S. Skogestad. ``Robust Control", /Practical Distillation Control/, W.L. Luyben (Editor), Van Nostrand Reinhold, 1992, 291-309.

14

Robust Control

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The objective of this chapter is to give the reader a basic knowledge of how robustness problems arise and what tools are available to identify and avoid them.

We first discuss possible sources of model uncertainty and look at the traditional methods for obtaining robust designs, such as gain margin, phase margin, and maximum peak criterion (M circles). However, these measures are difficult to generalize to multivariable systems.

As an introductory example to robustness problems in multivariable systems we then discuss two-point control of distillation columns using the LV configuration. Because of strong interactions in the plant, a decoupler is extremely sensitive to input gain uncertainty (caused by actuator uncertainty). These interactions are analyzed using singular value decomposition (SVD) and RGA analysis. We show that plants with large RGA elements are fundamentally difficult to control and that decouplers should not be used for such plants. It is shown that other configurations may be less sensitive to model uncertainty.

At the end of the chapter we look at uncertainty modelling in terms of norm bounded perturbations (Δs). It is shown that the structured singular value μ is a very powerful tool to analyze the robust stability and performance of multivariable control systems.

14-1 ROBUSTNESS AND UNCERTAINTY

A control system is robust if it is insensitive to differences between the actual system and the model of the system that was used to design the controller. Robustness problems are usually attributed to differences between the plant model and the actual plant (usually called model–plant mismatch or simply model uncertainty). Uncertainty in the plant model may have several origins:

1. There are always parameters in the linear model that are only known approximately or are simply in error.
2. Measurement devices have imperfections. This may give rise to uncertainty of the manipulated *inputs* in a distillation column because they are usually measured and adjusted in a cascade manner. In other cases, limited valve resolution may cause input uncertainty.
3. At high frequencies even the structure and the model order is unknown, and the

uncertainty will exceed 100% at some frequency.

4. The parameters in the linear model may vary due to nonlinearities or changes in the operating conditions.

Other considerations for robustness include measurement and actuator failures, constraints, changes in control objectives, opening or closing other loops, and so on. Furthermore, if a control design is based on an optimization, then robustness problems may also be caused by the mathematical objective function, that is, how well this function describes the real control problem.

In the somewhat narrow use of the term used in this chapter, we shall consider robustness with respect to model uncertainty and shall assume that a fixed (linear) controller is used. Intuitively, to be able to cope with large changes in the process, this controller has to be detuned with respect to the best response we might achieve when the process model is exact.

To consider the effect of model uncertainty, it needs to be quantified. There are several ways of doing this. One powerful method is the frequency domain (so-called *H*-infinity uncertainty description) in terms of norm-bounded perturbations (Δs). With this approach one also can take into account unknown or neglected high-frequency dynamics. This approach is discussed toward the end of this chapter. Readers who want to learn more about these methods than we can cover may consult the books by Maciejowski (1989) and Morari and Zafiriou (1989).

The following terms will be useful:

Nominal stability (NS). The system is stable with no model uncertainty.

Nominal performance (NP). The system satisfies the performance specifications with no model uncertainty.

Robust stability (RS). The system also is stable for the worst-case model uncertainty.

Robust performance (RP). The system also satisfies the performance specifications for the worst-case model uncertainty.

14-2 TRADITIONAL METHODS FOR DEALING WITH MODEL UNCERTAINTY

14-2-1 Single-Input–Single-Output Systems

For single-input–single-output (SISO) systems one has traditionally used gain margin (GM) and phase margin (PM) to avoid problems with model uncertainty. Consider a system with open-loop transfer function $g(s)c(s)$, and let $gc(j\omega)$ denote the frequency response. The GM tells by what factor the loop gain $|gc(j\omega)|$ may be increased before the system becomes unstable. The GM is thus a direct safeguard against steady-state gain uncertainty (error). Typically we require GM > 1.5. The phase margin tells how much negative phase we can add to $gc(s)$ before the system becomes unstable. The PM is a direct safeguard against time delay uncertainty: If the system has a crossover frequency equal to ω_c [defined as $|gc(j\omega_c)| = 1$], then the system becomes unstable if we add a time delay of $\theta = \mathrm{PM}/\omega_c$. For example, if PM = 30° and $\omega_c = 1$ rad/min, then the allowed time delay error is $\theta = (30/57.3)$ rad/1 rad/min = 0.52 min. It is important to note that decreasing the value of ω_c (lower closed-loop bandwidth, slower response) means that we can tolerate larger deadtime errors. For example, if we design the controller such that PM = 30° and expect a deadtime error up to 2 min, then we must design the control system such that $\omega_c < \mathrm{PM}/\theta = (30/57.3)/2 = 0.26$ rad/min, that is, the closed-loop time constant should be larger than 1/0.26 = 3.8 min.

Maximum Peak Criterion

In practice, we do not have pure gain and phase errors. For example, in a distillation column the time constant will usually increase when the steady-state gain increases. A more general way to specify stability margins is to require the Nyquist locus of $gc(j\omega)$ to stay outside some region of the -1 point (the "critical point") in the complex plane. Usually this is done by considering the maximum peak M_t of the closed-loop transfer function T:

$$M_t = \max_{\omega} |T(j\omega)|, \qquad T = gc(1 + gc)^{-1} \tag{14-1}$$

The reader may be familiar with M circles drawn in the Nyquist plot or in the Nichols chart. Typically, we require $M_t = 2$. There is a close relationship between M_t and PM and GM. Specifically, for a given M_t we are guaranteed

$$\text{GM} \geq 1 + \frac{1}{M_t},$$

$$\text{PM} \geq 2 \arcsin\left(\frac{1}{2M_t}\right) \geq \frac{1}{M_t} \text{ rad} \tag{14-2}$$

For example, with $M_t = 2$ we have GM > 1.5 and PM > 29.0° > $1/M_t$ rad = 0.5 rad.

Comment The peak value M_s of the sensitivity function $S = (1 + gc)^{-1}$ may be used as an alternative robustness measure. $1/M_s$ is simply the minimum distance between $gc(j\omega)$ and the -1 point. In most cases the values of M_t and M_s are closely related, but for some "strange" systems it may be safer to specify M_t rather than M_s. For a given value of M_s we are guaranteed GM > $M_s/(M_s - 1)$ and PM > $2\arcsin(1/2M_s) > 1/M_s$.

14-2-2 Multi-input–Multi-output Systems

The traditional method of dealing with robustness for multi-input–multi-output (MIMO) systems (e.g., within the framework of "optimal control," linear quadratic Gaussian (LQG), etc.) has been to introduce uncertain signals (noise and disturbances). One particular approach is the loop transfer recovery (LTR) method where unrealistic noise is added specifically to obtain a robust controller design. One may say that model uncertainty generates some sort of disturbance. However, this disturbance depends on the other signals in the systems, and thus introduces an element of feedback. Therefore, there is a fundamental difference between these sources of uncertainty (at least for linear systems): Model uncertainty may introduce instability, whereas signal uncertainty may not.

For SISO systems the main tool for robustness analysis has been GM and PM, and as previously noted, these measures are related to specific sources of model uncertainty. However, it is difficult to generalize GM and PM to MIMO systems. On the other hand, the maximum peak criterion may be generalized easily. The most common generalization is to replace the absolute value by the maximum singular value, for example, by considering

$$M_t = \max_{\omega} \bar{\sigma}(T(j\omega)),$$

$$T = GC(I + GC)^{-1} \tag{14-3}$$

The largest singular value is a scalar positive number, which at each frequency measures the magnitude of the matrix T. As shown later, this approach has a direct relationship to important model uncertainty descriptions and is used in this chapter.

Comment In Chapter 11, a different generalization of M_t to multivariable systems is used: First introduce the scalar function $W(j\omega) = \det(I + GC(j\omega)) - 1$ [for SISO systems $W(j\omega) = GC(j\omega)$] and then define $L_c = |W/(1 + W)|$. The maximum peak of $|L_c|$ (in decibels) is denoted $L_c^{\max}$ and is used as part of the biggest log-modulus tuning (BLT) method. For SISO systems $L_c^{\max} = M_t$.

Even though we may easily generalize the maximum peak criterion (e.g., $M_t \leq 2$) to multivariable systems, it is often not useful for the following three reasons:

1. In contrast to the SISO case, it may be insufficient to look at only the transfer function T. Specifically, for SISO systems $GC = CG$, but this does not hold for MIMO systems. This means that although the peak of T [in terms of $\bar{\sigma}(T(j\omega))$] is low, the peak of $T_I = CG(I + CG)^{-1}$ may be large. (We will see later that the transfer function T is related to relative uncertainty at the output of the plant, and T_I at the input of the plant.)
2. The singular value may be a poor generalization of the absolute value. There may be cases where the maximum peak criterion, for example, in terms of $\bar{\sigma}(T)$, is *not* satisfied, but in reality the system may be robustly stable. The reason is that the uncertainty generally has "structure," whereas the use of the singular value assumes unstructured uncertainty. As we will show, one should rather use the structured singular value, that is, $\mu(T)$.
3. In contrast to the SISO case, the response with model error may be poor (RP not satisfied), even though the stability margins are good (RS is satisfied) and the response without model error is good (NP satisfied).

In the next section we give a multivariable example where the maximum peak criterion is easily satisfied using a decoupling controller [in fact, we have $GC(s) = CG(s) = 0.7/sI$, and the values of M_t and M_s are both 1]. Yet, the response with only 20% gain error in each input channel is extremely poor. To handle such effects, in general, one has to define the model uncertainty and compute the structured singular value for RP.

The conclusion of this section is that most of the tools developed for SISO systems, and also their direct generalizations such as the peak criterion, are not sufficient for MIMO systems.

14-3 A MULTIVARIABLE SIMULATION EXAMPLE

This idealized distillation column example will introduce the reader to the deteriorating effect of model uncertainty, in particular for multivariable plants. The example is taken mainly from Skogestad, Morari, and Doyle (1988).

14-3-1 Analysis of the Model

We consider two-point (dual) composition control with the LV configuration as shown in Figure 14-1a. The overhead composition of a distillation column is to be controlled at $y_D = 0.99$ and the bottom composition at $x_B = 0.01$, with reflux L and boilup V as manipulated inputs for composition control, that is,

$$y = \begin{pmatrix} \Delta y_D \\ \Delta x_B \end{pmatrix}, \qquad u = \begin{pmatrix} \Delta L \\ \Delta V \end{pmatrix}$$

This choice is often made because L and V have an immediate effect on the product compositions. By linearizing the steady-state model and assuming that the dynamics may be approximated by first-order response with time constant $\tau = 75$ min, we derive the linear model in terms of deviation variables

$$\begin{pmatrix} \Delta y_D \\ \Delta x_B \end{pmatrix} = G^{LV} \begin{pmatrix} \Delta L \\ \Delta V \end{pmatrix},$$

$$G^{LV}(s) = \frac{1}{\tau s + 1} \begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix} \tag{14-4}$$

Here we have normalized the flows such that the feed rate $F = 1$. This is admittedly a poor model of a distillation column. Specifically, (a) the same time constant τ is used both for external and internal flow changes, (b) there should be a high-order lag in the transfer function from L to x_B to represent the liquid flow down to the column, and (c) higher-order composition dynamics should also be included. However,

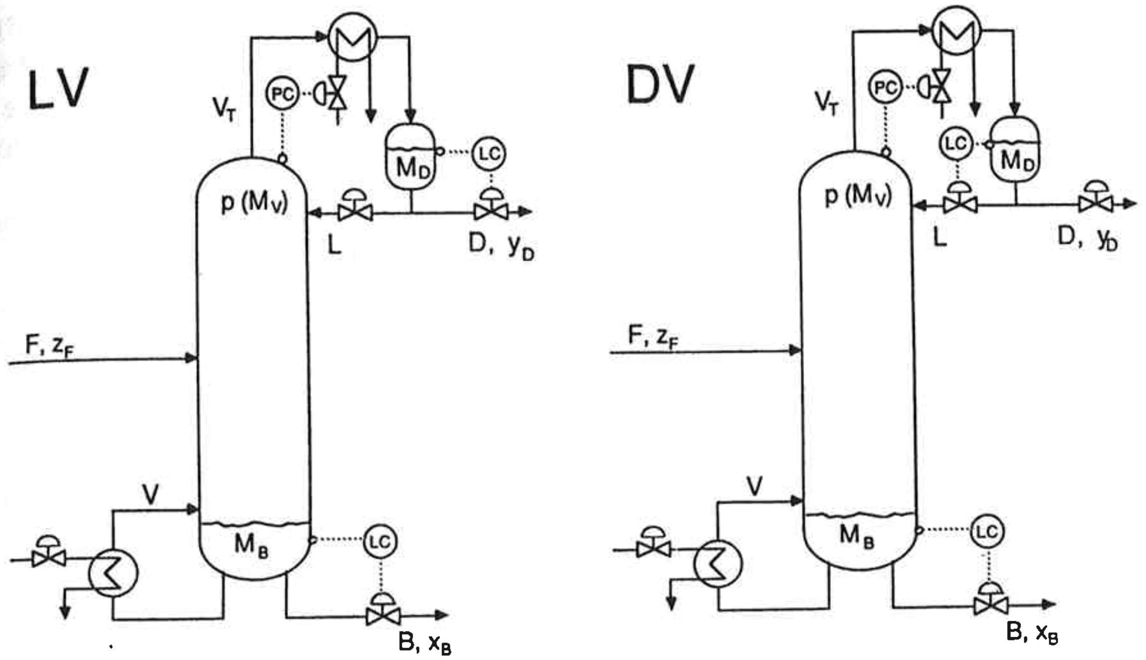

FIGURE 14-1. Control of distillation column with LV and DV configurations.

the model is simple and displays important features of the distillation column behavior. The RGA matrix for this model is at all frequencies:

$$\text{RGA}(G^{LV}) = \begin{pmatrix} 35.1 & -36.1 \\ -36.1 & 35.1 \end{pmatrix} \quad (14\text{-}5)$$

The large elements in this matrix indicate that this process is fundamentally difficult to control.

Interactions and Ill-Conditionedness

Consider the case with no composition control. The effect on top composition of a small change in reflux L with V constant is

$$\Delta y_D(s) = \frac{0.878}{75s + 1}\,\Delta L(s)$$

If we increase L by only 0.01 (that is, L/F is increased 0.4% from 2.7 to 2.701), then we see that the steady-state increase in y_D predicted from this linear model is 0.00878 (that is, y_D increases from 0.99 to 0.99878). This is a rather drastic change, and the reason is that the column operation is very dependent on keeping the correct product split D/F (with V constant the increase in L yields a corresponding decrease in D); that is, the column is very sensitive to changes in the *external flows* D and B.

Similarly, if we increase V by only 0.01 (with L constant), we see that the predicted steady-state change in y_D is -0.00864. Again, this is a very large change, but in the direction opposite that for the increase in L.

We therefore see that changes in L and V counteract each other, and if we increase L and V simultaneously by 0.01, then the overall steady-state change in y_D is only $0.00878 - 0.00864 = 0.00014$. The reason for this small change is that the compositions in the column are only weakly dependent on changes in the *internal flows* (i.e., changes in the internal flows L and V with the external flows D and B constant).

Summary Because both L and V affect both compositions y_D and x_B, we say that the process is *interactive*. Furthermore, the process is "ill-conditioned," that is, some combinations of ΔL and ΔV (corresponding to changing external flows) have a strong effect on the compositions, whereas other combinations of ΔL and ΔV (corresponding to changing internal flows) have a weak effect on the compositions. The condition number, which is the ratio between the gains in the strong and weak directions, is therefore large for this process (as seen in the following text, it is 141.7).

Singular Value Analysis of the Model

The preceding discussion shows that this column is an ill-conditioned plant, where the effect (the gain) of the inputs on the outputs depends strongly on the *direction* of the inputs. To see this better, consider the SVD of the steady-state gain matrix

$$G = U \Sigma V^T \tag{14-6}$$

or equivalently, because $V^T = V^{-1}$,

$$G\bar{v} = \bar{\sigma}(G)\bar{u}, \qquad G\underline{v} = \underline{\sigma}(G)\underline{u}$$

where $\bar{\sigma}$ denotes the maximum singular value and $\underline{\sigma}$ the minimum singular value. The singular values are

$$\Sigma = \text{diag}\{\bar{\sigma}, \underline{\sigma}\} = \text{diag}\{1.972, 0.0139\}$$

The output singular vectors are

$$V = (\bar{v}\ \underline{v}) = \begin{pmatrix} 0.707 & 0.708 \\ -0.708 & 0.707 \end{pmatrix}$$

The input singular vectors are

$$U = (\bar{u}\ \underline{u}) = \begin{pmatrix} 0.625 & 0.781 \\ 0.781 & -0.625 \end{pmatrix}$$

The large plant gain, $\bar{\sigma}(G) = 1.972$, is obtained when the inputs are in the direction $\begin{pmatrix} \Delta L \\ \Delta V \end{pmatrix} = \bar{v} = \begin{pmatrix} 0.707 \\ -0.708 \end{pmatrix}$, i.e., an increase in ΔL and a simutaneous decrease in ΔV. Because $\Delta B = -\Delta D = \Delta L - \Delta V$ (assuming constant molar flows and constant feed rate), this physically corresponds to the largest possible change in the *external* flows D and B. From the direction of the output vector $\bar{u} = \begin{pmatrix} 0.625 \\ 0.781 \end{pmatrix}$, we see that this change causes the outputs to move in the same direction, that is, it mainly affects the average composition $(y_D + x_B)/2$. All columns with both products of high purity are sensitive to changes in the external flows because the product rate D has to be about equal to the amount of light component in the feed. Any imbalance leads to large changes in product compositions (Shinskey, 1984).

The low plant gain, $\underline{\sigma}(G) = 0.0139$, is obtained for inputs in the direction $\begin{pmatrix} \Delta L \\ \Delta V \end{pmatrix} = \underline{v} = \begin{pmatrix} 0.708 \\ 0.707 \end{pmatrix}$ which corresponds to changing the *internal* flows only ($\Delta B = \Delta L - \Delta V \approx 0$). From the output vector $\underline{u} = \begin{pmatrix} 0.781 \\ -0.625 \end{pmatrix}$ we see that the effect is to move the outputs in different directions, that is, to change $y_D - x_B$. Thus, it takes a large control action to move the compositions in different directions, that is, to make both products purer simultaneously. The condition number of the plant, which is the ratio of the high and low plant gain, is then

$$\gamma(G) = \bar{\sigma}(G)/\underline{\sigma}(G) = 141.7 \tag{14-7}$$

The RGA is another indicator of ill-conditionedness, which is generally better than the condition number because it is scaling independent. The sum of the absolute value of the elements in the RGA (denoted $\|\text{RGA}\|_1 = \Sigma|\text{RGA}_{ij}|$) is approximately equal to the minimized (with respect to input and output scaling) condition number $\gamma^*(G)$. In our case we have $\|\text{RGA}\|_1 = 138.275$ and $\gamma^*(G) = 138.268$. We note that the minimized condition number is quite similar to the condition number in this case, but this does not hold in general.

14-3-2 Use of Decoupler

For "tight control" of ill-conditioned plants the controller should compensate for the strong directions by applying large input signals in the directions where the plant gain is low, that is, a "decoupling" controller similar to G^{-1} in directionality is desired. However, because of uncertainty,

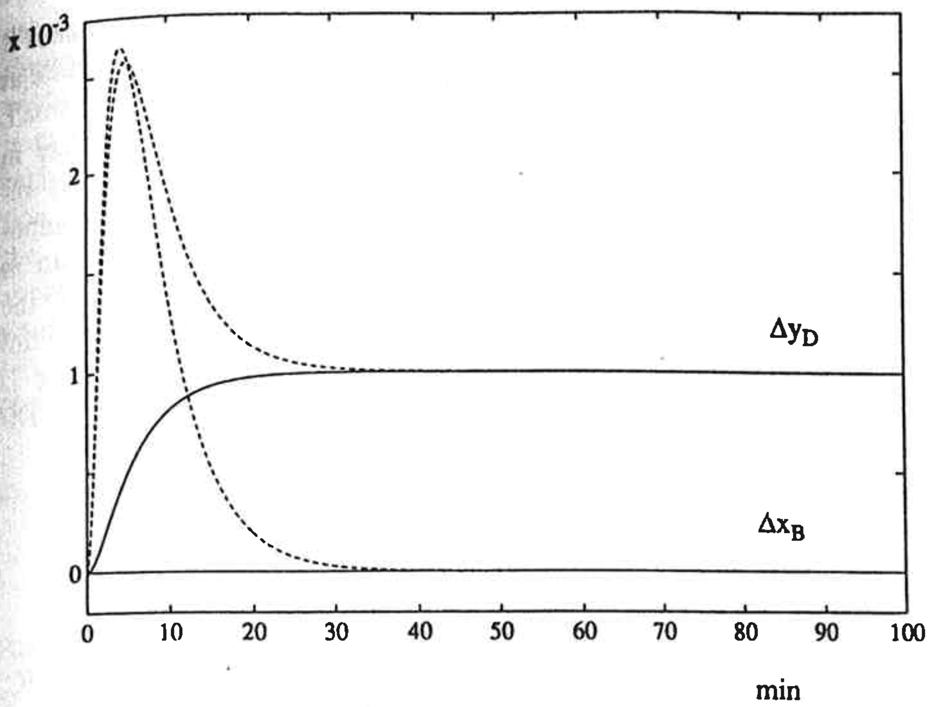

FIGURE 14-2. Response for decoupling controller using LV configuration. Setpoint change in $y_D = 10^{-3}/(5s+1)$. Solid line: No uncertainty; dotted line: 20% input gain uncertainty as defined in Equation 14-9.

the direction of the large inputs may not correspond exactly to the low plant gain direction, and the amplification of these large input signals may be much larger than expected. As shown in succeeding simulations, this will result in large values of the controlled variables y, leading to poor performance or even instability. Consider the following decoupling controller (a steady-state decoupler combined with a PI controller):

$$C_1(s) = \frac{k_1}{s} G^{LV-1}(s) = \frac{k_1(1+75s)}{s}\begin{pmatrix} 39.942 & -31.487 \\ 39.432 & -31.997 \end{pmatrix}, \quad k_1 = 0.7\ \text{min}^{-1} \quad (14\text{-}8)$$

We have $GC_1 = 0.7/sI$. In theory, this controller should counteract all the directions of the plant and give rise to two decoupled first-order responses with time constant $1/0.7 = 1.43$ min. This is indeed confirmed by the solid line in Figure 14-2, which shows the simulated response to a setpoint change in top composition.

14-3-3 Use of Decoupler When There is Model Uncertainty

In practice, the plant is different from the model, and the dotted lines in Figure 14-2 show the response when there is 20% error (uncertainty) in the gain in each input channel ("diagonal input uncertainty"):

$$\Delta L = 1.2\,\Delta L_c, \qquad \Delta V = 0.8\,\Delta V_c \quad (14\text{-}9)$$

ΔL and ΔV are the actual changes in the manipulated flow rates, whereas ΔL_c and ΔV_c are the desired values (what we believe the inputs are) as specified by the controller. It is important to stress that this diagonal input uncertainty, which stems from our inability to know the exact values of the manipulated inputs, is *always* present. Note that the uncertainty is on the *change* in the inputs (flow rates), not on their absolute values. A 20% error is reasonable for process control applications (some reduction may be possible, for example, by use of cascade control using flow measurements, but there will still be uncertainty because of measurement errors). Regardless, the main objective of this chapter is to demonstrate the effect of uncertainty, and its exact magnitude is of less importance.

The dotted lines in Figure 14-2 show the response with this model uncertainty. It differs drastically from the one predicted by the model, and the response is clearly not acceptable; the response is no longer decoupled, and Δy_D and Δx_B reach a value of about 2.5 before settling at their desired values of 1 and 0. In practice, for example, with a small time delay added at the outputs, this controller would give an unstable response.

There is a simple physical reason for the observed poor response to the setpoint change in y_D. To accomplish this change, which occurs mostly in the "bad" direction corresponding to the low plant gains, the inverse-based controller generates a *large* change in the internal flows (ΔL and ΔV), while trying to keep the changes in the external flows ($\Delta B = -\Delta D = \Delta L - \Delta V$) very *small*. However, uncertainty with respect to the values of ΔL and ΔV makes it impossible to make them both large while at the same time keeping their difference small. The result is a undesired *large* change in the external flows, which subsequently results in large changes in the product compositions because of the large plant gain in this direction. As we shall discuss in the following text, this sensitivity to input uncertainty may be avoided by controlling D or B directly, for example, using the DV configuration.

14-3-4 Alternative Controllers: Single-Loop PID

Unless special care is taken, most multivariable design methods (MPC, DMC, QDMC, LQG, LQG/LTR, DNA/INA, IMC, etc.) yield similar inverse-based controllers, and do generally not yield acceptable designs for ill-conditioned plants. This follows because they do not explicitly take uncertainty into account, and the optimal solution is then to use a controller that tries to remove the interactions by inverting the plant model.

The simplest way to make the closed-loop system insensitive to input uncertainty is to

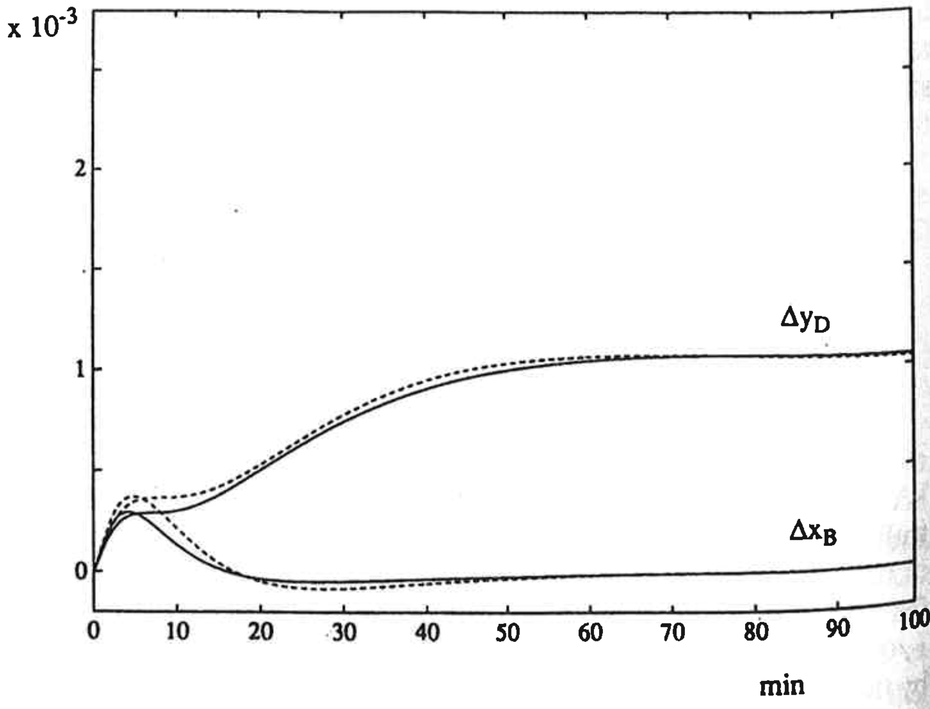

FIGURE 14-3. Response for PID controller using LV configuration.

use a *simple* controller (for example, two single-loop PID controllers) that does not try to make use of the details of the directions in the plant model. The problem with such a controller is that little or no correction is made for the strong interactions in the plant, and then even the nominal response (with no uncertainty) is relatively poor. This is shown in Figure 14-3 where we have used the following PID controllers (Lundström, Skogestad, and Wang, 1991):

$y_D - L$:
$$K_c = 162, \qquad \tau_I = 41 \text{ min}, \qquad \tau_D = 0.38 \text{ min} \tag{14-10}$$

$x_B - V$:
$$K_c = -39, \qquad \tau_I = 0.83 \text{ min}, \qquad \tau_D = 0.29 \text{ min} \tag{14-11}$$

The controller tunings yield a relatively fast response for x_B and a slower response for y_D. As seen from the dotted line in Figure 14-3, the response is not very much changed by introducing the model error in Equation 14-9.

In Figure 14-4 we show response for a so-called mu-optimal controller (see Lundström, Skogestad, and Wang, 1991), which is designed to optimize the worst-case response (robust performance) as discussed toward the end of this chapter. Although this is a multivariable controller, we note that the response is not too different from that with the simple PID controllers, although the response settles faster to the new steady state.

14-3-5 Alternative Configurations: *DV* Control

The process model considered in the preceding text, which uses the *LV* configuration, is fundamentally difficult to control irrespective of the controller. In such cases one should consider design changes that make the process simpler to control. One such change is to consider the *DV* configuration where L rather than D is used for condenser level control (Figure 14-1*b*). The independent variables left for composition control are then D and V:

$$\begin{pmatrix} \Delta y_D(s) \\ \Delta x_B(s) \end{pmatrix} = G^{DV}(s) \begin{pmatrix} \Delta D(s) \\ \Delta V(s) \end{pmatrix} \tag{14-12}$$

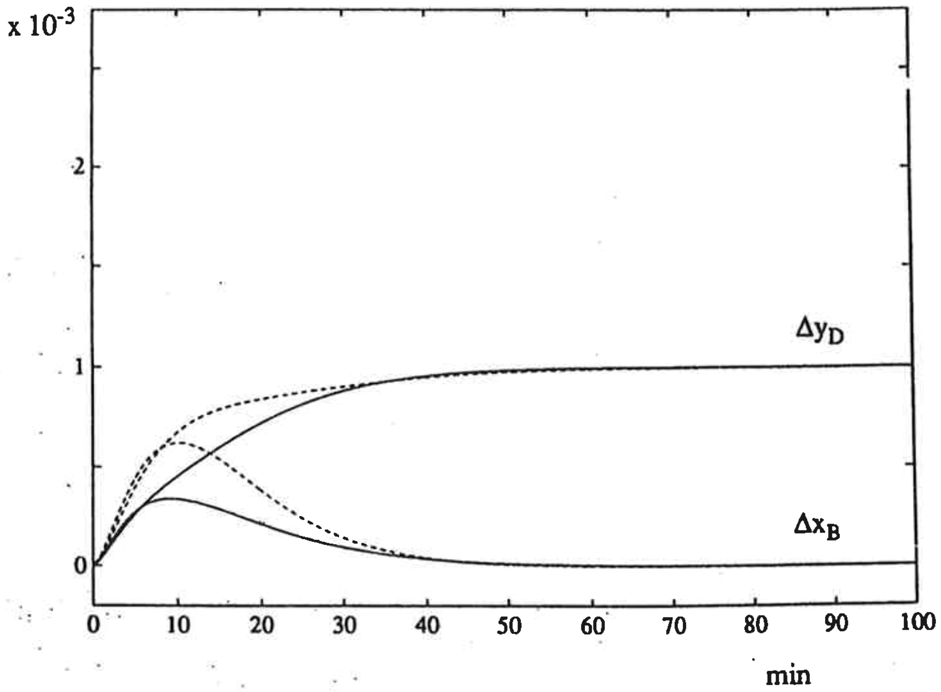

FIGURE 14-4. Response for μ-optimal controller using *LV* configuration.

Comment It is somewhat misleading to consider this a design change because the change from LV to DV configuration is accomplished by a change in the level *control* system. However, many engineers consider the level control system to be such an integral part of the process as to consider it part of the design, although this is, of course, not strictly true.

To derive a model for the DV configuration, assume constant molar flows and perfect control of level and pressure (these assumptions may easily be relaxed). Then $\Delta L = \Delta V - \Delta D$ and we have the following transformation between the two configurations:

$$\begin{pmatrix} \Delta L \\ \Delta V \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta D \\ \Delta V \end{pmatrix} \qquad (14\text{-}13)$$

and the following linear model is derived from Equation 14-4:

$$G^{DV}(s) = G^{LV}(s) \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \frac{1}{75s+1} \begin{pmatrix} 0.878 & -0.864 \\ 0.014 & -0.014 \end{pmatrix} \qquad (14\text{-}14)$$

This process is also ill-conditioned as $\gamma(G^{DV}) = 70.8$. However, the RGA matrix is

$$\text{RGA}(G^{DV}) = \begin{pmatrix} 0.45 & 0.55 \\ 0.55 & 0.45 \end{pmatrix} \qquad (14\text{-}15)$$

The diagonal elements are about 0.5, which indicates a strongly interactive system. However, in this case the RGA elements are not large and we may use a decoupler to counteract the interactions.

Simulations using a decoupler are shown in Figure 14-5. As expected the nominal response is perfectly decoupled. Furthermore, as illustrated by the dotted line in Figure 14-5, the decoupler also works well when there is model error. The reason why the model error does not cause problems in this case is that we have one manipulated variable (ΔD) that acts directly in the high-gain direction for external flows and another (ΔV) that acts in the low-gain direction for internal flows. We may then make large changes in the internal flows V without changing the external flows D. This was not possible with the LV configuration,

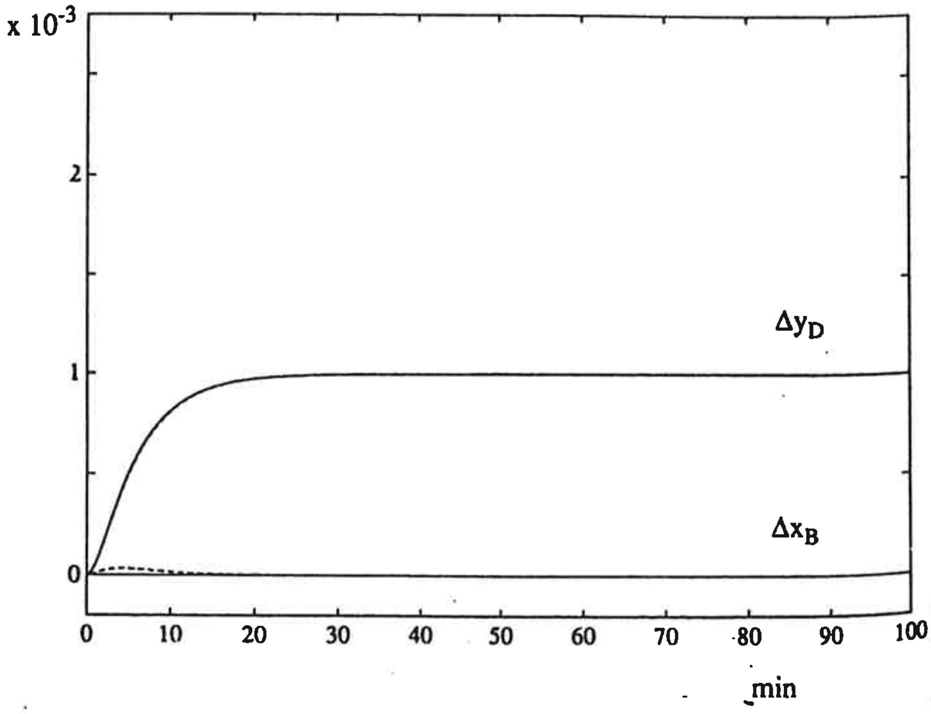

FIGURE 14-5. Response for decoupling controller using DV configuration.

where we had to increase both L and V in order to increase the internal flows.

The RGA behavior for various other configurations is treated in detail by Shinskey (1984) for the static case and by Skogestad, Lundström, and Jacobsen (1990) for the dynamic case.

14-3-6 Limitations with the Example: Real Columns

It should be stressed again that the column model used in the preceding text is not representative of a real column. In a real column the liquid lag θ_L(min), from the top to the bottom, makes the initial response for the LV configuration less interactive and the column is easier to control than found here. It turns out that the important parameter to consider for controllability is *not* the RGA at steady state (with exception of the sign), but rather the RGA at frequencies corresponding to the closed-loop bandwidth. For the LV configuration the RGA is large at low frequencies (steady state), but it drops at high frequencies and the RGA matrix becomes close to the identity matrix at frequencies greater than $1/\theta_L$.

Thus, control is simple, even with single-loop PI or PID controllers, if we are able to achieve very tight control of the column. However, if there are significant measurement delays (these are typically 5 min or larger for GC analysis), then we are forced to operate at low bandwidths and the responses in Figures 14-2 through 14-4 are more representative. Furthermore, it holds in general that one should *not* use a steady-state decoupler if the steady-state RGA elements are large (typically larger than 5).

In a real column one must pay attention to the level control for the DV configuration. This is because D does not directly affect composition; only indirectly, through its effect on reflux L through the level loop, does it have influence. In practice, it may be a good idea to let the condenser level controller set $L + D$ rather than L. In this case a change in D from the composition controller will immediately change L without having to wait for the level loop.

14-4 RGA AS A SIMPLE TOOL TO DETECT ROBUSTNESS PROBLEMS

14-4-1 RGA and Input Uncertainty

We have seen that a decoupler performed very poorly for the LV model. To understand this better consider the loop gain GC. The loop gain is an important quantity because it determines the feedback properties of the system. For example, the transfer function from setpoints y_s to control error $e = y_s - y$ is given by $e = Sy_s = (I + GC)^{-1}y_s$. We therefore see that large changes in GC due to model uncertainty will lead to large changes in the feedback response. Consider the case with diagonal input uncertainty Δ_I. Let Δ_1 and Δ_2 represent the relative uncertainty on the gain in each input channel. Then the actual ("perturbed") plant is

$$G_p(s) = G(s)(I + \Delta_I),$$

$$\Delta_I = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \qquad (14\text{-}16)$$

In the simulation example we had $\Delta_1 = 0.2$ and $\Delta_2 = -0.2$. The perturbed loop gain with model uncertainty becomes

$$G_pC = G(I + \Delta_I)C = GC + G\Delta_I C \qquad (14\text{-}17)$$

If a diagonal controller $C(s)$ (e.g., two PIs) is used, then we simply get (because Δ_I is also diagonal) $G_pC = GC(I + \Delta_I)$ and there is no particular sensitivity to this uncertainty. On the other hand, with a perfect decoupler (inverse-based controller) we have

$$C(s) = k(s)G^{-1}(s) \qquad (14\text{-}18)$$

where $k(s)$ is a scalar transfer function, for example, $k(s) = 0.7/s$, and we have $GC =$

$k(s)I$, where I is the identity matrix, and the perturbed loop gain becomes

$$G_pC = G_p(I + \Delta_I)C$$
$$= k(s)\left(I + G\Delta_I G^{-1}\right) \quad (14\text{-}19)$$

For the aforementioned LV configuration, the error term becomes

$$G^{LV}\Delta_I(G^{LV})^{-1} = \begin{pmatrix} 35.1\Delta_1 - 34.1\Delta_2 & -27.7\Delta_1 + 27.7\Delta_2 \\ 43.2\Delta_1 - 43.2\Delta_2 & -34.1\Delta_1 + 35.1\Delta_2 \end{pmatrix} \quad (14\text{-}20)$$

This error term is worse (largest) when Δ_1 and Δ_2 have opposite signs. With $\Delta_1 = 0.2$ and $\Delta_2 = -0.2$ as used in the simulations (Equation 14-9), we find

$$G^{LV}\Delta_I(G^{LV})^{-1} = \begin{pmatrix} 13.8 & -11.1 \\ 17.2 & -13.8 \end{pmatrix} \quad (14\text{-}21)$$

The elements in this matrix are much larger than 1, and the observed poor response with uncertainty is not surprising. Similarly, for the DV configuration we get

$$G^{DV}\Delta_I(G^{DV})^{-1} = \begin{pmatrix} -0.02 & 0.18 \\ 0.22 & 0.02 \end{pmatrix} \quad (14\text{-}22)$$

The elements in this matrix are much less than 1, and good performance is maintained even in the presence of uncertainty on each input.

The observant reader may have noted that the RGA elements appear on the diagonal in the matrix $G^{LV}\Delta_I(G^{LV})^{-1}$ in Equation 14-20. This turns out to be true in general because diagonal elements of the error term prove to be a direct function of the RGA (Skogestad and Morari, 1987):

$$(G\Delta G^{-1})_{ii} = \sum_{j=1}^{n} \lambda_{ij}(G)\Delta_j \quad (14\text{-}23)$$

Thus, if the plant has large RGA elements and an inverse-based controller is used, the overall system will be extremely sensitive to input uncertainty.

Control Implications

Consider a plant with large RGA elements in the frequency range corresponding to the closed-loop time constant. A diagonal controller (e.g., single-loop PIs) is robust (insensitive) with respect to input uncertainty, but will be unable to compensate for the strong couplings (as expressed by the large RGA elements) and will yield poor performance (even nominally). On the other hand, an inverse-based controller that corrects for the interactions may yield excellent nominal performance, but will be very sensitive to input uncertainty and will not yield robust performance. In summary, plants with large RGA elements around the crossover frequency are fundamentally difficult to control, and decouplers or other inverse-based controllers should never be used for such plants (the rule is never to use a *controller* with large RGA elements). However, one-way decouplers may work satisfactorily.

14-4-2 RGA and Element Uncertainty / Identification

Previously we introduced the RGA as a sensitivity measure with respect to input gain uncertainty. In fact, the RGA is an even better sensitivity measure with respect to element-by-element uncertainty in the matrix.

Consider any complex matrix G and let λ_{ij} denote the ijth element in its RGA matrix. The following result holds (Yu and Luyben, 1987):

> The (complex) matrix G becomes singular if we make a relative change $-1/\lambda_{ij}$ in its ijth element, that is, if a single element in G is perturbed from g_{ij} to $g_{pij} = g_{ij}(1 - 1/\lambda_{ij})$.

Thus, the RGA matrix is a direct measure of sensitivity to element-by-element uncertainty and matrices with large RGA values become singular for small relative errors in the elements.

Example 14-1

The matrix G^{LV} in Equation 14-4 is nonsingular. The 1,2 element of the RGA is $\lambda_{12}(G) = -36.1$. Thus the matrix G

becomes singular if $g_{12} = -0.864$ is perturbed to $g_{p12} = -0.864(1 - 1/(-36.1)) = -0.840$.

The preceding result is primarily an important algebraic property of the RGA, but it also has some important control implications:

Consider a plant with transfer matrix $G(s)$. If the relative uncertainty in an element at a given frequency is larger than $|1/\lambda_{ij}(j\omega)|$, then the plant may be singular at this frequency. This is of course detrimental for control performance. However, the assumption of element-by-element uncertainty is often poor from a physical point of view because the elements are usually always *coupled* in some way. In particular, this is the case for distillation columns. We know that the column will not become singular and impossible to control due to small individual changes in the elements. The importance of the previous result as a "proof" of why large RGA elements imply control problems is therefore not as obvious as it may first seem.

However, for process identification the result is definitely useful. Models of multivariable plants $G(s)$ are often obtained by identifying one element at the time, for example, by using step or impulse responses. From the preceding result it is clear this method will most likely give meaningless results (e.g., the wrong sign of the steady-state RGA) if there are large RGA elements within the bandwidth where the model is intended to be used. Consequently, identification must be combined with first principles modelling if a good multivariable model is desired in such cases.

Example 14-2

Assume the true plant model is

$$G = \begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix}$$

By extremely careful identification we obtain the model

$$G_p = \begin{pmatrix} 0.87 & -0.88 \\ 1.09 & -1.08 \end{pmatrix}$$

This model seems to be very good, but is actually useless for control purposes because the RGA elements have the wrong sign (the 1, 1 element in the RGA is -47.9 instead of $+35.1$). A controller with integral action based on G_p would yield an unstable system.

Comment The statement that identification is very difficult and critical for plants with large RGA elements may *not* be true if we use decentralized control (single-loop PI or PID controllers). In this case we usually do not use the multivariable model, but rather tune the controllers based on the diagonal elements of G only, or by trial-and-error under closed loop. However, if we decide on pairings for decentralized control based on the identified model, then pairing on the wrong elements (e.g., corresponding to negative RGA) may give instability.

The implication for distillation columns is that one must be careful about using the LV configuration when identifying a model for the column. Rather, one may perform test runs with another configuration, for example, the DV configuration, at least for obtaining the steady-state gains. The gains for the LV configuration may subsequently be derived using consistency relationships between various configurations (recall Equation 14-14). Alternatively, the steady-state gains for the LV model could be obtained from simulations, and test runs for changes in L and V are used only to determine the initial dynamic response (in this case one has the additional advantage that it is not necessary to wait for the responses to settle in order to obtain the gain).

14-5 ADVANCED TOOLS FOR ROBUST CONTROL: μ ANALYSIS

So far in this chapter we have pointed out the special robustness problems encountered for MIMO plants and we have used the RGA as our main tool to detect these robustness problems. We found that plants with *large* RGA elements are (a) fundamentally difficult to control because of sensitivity to input gain uncertainty (and, therefore, decouplers should not be used) and (b) are very difficult to identify because of element-by-element uncertainty.

We have not yet addressed the problem of analyzing the robustness of a given system with plant $G(s)$ and controller $C(s)$. In the beginning of this chapter we mentioned that the peak criterion in terms of M were useful for robustness analysis for SISO systems both in terms of stability (RS) and performance (RP). However, for MIMO systems things are not as simple. We shall first consider uncertainty descriptions and robust stability and then move on to performance. The calculations and plots in the remainder of this chapter refer to the simplified LV model of the distillation column, using the controller with steady-state decoupler plus PI control.

14-5-1 Uncertainty Descriptions

To illustrate that most sources of uncertainty may be represented as norm-bounded perturbations with frequency-dependent magnitudes ("weights"), we shall consider a SISO plant with nominal transfer function

$$g(s) = k\frac{e^{-\theta s}}{1 + \tau s} \tag{14-24}$$

The parameters k, θ, and τ are uncertain and/or may vary with operating conditions. Assume that the relative uncertainty in these three parameters is given by, r_1, r_2, and r_3, respectively. A general way to represent model uncertainty is in terms of norm-bounded perturbations Δ_i. Then the set of possible (or "perturbed") values of the parameters is given by

$$k_p = k(1 + r_1\Delta_1), \qquad |\Delta_1| \le 1 \tag{14-25}$$

$$\theta_p = \theta(1 + r_2\Delta_2), \qquad |\Delta_2| \le 1 \tag{14-26}$$

$$\tau_p = \tau(1 + r_3\Delta_3), \qquad |\Delta_3| \le 1 \tag{14-27}$$

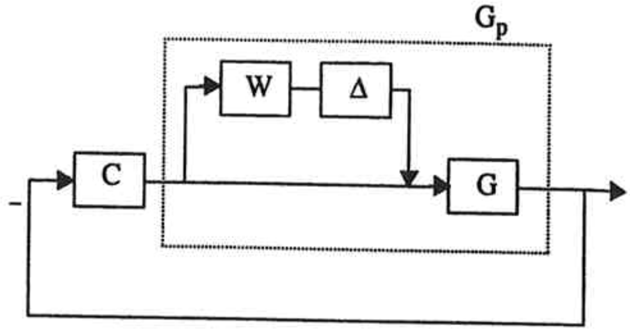

FIGURE 14-6. Multiplicative input uncertainty.

Note that the Δ_i's in the remainder of this chapter are normalized to be less than 1 in magnitude.

First consider the gain uncertainty. For example, assume that k may vary $\pm 20\%$, so that $r_1 = 0.2$. Note that the perturbation on k given by Equation 14-25 may be represented as a relative (or multiplicative) uncertainty as shown in Figure 14-6 with the weight $w = r_1 = 0.2$. In general, the magnitude of the weight varies with frequency, but in this case it is constant only with gain uncertainty.

Now, consider the time delay uncertainty in Equation 14-26. We also want to represent this uncertainty as a relative perturbation. To this purpose use the approximation $e^{-x} \approx 1 - x$ (which is good for small x) and derive

$$e^{-\theta_p s} = e^{-\theta s}e^{-\theta r_2 \Delta_2 s} \approx e^{-\theta s}(1 - r_2\theta s\Delta_2)$$

or, because the sign of Δ_2 may be both positive or negative,

$$e^{-\theta_p s} \approx e^{-\theta s}(1 + w_2\Delta_2), \qquad w_2(s) = r_2\theta s \tag{14-28}$$

w_2 is the weight for the relative error generated by the time delay uncertainty. With this approximation, $w_2 = 0$ at steady state, reaches 1 (100%) at the frequency $1/(\theta r_2)$ (which is the inverse of the time delay uncertainty), and goes to infinity at high frequencies.

Comment It is also possible to make other approximations for the time delay uncertainty. Skogestad, Morari, and Doyle (1988) and Lundström, Skogestad, and Wang (1991) use an approach where one considers numerically the relative uncertainty generated by the time delay. This results in a complex perturbation Δ_2, but otherwise in the same w_2, except that it levels off at 2 at high frequencies. This approach is used in the computations that follow. (However, for other reasons we would probably have preferred to *not* let w_2 level off at 2 if we had redone this work today.)

We now have two sources of relative uncertainty. Combining them gives an overall 2×2 (real) perturbation block Δ, with Δ_1 and Δ_2 on its diagonal. To simplify, we may include the combined effect of the gain and time delay uncertainty using a single (complex) perturbation by adding their magnitudes together, that is, $w = |r_1| + |w_2|$, or approximately

$$w(s) = r_1 + r_2 \theta s \qquad (14\text{-}29)$$

For example, with a 20% gain uncertainty and a time delay uncertainty of ± 0.9 min ($\theta r_2 = 0.9$ min), we obtain $w(s) = 0.2 + 0.9s$.

We may also model the time constant uncertainty with norm-bounded perturbations, but we should preferably use an "inverse" perturbation. For example, we may write

$$\frac{1}{1 + \tau_p s} = \frac{1}{1 + \tau s}(1 + w_3 \Delta_3)^{-1},$$

$$w_3(s) = \frac{r_3 \tau s}{1 + \tau s} \qquad (14\text{-}30)$$

For distillation columns, the time constant τ often varies considerably, but we shall not include this uncertainty here. The reason is that the time constant uncertainty (variations) is generally strongly coupled to the gain uncertainty such that the ratio k_p/τ_p stays relatively constant. Note that k_p/τ_p yields the slope of the initial response and is therefore of primary interest for feedback control.

14-5-2 Conditions for Robust Stability

By robust stability (RS) we mean that the system is stable for all possible plants as defined by the uncertainty set (using the Δ_i's as previously discussed). This is a "worst case" approach, and for this reason one must be careful to not include unrealistic or impossible parameter variations. This is why it is recommended not to include large individual variations in the gain k and the time constant τ for a distillation column model.

Now, consider the distillation column example with combined gain and time delay uncertainty. For multivariable plants it makes a difference whether the uncertainty is at the input or the output of the plant. We will here consider input uncertainty, and the weight w_I then represents, for example, variations in the input gain and neglected valve dynamics. We assume the same magnitude of the uncertainty for each input. The set of possible plants is given by

$$G_p(s) = G(I + w_I \Delta_I),$$

$$\Delta_I = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \qquad (14\text{-}31)$$

where Δ_i represents the independent uncertainty in each input channel. This is identical to Equation 14-16, except that w_I yields the magnitude because Δ_i is now normalized to be less than 1. Note that Δ_I is a diagonal matrix (it has "structure"). We assume that the system without uncertainty is stable (we have NS). Instability may then only be caused by the "new" feedback paths caused by the Δ_I block. Therefore, to test for RS we rearrange Figure 14-6 into the standard form in Figure 14-7 where Δ in our case is the matrix Δ_I and $M = -w_I C(I + GC)^{-1} G = -w_I T_I$. M is the transfer function from the output to the input of the Δ_I block. To test for stability we make use of the "small gain theorem". Because the Δ block is normalized to be less than 1 at all frequencies, this theorem says that the system is stable if the M block

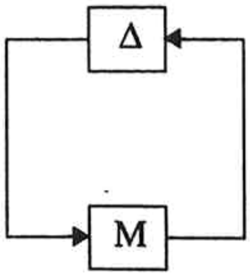

FIGURE 14-7. General block diagram for studying robust stability.

is less than 1 at all frequencies. We use the singular value (also called spectral norm) to compute the magnitude (norm) of M. Robust stability is then satisfied if at all frequencies ω,

$$\bar{\sigma}(M) = \bar{\sigma}(w_I T_I(j\omega)) < 1 \quad \text{(14-32)}$$

However, Equation 14-32 is generally conservative for the following reasons:

1. It allows for Δ to be complex.
2. It allows for Δ to be a full matrix.

It is actually the second point that is the main problem in most cases.

The structured singular value $\mu(M)$ of Doyle (e.g., see Skogestad, Morari, and Doyle, 1988) is defined to overcome these difficulties, and we have that RS is satisfied *if and only if* at all ω,

$$\mu_\Delta(M) = \mu_\Delta(w_I T_I) < 1 \quad \text{(14-33)}$$

This is a tight condition provided the uncertainty description is tight. Note that for computing μ we have to specify the block *structure* of Δ and also if Δ is real or complex. Today there exists very good software for computing μ when Δ is complex. The most common method is to approximate μ by a "scaled" singular value:

$$\mu_\Delta(M) \le \min_D \bar{\sigma}(DMD^{-1}) \quad \text{(14-34)}$$

where D is a real matrix with a block-diagonal structure such that $D\Delta = \Delta D$. This upper bound is exact when Δ has three or fewer "blocks" (in our example, Δ_I has two blocks).

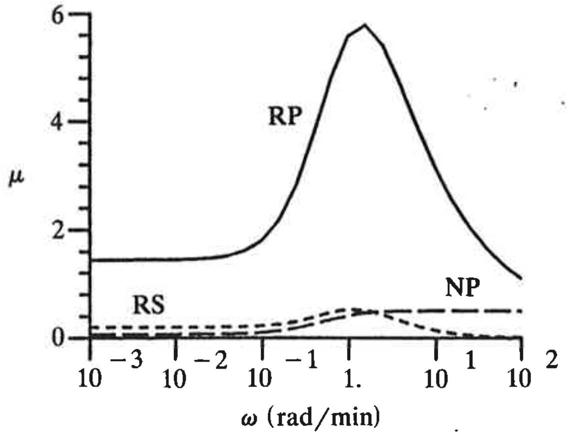

FIGURE 14-8. μ plots for decoupling controller using LV configuration.

As an example, consider the following input uncertainty in each of the two input channels

$$w_I(s) = 0.2 + \frac{0.9s}{0.5s+1} = 0.2\frac{5s+1}{0.5s+1} \quad \text{(14-35)}$$

This corresponds to 20% gain error and a neglected time delay of about 0.9 min. The weight levels off at 2 (200% uncertainty) at high frequency. Figure 14-8 shows $\mu(M) = \mu(w_I T_I)$ for RS with this uncertainty using the decoupling controller. The μ plot for RS shows the inverse of the margin we have with respect to our stability requirement. For example, the peak value of $\mu_{\Delta_I}(M)$ as a function of frequency is about 0.53. This means that we may increase the uncertainty by a factor $1/\mu = 1.89$ before the worst-case model yields instability. This means that we tolerate about 38% gain uncertainty and a time delay of about 1.7 min before we get instability.

Comment For this decoupling controller we have $GC = (0.7/s)I$ and $T_I = T = 1/(1.43s + 1)I$. For this particular case the structure of Δ does not matter and we get a simple analytic expression for μ for robust stability:

$$\mu_\Delta(M) = \bar{\sigma}(w_I T_I)$$

$$= \left| 0.2 \frac{5s + 1}{(0.5s + 1)(1.43s + 1)} \right| .$$

14-5-3 Definition of Performance

To define performance we shall use the frequency domain and define an upper bound on the sensitivity function S. The sensitivity function gives the change in the response caused by feedback and is probably the best and simplest function to consider when defining performance in the frequency domain. At each frequency we require

$$|S(j\omega)| = |(1 + GC(j\omega))^{-1}| < |w_P^{-1}(j\omega)| \quad (14\text{-}36)$$

or equivalently that the weighted sensitivity is less than 1:

$$\text{NP: } |w_P S(j\omega)| < 1 \quad \text{at all } \omega \quad (14\text{-}37)$$

The peak value (with respect to frequency) of the weighted sensitivity $w_P S$ is also called the H_∞ norm. $w_P(s)$ is the performance weight. Typically, we use the weight

$$w_P(s) = \frac{1}{M_s} \frac{\tau_{cl}s + M_s}{\tau_{cl}s} \quad (14\text{-}38)$$

This requires (a) integral action, (b) that the peak value of $|S|$ should be less than M_s (typically $M_s = 2$), and (c) that the closed-loop response time should be less than τ_{cl} (i.e., the bandwidth should at least be $\omega_B = 1/\tau_{cl}$).

For multivariable systems, the largest singular value of S, $\bar{\sigma}(S)$, is used instead of the absolute value $|S|$. In the introduction we mentioned that the maximum peak on S may be used as a robustness criterion. However, here we are restricting the peak of S at high frequencies primarily to get a good response (without too much oscillation and overshoot). The robustness issues are taken care of much more directly by specifying the allowed uncertainty: see the RS and RP conditions.

NP Specification for our Example

At each frequency the value of $\bar{\sigma}(w_P S)$ should be less than 1. We have selected

$$w_P(s) = \frac{1}{2} \frac{20s + 2}{20s} \quad (14\text{-}39)$$

This requires integral action, a maximum closed-loop time constant of approximately $\tau_{cl} = 20$ min (which of course is relatively slow when the allowed time delay is only about 0.9 min) and a maximum peak for $\bar{\sigma}(S)$ of $M_s = 2$.

As expected, we see from the plot in Figure 14-8 that the NP condition is easily satisfied with the decoupling controller. $\bar{\sigma}(w_P S)$ approaches $1/M_s = 0.5$ at high frequency because of the maximum peak requirement on $\bar{\sigma}(S)$.

14-5-4 Conditions for Robust Performance

Robust performance (RP) means that the performance specification is satisfied for the worst-case uncertainty. The most efficient way to test for RP is to compute μ for RP. If this μ value is less than 1 at all frequencies, then the performance objective is satisfied for the *worst case*. Although our system has good robustness margins and excellent nominal performance, we know from the simulations in Figure 14-2 that the performance with uncertainty (RP) may be extremely poor. This is indeed confirmed by the μ curve for RP in Figure 14-8, which has a peak value of about 6. This means that even with 6 times less uncertainty, the performance will be about 6 times poorer than what we require. Because of a property of μ we may therefore define μ for NP as $\mu_{\Delta_P}(w_P S) = \bar{\sigma}(w_P S)$ where Δ_P is a "fake" uncertainty matrix. Δ_P is a "full" matrix, that is, the off-diagonal elements may be nonzero. μ for robust performance is computed as $\mu_\Delta(N)$ where the matrix Δ in this case has a block-diagonal structure with Δ_I (the true uncertainty) and Δ_P (the fake uncertainty stemming from the performance specification) along the main diagonal and

$$N = \begin{pmatrix} -w_I T_I & -w_I CS \\ w_P SG & w_P S \end{pmatrix} \tag{14-40}$$

The derivation of N is given in, for example, Skogestad, Morari, and Doyle (1988).

The μ-optimal controller is the controller that minimizes μ for RP. For our example we are able to press the peak of μ down to about 0.978 (Lundström, Skogestad, and Wang, 1991). The simulation in Figure 14-4 shows that the response even with this controller is relatively poor. The reason is that the combined effect of large interactions (as seen from the large RGA values) and input uncertainty makes this plant fundamentally difficult to control.

Comment In the time domain our RP-problem specification may be formulated *approximately* as follows: Let the plant be

$$G_p^{LV}(s) = G^{LV}(s)\begin{pmatrix} k_1 e^{-\theta_1 s} & 0 \\ 0 & k_2 e^{-\theta_2 s} \end{pmatrix} \tag{14-41}$$

where $G^{LV}(s)$ is given in Equation 14-4. Let $0.8 \le k_1 \le 1.2$, $0.8 \le k_2 \le 1.2$, $0 \le \theta_1 \le 0.9$ min, and $0 \le \theta_2 \le 0.9$ min. The response to a step change in setpoint should have a closed-loop time constant less than about 20 min. Specifically, the error of each output to a unit setpoint change should be less than 0.37 after 20 min, less than 0.13 after 40 min, and less than 0.02 after 80 min, and with no large overshoot or oscillations in the response.

14-6 NOMENCLATURE

B	Bottom product flow (kmol/min)
D	Distillate product flow (kmol/min)
G	Nominal plant model
L	Reflux flow (kmol/min)
M	Matrix used to test for robust stability
M_t	Maximum peak of T
M_s	Maximum peak of S
RGA	Matrix of relative gains
s	Laplace variable ($s = j\omega$ yields the frequency response)
$S = (I + GC)^{-1}$	Sensitivity function
$T = GC(I + GC)^{-1}$	Closed-loop transfer function
$T_I = CG(I + CG)^{-1}$	Closed-loop transfer function at the input
U	Unitary matrix of output singular vectors
V	Unitary matrix of input singular vectors
V	Boilup (kmol/min)
x_B	Bottom composition (mole fraction)
y_D	Distillate composition (mole fraction)
w	Frequency-dependent weight function

Greek letters

Δ	Overall perturbation block used to represent uncertainty
Δ_I	Overall perturbation block for input uncertainty
$\Delta_i, \Delta_1, \Delta_2, \Delta_3$	Individual scalar perturbations
$\Delta L, \Delta y_D$, etc.	Deviation variables for reflux, top composition, etc.
$\gamma(A) = \bar{\sigma}(A)/\underline{\sigma}(A)$	Condition number of matrix A
$\mu(A)$	Structured singular value of matrix A
ω	Frequency (rad/min)
$\bar{\sigma}(A)$	Maximum singular value of matrix A
$\underline{\sigma}(A)$	Minimum singular value of matrix A

Subscripts

p	Perturbed (with model uncertainty)
P	Performance

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