74

Pairing Criteria for Unstable Plants

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Abstract

For decentralized control the pairing criteria involving the Niederlinski Index and the steady state Relative Gain Array are extended to open loop unstable plants. It is found that for unstable plants it will in some cases be preferable to choose pairings giving negative Niederlinski Index and/or RGA.

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1 Introduction

Decentralized control remains popular in the industry, despite developments of advanced controller synthesis procedures leading to full multivariable controllers. Some reasons for the continued popularity are:

1. Decentralized controllers are easy to implement.
2. They are easy for operators to understand.
3. The operators can easily retune the controllers to take account of changing process conditions (as a result of 2 above).
4. Some measurements or manipulated variables may fail. Tolerance of such failures are more easily incorporated into the design of decentralized controllers than full controllers.
5. The control system can be brought gradually into service during process startup and taken gradually out of service during shutdown.

The design of a decentralized control system consists of two main steps:

1) Control structure selection, that is, pairing inputs and outputs.

2) Design of a single-input single-output (SISO) controller for each loop.

In this paper we consider the pairing problem (Step 1). A significant amount of work has been done on the choice of pairings for decentralized control of stable plants, e.g. Bristol (1966), Niederlinski (1971), McAvoy (1983), Grosdidier and Morari (1986), Skogestad and Morari (1988), to reference a few. Most of the proposed pairing conditions use steady-state information, including the Relative Gain Array (RGA) (Bristol, 1966) and the Niederlinski Index (N_I) (Niederlinski, 1971), and the rule is to avoid pairings corresponding to negative steady-state values. However, these pairing criteria were derived for open loop stable plants, and their application or relevance for open loop unstable plants have not been clarified previously. The objective of this paper is to generalize these "steady-state" pairing criteria to unstable plants.

Remark: We will in this paper avoid using the term "steady state" since it does not exist for an unstable system in open loop. Instead we refer to conditions at zero frequency ($\omega = 0$ or $s = 0$), and use, for example, the term "RGA(0)" instead of "steady-state RGA".

2 Tools for pairing selection

In the paper $G(s)$ denotes a square $n \times n$ plant and $C(s)$ the decentralized controller. Without loss of generality we assume that $C(s)$ is diagonal (after possibly rearranging the order of columns and rows in $G(s)$). We also assume that $GC(s)$ contains integrators in all channels, and that $C(s)$ otherwise is stable. G may be unstable and we let P

denote the number of unstable poles in the open right half plane (RHP), not counting poles at $s = 0$ (integrators). Negative feedback is assumed throughout the paper, amd we assume that $G(s)C(s)$ has no internal pole-zero cancellations in the right half plane. We then have that closed-loop stability is guaranteed if and only if $(I + G(s)C(s))^{-1}$ has no RHP-poles. The Laplace variable s will be dropped where it is not needed for clarity.

2.1 Niederlinski Index

The Niederlinski Index, denoted N_I, is defined as

$$N_I(s) = \det(G)/\prod_{i=1}^{n} g_{ii} = \det G/\det \tilde{G} \tag{1}$$

where

$$\tilde{G} = \mathrm{diag}\{g_{ii}\} = \begin{pmatrix} g_{11} & & & \\ & g_{22} & & \\ & & \ddots & \\ & & & g_{nn} \end{pmatrix} \tag{2}$$

For later use let P denote the number of unstable poles in G, and $\tilde{P}$ denote the number of unstable poles in $\tilde{G}$. Note that $\tilde{P}$ is equal to the sum of unstable poles in the diagonal elements of $G(s)$. Thus in most cases $\tilde{P} > P$.

2.2 Relative Gain Array

The RGA-elements, λ_{ij}, are defined as follows

$$\lambda_{ij}(s) = g_{ij}[G^{-1}]_{ji} \tag{3}$$

where $[G^{-1}]_{ji}$ represents the ji'th element of G^{-1}. We have $[G^{-1}]_{ji} = c_{ij}/\det(G)$ where $c_{ij} = (-1)^{i+j}\det(G^{ij})$ denotes the ij'th cofactor of G, and G^{ij} is obtained from G by removing row i and column j.

We will here consider the diagonal RGA-elements for which we derive

$$\lambda_{ii}(s) = g_{ii}[G^{-1}]_{ii} = \frac{g_{ii}\det G^{ii}}{\det G} = \det G'_{ii}/\det G \tag{4}$$

where

$$G'_{ii} = \begin{pmatrix} g_{ii} & 0 \\ 0 & G^{ii} \end{pmatrix} \tag{5}$$

For later use let P'_{ii} denote the number of unstable poles in G'_{ii}. For plants with many inputs and outputs we usually have $\tilde{P} > P'_{ii} > P$. For example, consider a $n \times n$ plant G with a single unstable pole which appears in all the elements of G. In this case $\tilde{P} = n$, $P'_{ii} = 2, \forall i$ and $P = 1$. However, note that P'_{ii} in general may be different for different loops i.

For the special case of 2×2 plants we get $\tilde{G} = G'_{ii}$, so $P_{ii} = \tilde{P}$, and we also see from (1) and (4) that $N_I = 1/\lambda_{ii}$. Consequently, for 2×2 plants the RGA and N_I contain the same information. We shall at the end of the paper discuss how they differ.

Example 1.

Consider the plant $G(s) = C(sI - A)^{-1}B$, with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}; \quad B = \begin{pmatrix} 5 & -8 \\ 4 & 10 \\ 2 & -8 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \tag{6}$$

Clearly, there is only one unstable pole so $P = 1$. For $\tilde{G}$ we get $\tilde{P} = 2$ because the RHP pole appears in both diagonal elements of G. This is seen from the transfer matrix

$$G(s) = \begin{bmatrix} \frac{9s+1}{(-s+1)(s+1)} & \frac{-2s-18}{(-s+1)(s+1)} \\ \frac{-1.5s-6}{(-s+1)(0.5s+1)} & \frac{12}{(-s+1)(0.5s+1)} \end{bmatrix} \tag{7}$$

At $s = 0$ we get

$$G(0) = \begin{pmatrix} 1 & -18 \\ -6 & 12 \end{pmatrix} \tag{8}$$

The pairing $(y_1 - u_1, y_2 - u_2)$ indicated by Eq. (8) we term "pairing 1", and the opposite pairing we term "pairing 2". We get for pairing 1 the RGA-matrix (denoted Λ) and the Niederlinski Index

$$N_I(0) = -8, \quad \Lambda(0) = \begin{pmatrix} -0.13 & 1.13 \\ 1.13 & -0.13 \end{pmatrix}$$

Thus, if the plant were stable we would from the conventional pairing rules based on the Niederlinski Index and the RGA prefer pairing 2, but as we will show, the opposite conclusion is correct for this unstable plant since $\tilde{P} - P = 1$ is odd.

3 Nyquist stability conditions

We showed above that $N_I(s) = \det G / \det \tilde{G}$ and $\lambda_{ii}(s) = \det G'_{ii} / \det G$, that is, both the N_I and the RGA-elements are given by the ratio of two determinants, one involving the plant model (G), and one involving some "alternative" model ($\tilde{G}$ or G'_{ii}). The following derivation makes it possible to derive the results for these two measures in a unified manner.

Let $\hat{G}$ denote some alternative model of the plant G. Throughout this paper we assume that both that both of the square matrices $G(0)$ and $\hat{G}(0)$ are non-singular. Introduce the complementary sensitivity function in terms of $\hat{G}$

$$\hat{H} = \hat{G}C(I + \hat{G}C)^{-1} \tag{9}$$

and the relative "error"

$$E = (G - \hat{G})\hat{G}^{-1} \tag{10}$$

Then the following factorization applies (e.g. Grosdidier and Morari, 1986)

$$(I + GC) = (I + E\hat{H})(I + \hat{G}C) \tag{11}$$

Proof: Trivial by noting that $\hat{H} = \hat{G}C(I+\hat{G}C)^{-1} = (I+\hat{G}C)^{-1}\hat{G}C$.

The following Lemma then follows directly from the multivariable Nyquist Theorem. For completeness a proof is given in Appendix, but similar results, at least for stable plants, have been presented before (e.g., Grosdidier and Morari, 1986, Nwokah and Perez, 1991).

Lemma 1 *Let the number of open loop unstable poles of $G(s)C(s)$ and $\hat{G}(s)C(s)$ be P and $\hat{P}$, respectively. Assume that the closed loop system consisting of $\hat{G}(s)C(s)$ is stable. Then the closed loop system consisting of $G(s)C(s)$ is stable if and only if*

$$\mathcal{N}(\det(I+E\hat{H})) = \hat{P} - P \tag{12}$$

where $\mathcal{N}$ denotes the number clockwise encirclements of the origin as s traverses the Nyquist D-contour.

In other words, (12) says that for stability $\det(I+E\hat{H})$ must provide for the difference in the number of required encirclements between $\det(I+GC)$ and $\det(I+\hat{G}C)$. If this is not the case then at least one of the systems consisting of GC or $\hat{G}C$ must be unstable.

We will now see that information about what happens at $s=0$ may provide useful information about the number of encirclements.

Lemma 2 *Let the number of open loop unstable poles (excluding poles at $s=0$) of $G(s)C(s)$ and $\hat{G}(s)C(s)$ be P and $\hat{P}$, respectively. Assume that the controller C is such that $\hat{G}C$ has integral action in all channels, and that the transfer functions GC and $\hat{G}C$ are strictly proper. Then if*

$$\det G(0)/\det\hat{G}(0)\begin{cases} <0 & \textit{for } \hat{P}-P \textit{ even} \\ >0 & \textit{for } \hat{P}-P \textit{ odd}\end{cases} \tag{13}$$

at least one of the following instabilities will occur: a) The closed-loop system consisting of GC is unstable. b) The closed-loop system consisting of $\hat{G}C$ is unstable.

Proof: For stability of both $(I+GC)^{-1}$ and $(I+\hat{G}C)^{-1}$ we know from Lemma 1 that $\det(I+E(s)\hat{H}(s))$ needs to encircle the origin $\hat{P}-P$ times as s traverses the Nyquist D-contour. $\hat{H}(0) = I$ because of the requirement for integral action in all channels of $\hat{G}C$. Also, since GC and $\hat{G}C$ are strictly proper, $E\hat{H}$ is strictly proper, and hence $E(s)\hat{H}(s)\to 0$ as $s\to\infty$. We therefore have

$$\lim_{s\to\infty} \det(I+E(s)\hat{H}(s)) = 1 \tag{14}$$

$$\lim_{s\to 0} \det(I+E(s)\hat{H}(s)) = \lim_{s\to 0}\det(G(s)\hat{G}^{-1}(s)) \tag{15}$$

Thus, the map of $\det(I+E\hat{H})$ starts at $\det G(0)/\det\hat{G}(0)$ (for $s=0$) and ends at 1 (for $s=\infty$). A more careful analysis of the Nyquist plot reveals that the number of encirclements will be odd for $\det G(0)/\det\hat{G}(0) > 0$, and even for $\det G(0)/\det\hat{G}(0) < 0$. Thus, if this parity (odd or even) does not match that of $\hat{P}-P$ we will get instability, and the theorem follows. QED.

The results in the next section follow directly from Lemma 1 since $\det G(0)/\det\tilde{G}(0) = N_I(0)$ and $\det G(0)/\det G'_{ii}(0) = 1/\lambda_{ii}(0)$.

4 Pairing Criteria for Unstable Plants

Theorem 1 Niederlinski Index. *Let the number of open loop unstable poles (excluding poles at* $s = 0$*) of* $G(s)$ *and* $\tilde{G}(s) = diag\{g_{11}(s), g_{22}(s), \ldots, g_{nn}(s)\}$ *be* P *and* $\tilde{P}$*, respectively. Assume that the controller* C *is such that* $\tilde{G}C$ *has integral action in all channels and is otherwise stable, and assume that* GC *is strictly proper. Then if* $N_I(0)$ *has the "wrong" sign, that is, if*

$$N_I(0) \begin{cases} < & 0 \quad \text{for } \tilde{P} - P \text{ even} \\ > & 0 \quad \text{for } \tilde{P} - P \text{ odd} \end{cases} \tag{16}$$

at least one of the following instabilities will occur:
a) The overall system is unstable, i.e., $(I + GC)^{-1}$ *is unstable.*
b) At least one of the loops is unstable by itself, i.e., $(I + \tilde{G}C)^{-1}$ *is unstable.*

Proof: Let $\hat{G} = \tilde{G}$ in Lemma 2 and use the definition of N_I in (1).
Remarks:

1. It is generally undesirable for the loops to be unstable by themselves, and one should at least not close such a loop loop first when the system is brought into service.

2. Theorem 2 is a direct generalization of the Niederlinski pairing criterion (see Theorem 3 in Grosdidier et al., 1985) to unstable plants.

3. The matrix $G\tilde{G}^{-1}$ which appears in the Niederlinski Index is the inverse of the PRGA matrix introduced by Hovd and Skogestad (1992). The diagonal elements of the PRGA are identical to the diagonal elements of the RGA.

Theorem 2 Relative Gain Array. *Let the number of open loop unstable poles (excluding poles at* $s = 0$*) of* $G(s)$ *and* $G'_{ii}(s) = diag\{g_{ii}(s), G^{ii}(s)\}$ *be* P *and* P'_{ii}*, respectively. Assume that the controller* C *is such that* $G'_{ii}C$ *has integral action in all channels and is otherwise stable, and assume that* GC *is strictly proper. Then if* $\lambda_{ii}(0)$ *has the "wrong" sign, that is, if*

$$\lambda_{ii}(0) \begin{cases} < & 0 \quad for\ P'_{ii} - P\ even \\ > & 0 \quad for\ P'_{ii} - P\ odd \end{cases} \tag{17}$$

at least one of the following instabilities will occur
a) The overall system is unstable, i.e,, $(I + GC)^{-1}$ *is unstable.*
b) Loop i *is unstable by itself, i.e.,* $(1 + g_{ii}c_i)^{-1}$ *is unstable.*
c) The system is unstable as loop i *is removed, i.e,* $(I + G^{ii}C^{ii})^{-1}$ *is unstable.*

Proof: Let $\hat{G} = G'_{ii}$ in Lemma 2 and use the definition of RGA in (4).

Remarks:

1. All of these instabilities are undesirable. Of course, the worst is that the overall system is unstable. The last possibility for instability is also most undesirable. It will imply instability of the system should loop i become inactive, for example, due to input saturation (in which case u_i is constant).

2. Theorem 2 is a direct generalization of the widely used RGA pairing criterion (see Theorem 6 in Grosdidier et al., 1985) to unstable plants.

3. P'_{ii} is equal to the sum of unstable poles in g_{ii} and G^{ii}.

4. For 2×2 plants $N_I(0) = 1/\lambda_{ii}(0)$, but for larger systems the measures contain different information. Specifically, consider a plant of size 3×3 or larger where the overall closed loop system (consisting of GC) is stable. Then if $\lambda_{ii}(0)$ has the wrong sign, we know that loop i by itself is unstable or that the system with loop i removed is unstable, both of which are undesirable. Thus, pairing such that $\lambda_{ii}(0)$ has the wrong sign does not mean that any of the individual loops are unstable. On the other hand, if $N_I(0)$ has the wrong sign we know that (at least) one of the individual loops are unstable.

5. Since $\lambda_{ii}(0)$ and $N_I(0)$ contain different information for plants larger than 2×2, we may have cases where all $\lambda_{ii}(0)$'s $(i = 1, \ldots, n)$ have the right sign, whereas $N_I(0)$ has the wrong sign, and *vice versa*.

Summary of Pairing Rules

- For *stable* plants one should select pairings corresponding to *positive* values of the Niederlinski Index and RGA(0).

- For the special case of a $n \times n$ plant with *one* unstable pole which appears in *all* the elements of $G(s)$ we have $\tilde{P} = P$ and $P'_{ii} = 2$. In this case Theorems 1 and 2 yield:

  1. Select a set of pairings such that $N_I(0)$ is positive if n is odd and negative if n is even.
  2. Pair on *negative* RGA elements (i.e, $\lambda_{ii}(0) < 0$).

- For the special case of a $n \times n$ plant with P unstable poles which appear in *all* the elements of $G(s)$ we have $\tilde{P} = nP$ and $P'_{ii} = 2P$. In this case Theorems 1 and 2 yield:

  1. Select a set of pairings such that $N_I(0)$ is positive if $(n-1)P$ is even and negative if $(n-1)P$ is odd.
  2. Pair on positive RGA elements (i.e, $\lambda_{ii}(0) > 0$) if P is even and on negative RGA-elements if P is odd.

4.1 Example, continued

For the 2×2 example with one unstable pole introduced before we get both from Theorem 1 and Theorem 2 that some kind of undesirable instability will occur for pairing 2. On this basis pairing 1, which corresponds to pairing on negative values of $N_I(0)$ and RGA(0), is preferable.

Nyquist plots. For pairing 1 we use the controllers

$$c_1(s) = -\frac{s+1}{s}, \quad c_2(s) = -\frac{(s+1)(0.1s+1)}{s(0.01s+1)}$$

Fig. 1 shows Nyquist plots of $1 + g_{11}c_1$, $1 + g_{22}c_2$ and $I + E_H\tilde{H}$ for this choice. We see that $1 + g_{11}c_1$ and $1 + g_{22}c_2$ each make one counterclockwise encirclement of the origin and we get from the Nyquist stability criterion that the individual loops are stable. Furthermore, $\det(I + E_H\tilde{H})$ makes one clockwise encirclement of the origin, which is equal to $\tilde{P} - P$, and the system is closed-loop stable according to Lemma 1. This confirms the predictions based on the Niederlinski Index and RGA(0).

Comment. It may be of interest to check the result using the Nyquist stability condition. From (11) we have

$$\det(I + GC) = \det(I + E\hat{H}) \cdot \prod_{i=1}^{n}(1 + g_{ii}c_i)$$

and the number of clockwise encirclements of the origin for $\det(I + GC)$ is therefore $\mathcal{N}(\det(I + GC)) = +1 - 1 - 1 = -1 = -P$, and the system is stable according to the generalized Nyquist stability theorem.

5 RGA as a special case of the Niederlinski Index

As discussed above, the Niederlinski Index, $N_I(0)$, and the RGA-elements, $\lambda_{ii}(0)$, contain different information for $n \times n$ plants with $n > 3$. Nevertheless, it is clear from (1) and (4) that the two quantities are closely related, and we will show that if we consider the Niederlinski Index also of the subsystems then the RGA is strictly not needed.

Define the Niederlinski Index for subsystem ii (that is, with loop i removed) as

$$N_I^{ii}(s) = \det(G^{ii}) / \prod_{j \neq i} g_{jj} \tag{18}$$

Then we get from (1) and (4)

$$\lambda_{ii} = N_I^{ii} / N_I \tag{19}$$

(this expression applies to the diagonal RGA-elements, and not to the off-diagonal ones). It is then clear that if we evaluate the Niederlinski Index also of the subsystems then we will have all the information given by the diagonal RGA-elements (and more).

Still the RGA-matrix is very useful because it has nice algebraic properties, since it can be evaluated once for all possible choices of pairings (if one computes the entire RGA-matrix and not only the diagonal elements), and because one can often with one "glance" decide on an appropriate set of pairings.

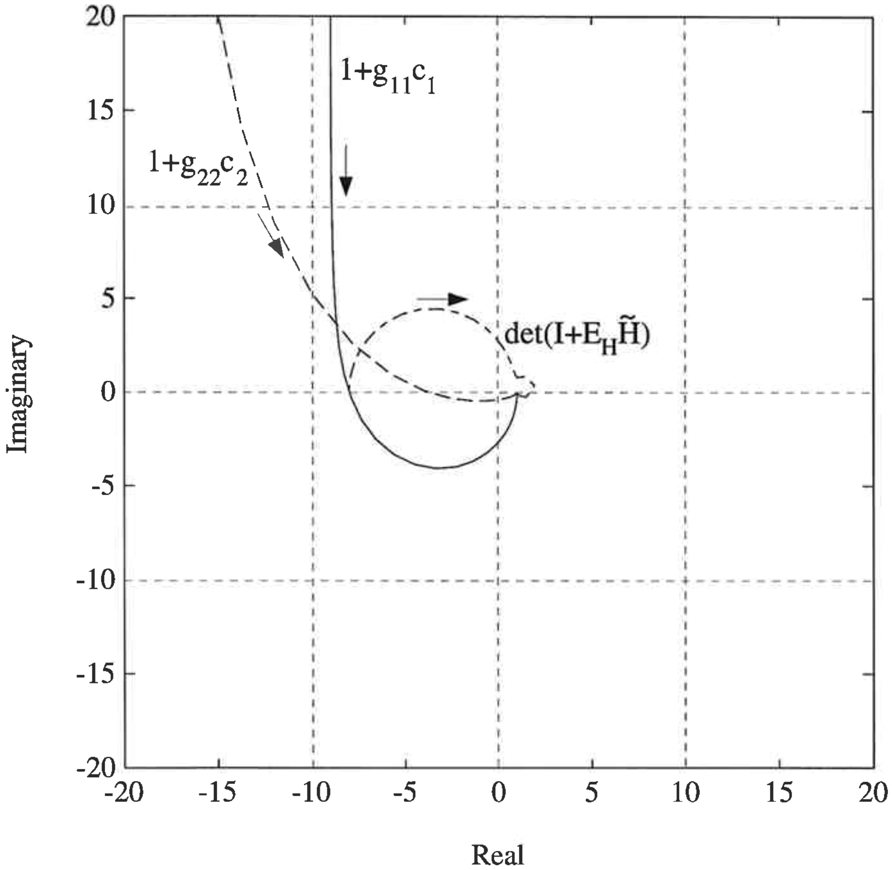

Figure 1: Nyquist plots for $1 + g_{11}c_1$, $1 + g_{22}c_2$ and $I + E_H\tilde{H}$. The arrows show the direction of increasing frequency. Only positive frequencies are shown.

For large systems it is very useful to evaluate the Niederlinski Index of the susbsystems in the order one plans to close the loops when the system is brought into service. One should not close the loops in an order where the Niederlinski index of any of these "increasing" submatrices has the wrong sign.

6 Conclusions

- The pairing criteria based on the Niederlinski Index and RGA for systems under decentralized integral control have been generalized to hold for open loop unstable plants (Theorems 1 and 2).
- By evaluating the Niederlinski Index of the subsystems the RGA is strictly not needed (Eq.19), but in practical use the RGA-matrix is still very useful.

Nomenclature

C - Controller transfer function matrix.
c_i - ii'th element of controller C (for diagonal C).
E - $(G - \hat{G})\hat{G}^{-1}$.

G - Plant transfer function matrix.
$\hat{G}$ - Simplified or alternative model for G.
G^{ii} - G with row i and column i deleted.
$\tilde{G}$ - Matrix consisting of the diagonal elements of G, $\mathrm{diag}\{g_{ii}\}, i = 1, n$.
G'_{ii} - $\mathrm{diag}\{g_{ii}, G^{ii}\}$.
g_{ij} - ij'th element of G.
$\hat{H}$ - complementary sensitivity matrix corresponding to $\hat{G}$, $\hat{G}C(I+\hat{G}C)^{-1}$.
$\tilde{H}$ - matrix of complementary sensitivity functions for individual loops, $\tilde{G}C(I+\tilde{G}C)^{-1}$.
N_I - Niederlinski Index, $\det G(0)/\prod_i g_{ii}(0)$.
P - number of open loop unstable poles of GC.
$\hat{P}$ - number of open loop unstable poles of $\hat{G}C$.
$\tilde{P}$ - number of open loop unstable poles of $\tilde{G}C$.
P'_{ii} - number of open loop unstable poles of $G'_{ii}C$.
RGA - Relative Gain Array matrix, $G \times [G^{-1}]^T$, $\times$ denotes element-by-element multiplication.
s - Laplace variable.

Greek symbols:
λ_{ij} - ij'th element of RGA.

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7 Appendix

Proof of Lemma 1

Let us first state the multivarieble Nyquist theorem.

Theorem 3 The multivariable Nyquist theorem. *Let $\mathcal{N}(f(s))$ denote the number of clockwise encirclements of the map of the Nyquist D contour under the function $f(s)$. Let the number of open loop unstable poles of $G(s)C(s)$ be P. Then the closed loop system is stable if and only if*

$$\mathcal{N}(\det(I + GC)) = -P \tag{20}$$

Proof: The theorem has been proved several times, see Maciejowski (1989).
Remarks:

1. In this paper we define "unstable poles" or "RHP-poles" as poles in the open RHP, excluding the $j\omega$-axis.

2. The Nyquist D-contour follows the $j\omega$-axis and encircles the entire RHP, but must avoid locations where $f(s)$ has poles. This means that the Nyquist D contour should make an indention into the RHP at locations where $f(s)$ has $j\omega$-axis poles. In practice, this is not a problem in this paper, since the function we consider, $f(s) = \det(I + E(s)\hat{H}(s))$, does not generally have $j\omega$-axis poles.

For the encirclements of the product of two functions we have

$$\mathcal{N}(f_1 f_2) = \mathcal{N}(f_1) + \mathcal{N}(f_2) \tag{21}$$

From (11) and the fact $\det(AB) = \det A \cdot \det B$ we then get

$$\mathcal{N}(\det(I + GC)) = \mathcal{N}(\det(I + E\hat{H})) + \mathcal{N}(\det(I + \hat{G}C)) \tag{22}$$

Lemma 1 now follows from Theorem 3 and Eq.(22).

ARKIV.

PAIRING CRITERIA FOR UNSTABLE PLANTS

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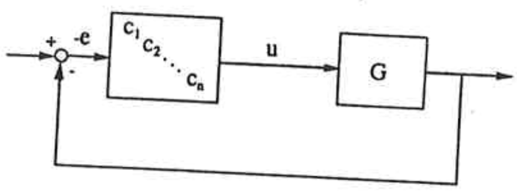


TOOLS FOR PAIRING SELECTION

1. **Niederlinski Index** (1971)

$$N_I(s) = \det G / \prod_{i=1}^{n} g_{ii} = \frac{\det G}{\det \widetilde{G}} \qquad (1)$$

where $\widetilde{G} = \text{diag}\{g_{ii}\}$ - diagonal elements of G

2. **Relative Gain Array** (Bristol, 1966).

$$\lambda_{ii}(s) = g_{ii}[G^{-1}]_{ii} = \frac{g_{ii}\det G^{ii}}{\det G} = \frac{\det G'_{ii}}{\det G} \qquad (2)$$

where

$$G'_{ii} = \begin{pmatrix} g_{ii} & \\ & G^{ii} \end{pmatrix}$$

G^{ii}: remove column i and row i

Special case 2 × 2 systems: $\widetilde{G} = G'_{ii}$ and

$$N_I = 1/\lambda_{ii}$$

$n \times n$ system with $n > 3$: Not the same.

EXAMPLE

2 × 2 plant with one unstable pole

$$G(s) = \begin{bmatrix} \frac{9s+1}{(-s+1)(s+1)} & \frac{-2s-18}{(-s+1)(s+1)} \\ \frac{-1.5s-6}{(-s+1)(0.5s+1)} & \frac{12}{(-s+1)(0.5s+1)} \end{bmatrix}$$

Evaluate at $s = 0$

$$G(0) = \begin{pmatrix} 1 & -18 \\ -6 & 12 \end{pmatrix}$$

For **stable** plants: Pair on positive steady-state NI and RGA

$$N_I = -8; \quad \text{RGA} = \begin{pmatrix} -0.13 & 1.13 \\ 1.13 & -0.13 \end{pmatrix}$$

LL SHOW: MUST PAIR ON NEGATIVE RG

- G has $P = 1$ unstable poles.
- $\widetilde{G}$ has $\widetilde{P} = 2$ unstable poles.

MULTIVARIABLE NYQUIST THEOREM

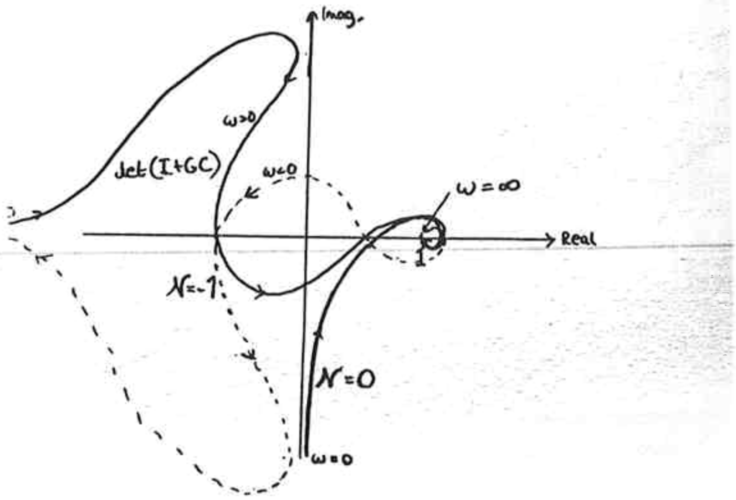
Stability of closed-loop system: Consider $(I + GC)^{-1}$.

P: no. of unstable poles in GC

$\mathcal{N}$: No. of Nyquist plot encirclements of origin.

Theorem. The closed loop system is stable if and only if

$$\mathcal{N}(\det(I + GC)) = -P$$



ENCIRCLEMENTS OF $\det(I + E\hat{H})$

$G(s)$ - model with P unstable poles

$\hat{G}$ - "alternative" model ($\widetilde{G}$ or G'_{ii}) with $\hat{P}$ unstable poles.

$\hat{H} = \hat{G}C(I + \hat{G}C)^{-1}$ - complementary sensitivity

$E = (G - \hat{G})\hat{G}^{-1}$ - relative "error"

Factorization of return difference

$$\underbrace{(I + GC)}_{-P \text{ encirclements}} = (I + E\hat{H})\ \underbrace{(I + \hat{G}C)}_{-\hat{P} \text{ encirclements}}$$

$\det(I + E\hat{H})$: must encircle $\hat{P} - P$ times

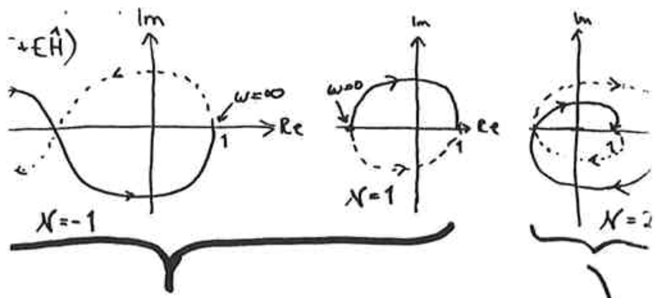
Otherwise: $(I + GC)^{-1}$ or $(I + \hat{G}C)^{-1}$ unstable.

5

CONDITIONS AT $s = 0$

- **$\det(I + E\hat{H})$: encircle $\hat{P} - P$ times.**
- Integral action: $\hat{H}(0) = I$.
- Strictly proper: $\hat{H}(\infty) = 0$.
- $E = G\hat{G}^{-1} - I$
- Get:

$$\det(I + E\hat{H}) \rightarrow \begin{cases} 1 & s \rightarrow \infty \\ \det G(0)/\det \hat{G}(0) & s \rightarrow 0 \end{cases}$$



- $\det G(0)/\det \hat{G}(0) < 0$: odd no. of encirclements
- $\det G(0)/\det \hat{G}(0) > 0$: even no. of encirclements
- If parity (odd/even) does not match $\hat{P} - P$: **Unstable**

6

NIEDERLINSKI INDEX (set $\hat{G} = \widetilde{G}$)

Assume:

1. G has P unstable poles
2. $\widetilde{G}(s) = \{g_{11}(s), g_{22}(s), \ldots, g_{nn}(s)\}$ has $\widetilde{P}$ unstable poles
3. Integral action and GC strictly proper

Theorem. If $N_I(0)$ has the "wrong" sign, that is, if

$$N_I(0) \begin{cases} < 0 & \text{for } \widetilde{P} - P \text{ even} \\ > 0 & \text{for } \widetilde{P} - P \text{ odd} \end{cases}$$

then at least one of the following instabilities will occur:

a) The overall system is unstable, i.e., $(I + GC)^{-1}$ is unstable.

b) At least one of the loops is unstable by itself, i.e., $(I + \widetilde{G}C)^{-1}$ is unstable.

7

RELATIVE GAIN ARRAY (set $\hat{G} = G'_{ii}$)

Assume

1. G has P unstable poles
2. $G'_{ii}(s) = \{g_{ii}(s), G^{ii}(s)\}$ has P'_{ii} unstable poles
3. Integral action and GC strictly proper

Theorem. If $\lambda_{ii}(0)$ has the "wrong" sign, that is, if

$$\lambda_{ii}(0) \begin{cases} < 0 & \text{for } P'_{ii} - P \text{ even} \\ > 0 & \text{for } P'_{ii} - P \text{ odd} \end{cases} \quad (3)$$

then at least one of the following instabilities will occur

a) The overall system is unstable, i.e,, $(I + GC)^{-1}$ is unstable.

b) Loop i is unstable by itself, i.e., $(1 + g_{ii}c_i)^{-1}$ is unstable.

c) The system is unstable as loop i is removed, i.e, $(I + G^{ii}C^{ii})^{-1}$ is unstable.

8

SUMMARY OF PAIRING RULES

- Stable plants: Pair on **positive** $N_I(0)$ and RGA(0).
- Special case: $n \times n$ plant with **one** unstable pole which appears in **all** elements of $G(s)$:
  1. Want $N_I(0)$ positive if n is odd (negative if even).
  2. Want **negative** RGA elements.
- Special case: $n \times n$ plant with P unstable poles which appear in **all** elements of $G(s)$:
  1. Want $N_I(0)$ positive if $(n-1)P$ is even (negative if odd).
  2. Want positive RGA elements if P is even (negative if odd).

Need both RGA and NI?

Define Niederlinski Index for subsystem with loop i removed

$$N_I^{ii}(s) = \det(G^{ii}) / \prod_{j \neq i} g_{jj} \tag{4}$$

Definitions of RGA and NI yield

$$\lambda_{ii} = N_I^{ii} / N_I$$

That is, do not need RGA.

- But RGA
  1. Has nice algebraic properties
  2. Only one evaluation needed
  3. One "glance" yields pairings

CONCLUSIONS

- **Pairing criteria based on the NI and RGA have been generalized to open loop unstable plants.**
- **By evaluating the NI of subsystems the RGA is strictly not needed, but in practice the RGA-matrix is very useful.**