assume in this case that we select the temperature loop, $T \leftrightarrow T_C$ (loop 1). This loop is then closed that it is stable by itself. Next we select to close the loop $c_B \leftrightarrow F$ (loop 2) such that the overall 2×2 system is stable. We recompute the RGA for the corresponding 2×2 plant (it cannot be obtained directly from the previous RGA matrix in eq 19) and find that it corresponds to pairing on a RGA value of 26, which is the "wrong" sign since there is one unstable pole. Thus, since loop 1 is stable by itself, and since the overall 2×2 system is stable, we have from theorem 2 that loop 2 must be unstable by itself. Finally, we close the loop $c_A \leftrightarrow c_{AF}$ (loop 3) which corresponds to pairing on a positive RGA element in eq 19. This is again the "wrong" sign, and since the overall system is stable and the 2×2 system with loop 3 removed is stable, we have that loop 3 must be unstable by itself. In conclusion, when we close the loops sequentially such that the system is stable after closing each loop, both loops 2 and 3 are unstable by themselves. In addition, the remaining system when loop 1 is removed is unstable (this follows since loop 1 corresponds to pairing on a positive RGA-element ("wrong sign") in eq 19). These three instabilities all occur when loop 1 is removed, so if we can assure that loop 1 is always functioning, we can design a control system with no integrity problems. As an example, with the "conventional pairing" the following controllers yield a stable closed-loop system provided loop 1 is in service:

$$c_{c1} = -40\frac{10s+1}{10s}; \quad c_{c2} = -10\frac{10s+1}{10s}; \quad c_{c3} = 10\frac{10s+1}{10s} \quad (23)$$

This controller is of the same complexity as the proposed controller for the "pairing for integrity" in eq 20, but gives far superior closed-loop performance, as can be seen from the solid lines in Figure 3.

Decentralized Detuning. In this paper we have derived conditions for stability of subsystems (*integrity*) under decentralized integral control by use of the generalized Nyquist condition in terms of the determinant, $\det(I + GC)$.

A related problem is *detuning*. The simplest case is when each loop is detuned by the same factor. We say that a system is *detunable* if the controller controller $C' = \delta C$ stabilizes the system for all $0 \leq \delta \leq 1$ (clearly, the concept of detunability only applies to stable plants). In this case introducing $\det(I + GC') = \prod_i \lambda_i(I + GC')$ proves useful. It yields the generalized Nyquist condition in terms of characteristic loci where for stability we consider the combined encirclements of the $\lambda_i(GC)$'s around the point $-1/\delta$. This leads to the results for integral controllability (IC) presented by Grosdidier et al. (1985; Theorem 7). (More precisely, a plant is IC if there *exists* a controller yielding a detunable system.)

A further generalization to the case where each loop may be detuned independently leads to the notation of *decentralized detunability* (*DD*): *A system possesses DD if the controller* $C' = \Delta C$ stabilizes *the system for all* $\Delta = diag\{\delta_1,...,\delta_n\}$ *with* $\delta_i \in [0,1]$. (For unstable plants one needs to place some additional restriction on the allowed detuning to get useful results, for example, by allowing detuning only of one loop at the time.) The related existence condition called decentralized integral controllability (DIC) was introduced by Skogestad and Morari (1988, 1992). (More precisely, a plant is DIC if there *exists* a controller yielding DD.) DIC may be shown to be equivalent to requiring D stability (Yu and Fan, 1990).

Clearly, integrity is a prequisite for DD, and satisfying the conditions in terms of the sign of RGA and Niederlinski index at $s = 0$ (theorems 1 and 2) is a prerequisite for DIC. Loosely speaking, one can say that the RGA and Niederlinski index provide a means of checking "corner values" of the DIC condition.

7. Conclusions

The pairing criteria based on the Niederlinski index and RGA for systems under decentralized integral control have been generalized to hold for open-loop unstable plants (theorems 1 and 2).

By evaluating the Niederlinski index of the subsystems the RGA is strictly not needed (eq 22), but in practical use the RGA matrix is still very useful.

Nomenclature

C = controller transfer function matrix
c_i = iith element of controller C (for diagonal C)
$E = (G - \hat{G})\hat{G}^{-1}$
G = plant transfer function matrix
$\hat{G}$ = simplified or alternative model for G
G^{ii} = G with row i and column i deleted
$\tilde{G}$ = matrix consisting of the diagonal elements of G, diag$\{g_{ii}\}$, $i = 1, n$
G'_{ii} = diag$\{g_{ii}, G^{ii}\}$
g_{ij} = ijth element of G
$\hat{H}$ = complementary sensitivity matrix corresponding to $\hat{G}$, $\hat{G}C(I + \hat{G}C)^{-1}$
$\tilde{H}$ = matrix of complementary sensitivity functions for individual loops, $\tilde{G}C(I + \tilde{G}C)^{-1}$
N_I = Niederlinski index, det $G(0)/\Pi_i g_{ii}(0)$
P = number of open loop unstable poles of GC
$\hat{P}$ = number of open loop unstable poles of $\hat{G}C$
$\tilde{P}$ = number of open loop unstable poles of $\tilde{G}C$
P'_{ii} = number of open loop unstable poles of $G'_{ii}C$
RGA = relative gain array matrix, $G \times [G^{-1}]^T$, $\times$ denotes element-by-element multiplication
S = sensitivity function, $(I + GC)^{-1}$
s = Laplace variable

Greek Symbols

λ_i = ith eigenvalue
λ_{ij} = ijth element of RGA

Appendix 1

Proof of Lemma 1. Let us first state the multivariable Nyquist theorem.

Theorem 3: The Multivariable Nyquist Theorem. *Let $\mathcal{N}(f(s))$ denote the number of clockwise encirclements of the map of* the *Nyquist D contour under the function $f(s)$. Let the number of open loop unstable poles of $G(s)C(s)$ be P. Then the closed loop system is stable if and only if*

$$\mathcal{N}(\det(I + GC)) = -P \tag{24}$$

Proof: The theorem has been proved several times; see Maciejowski (1989).

Remarks:

1. In this paper we define "unstable poles" or "RHP poles" as poles in the open RHP, excluding the $j\omega$-axis.

2. The Nyquist D contour follows the $j\omega$-axis and encircles the entire RHP, but must avoid locations where $f(s)$ has poles. This means that the Nyquist D contour should make an indentation into the RHP at locations where $f(s)$ has $j\omega$-axis poles (alternatively one may replace term $1/s$ by $1/(s + \epsilon)$ where ϵ is a small positive number). In practice, this is not a problem in this paper, since the function we consider, $f(s) = \det(I + E(s)\hat{H}(s))$, does not generally have $j\omega$-axis poles. For the encirclements of the product of two functions we have

$$\mathcal{N}(f_1 f_2) = \mathcal{N}(f_1) + \mathcal{N}(f_2) \tag{25}$$

From (9) and the fact $\det(AB) = \det A \cdot \det B$ we then get

$$\mathcal{N}(\det(I + GC)) = \mathcal{N}(\det(I + E\hat{H})) + \mathcal{N}(\det(I + \hat{G}C)) \tag{26}$$

Lemma 1 now follows from theorem 3 and (26).

Appendix 2

The model for the CSTR in the example is given by

$$\frac{dc_A}{dt} = (c_{AF} - c_A)\frac{F}{V} - r \tag{27}$$

$$\frac{dc_B}{dt} = -c_B\frac{F}{V} + r \tag{28}$$

$$\frac{dT}{dt} = (T_F - T)\frac{F}{V} - \frac{\Delta Hr}{\rho C_p} - \frac{hA}{\rho V C_p}(T - T_C) \tag{29}$$

where the rate of reaction, r, is given by

$$r = kc_A e^{-E/RT} \tag{30}$$

$c_A = 10$ kmol/m^3, $c_{AF} = 15.61$ kmol/m^3, $c_B = 5.62$ kmol/m^3, $C_p = 1.8$ kJ/(kg K), $\Delta H = -20\,000$ kJ/kmol, $E = 80\,000$ kJ/kmol, $F = 5$ m^3/min, $R = 8.314$ kJ/(kg K), $T = 590$ K, $T_C = 401.6$ K, $T_F = 580$ K, $hA = 2600$ kJ/(min K), $k = 680\,000$ min^{-1}, $V = 50$ m^3, and $\rho = 800$ kg/m^3.

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