Inconsistencies in Dynamic Models for Ill-Conditioned Plants - with Application to Low-Order Models of Distillation Columns.

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Abstract

The open-loop responses of ill-conditioned processes often take the form of almost pure first-order dynamics. Physically, a single slow pole (state), resulting from interactions, is dominating all the individual responses, and it is difficult to identify the other poles of the process. Attempting to identify a model based on fitting the individual transfer-function elements of an $n \times n$ process, such that they all contain the dominant time-constant of the process, results in a model with a minimal realization which incorrectly contains at least n poles equal to the dominant pole of the process. It is shown that this model with excessive slow poles, although a reasonable approximation for open-loop dynamics, yields a poor prediction of the closed-loop behavior of the process.

The emphasis of the paper is on high-purity distillation. However, the results are relevant also for other ill-conditioned processes and the paper includes a heat-exchanger example.

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1 Introduction

When obtaining dynamic models for process control one usually seeks simple linear models. The traditional approach has been to obtain a low-order transfer-function model, e.g. of the kind first-order-plus-deadtime, by fitting the open-loop response obtained from plant data or from simulations with a detailed dynamic model. Similarly, for multivariable processes the transfer matrix has usually been obtained by fitting the individual elements *independently.*

However, this is often a poor approach. Skogestad and Morari (1988) argue that it may easily lead to poor models for ill-conditioned processes unless one explicitly takes into account the coupling between the gains of the different elements. Skogestad and Hovd (1990) show that a plant, $G(s)$, with large values of the Relative Gain Array (RGA) (Bristol, 1966)

$$RGA = G(s) \times (G(s)^{-1})^T$$

(where $\times$ denotes element-by-element multiplication), is very sensitive to errors in the individual elements, and they conclude that identification must be combined with first principles modelling if a good multivariable model is desired in such cases. In particular, one is not able to obtain a good model of the low-gain direction of the plant (Skogestad and Morari, 1988, Andersen et al., 1989), and the model will easily have the wrong sign of the determinant at steady-state.

Another problem with this identification approach, is that the model may be inconsistent in that a single physical state is repeated in the model. This issue is the main topic of this paper. Ill-conditioned plants often have a *single* dominating "slow" pole (large time constant) which is a result of interactions in the process, and is thus shared by all the transfer matrix-elements. However, by fitting the elements of an $n \times n$ process *independently*, such that they all contain the dominant pole, one may get an inconsistent model with at least n poles similar to the single dominating pole of the process. As shown in this paper, the inconsistency will usually result in a poor prediction of the process behavior under partial feedback control.

In many cases such transfer-matrices are fitted using different time-constants in the different elements, without considering whether the time-constants actually originates from a single state (pole). This is for instance common practice in the distillation control literature (e.g., Shunta and Luyben, 1972, Hammarström et al., 1982, Waller et al., 1988). Such an approach will again result in a model with a minimal realization containing n^2 large time-constants, all in the order of the dominant time-constant of the process, and thus an inconsistent model.

The general literature on identification theory has so far not focused very much on multivariable issues, and the particular problems mentioned above that may be encountered when identifying ill-conditioned plants, do not seem to have been discussed.

We start the paper with an example of an inconsistent low-order model of a heat exchanger. The model, although seemingly a good open-loop description of the plant, is shown to yield unexpected behavior when one control loop is closed ("one-point control"). The results in this example are subsequently explained using analytical results. We then briefly discuss what types of processes that are likely to be modelled with an excessive number of slow poles. The last part of the paper is devoted to the specific problem of obtaining low-order models of distillation columns.

All the results presented in this paper are for 2×2 processes, i.e., two inputs and two outputs. However, the results are of relevance also for higher dimensional processes.

2 Introductory Example

Example 1. Heat-exchanger. Consider a heat-exchanger modelled using a single mixing tank for both the hot and cold side (see Figure 1). Neglecting the heat accumulated in the walls yields a model with two states. The model is derived in Appendix, and data for the example are given in Table 1. In the following we only use the linearized form, $y(s) = G(s)u(s)$, of the model. Here $y = [y_1 \quad y_2]^T = [T_C \quad T_H]^T$ is the cold and hot outlet temperature and $u = [u_1 \quad u_2]^T = [q_C \quad q_H]^T$ is the cold and hot inlet flow rates. The exact linear model is

$$G(s) = \frac{1}{(1+\tau_1 s)(1+\tau_2 s)} \begin{pmatrix} k_{11}(1+4.76s) & k_{12} \\ k_{21} & k_{22}(1+4.76s) \end{pmatrix} \quad (1)$$

$$\tau_1 = 100 \quad ; \tau_2 = 2.44 \quad ; k_{11} = -k_{22} = -1874 \quad ; k_{12} = -k_{21} = 1785$$

The model is strongly ill-conditioned and has a steady-state condition number of 41 and diagonal steady-state RGA-values of 10.8. The physical explanation for the ill-conditioning is simply that the heat transfer is very effective such that the two outlet temperatures (outputs) are almost the same (61.59°C and 63.41°C in our case), and it is very difficult to change them independently. In particular, it is difficult to make them closer (this is the "weak" or "difficult" or "low-gain" direction of the plant), whereas we may easily make them *both* hotter or colder (this is the "strong" or "high-gain" direction of the plant).

Open-loop responses in the outlet temperatures to a 10% step change in hot inlet flow u_2 obtained from the model (1) are shown by the solid lines in Figure 2 (similar responses, but with opposite signs, are obtained for changes in u_1). From the figure we observe that the responses in both outputs are close to first-order with a time-constant around 100 minutes. We also note that the smallest time constant, $\tau_2 = 2.44$ minutes, which we later show is associated with the low-gain direction of the plant, is very difficult to observe from the open-loop responses. Indeed, as seen from the dashed lines in Figure 2, an excellent fit is obtained with the following model

$$G(s) = \frac{1}{1+\tau_1 s} \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \quad (2)$$

Although it may seem like the model (2) only has a single time constant $\tau_1 = 100$ minutes, the state-space realization contains two poles at $-1/\tau_1$.

We now want to study the behavior of the process under partial ("one-point") feedback control, i.e., controlling one of the outlet temperatures. The cold outlet temperature T_C (y_1) is controlled with the cold inlet flow q_C (u_1) using a P-controller with gain $K = 0.015$ which yields a closed-loop time-constant for this loop of about 3.5 minutes. Figure 3 shows the responses to a 10% step change in hot inlet flow q_H (u_2) with this loop closed.

The solid lines are obtained with the "full" linear model (1), whereas the dashed lines are obtained with the fitted model (2). For the response in the uncontrolled output, T_H (y_2), there is a significant difference between the two models. The full model yields a "fast" response in T_H (similar to that of the controlled output T_C), whereas the fitted model yields a slow settling towards the new steady-state. The reason for the large difference in behavior is, as we shall see, the different number of slow poles in the two models.

3 Minimum number of states and inconsistency.

Consider a linear system described by the model

$$\dot{x} = Ax + Bu; \quad y = Cx + Du \tag{3}$$

Here x denotes states, u inputs, y outputs and $\dot{x}$ the time derivative of x. Laplace transformation of (3) yields the transfer-matrix

$$G(s) = C(sI - A)^{-1}B + D \tag{4}$$

For a system with n states, m inputs and p outputs we have $dim(A) = n \times n$, $dim(B) = n \times m$, $dim(C) = p \times n$ and $dim(D) = p \times m$. The maximum rank of $G(s)$ is $r_{max} = min(p, m)$. Assume that $G(0)$ has rank $r > 1$. With $D \neq 0$ we may define a model with a single state (time-constant) by letting the dynamic part of the model, $C(sI - A)^{-1}B$, have rank equal to 1 and use D to make the rank of $G(0) = r$. However, such a model yields a very poor initial response for most processes and is therefore not considered. With $D = 0$, which is more reasonable from a physical point of view, it is easily seen from (4) that we need at least r states for $G(0)$ to have rank r.

Example 1, continued. In the heat-exchanger example we had a non-singular steady-state matrix $G(0)$ with rank $r = 2$, and consequently we need at least two states to describe the system using a state-space description with $D = 0$. Thus, when attempting to describe the system using only one time-constant we obtained the simplified model (2) with two poles at $-1/\tau_1$.

Some readers might believe that also the full model has two poles at $-1/\tau_1 = -1/100$ since there are two mixing tanks which isolated would have a time-constant of $V/q = 100$ minutes each. However, an analysis of the full model (1) reveals that there is a multivariable zero that cancels one of the apparent poles at $-1/\tau_1$.

The single pole at $-1/\tau_1$, which is shared by all the transfer function elements, is a result of the interactions between the two sides of the heat exchanger. In addition the full model has a significantly faster pole corresponding to a time-constant $\tau_2 = 2.44$ min. Applying one-point feedback control to the full model (1) causes the shared pole $-1/\tau_1$ to move, and also the uncontrolled response to become fast. However, this is not the case when the simplified model (2) is used, because here only one of the two poles at $-1/\tau_1$ is moved. This is shown in the next section.

Example 2. PI-control of Wahl and Harriot column (1970).

High-purity distillation columns operating with reflux L and boilup V as independent variables (see Figure 5) may be strongly ill conditioned. Furthermore, it is well known that the individual open-loop responses may be well approximated using only one dominating time-constant. This has been shown both from plant data (McNeill and Sachs, 1969) and in several theoretical papers (e.g., Davidson, 1956; Moczek et al., 1965; Wahl and Harriot, 1970; Kim and Friedly, 1974; Skogestad and Morari, 1987a). Due to this, first order models are commonly used in the distillation control literature.

Wahl and Harriot (1970) used a simple low-order model to study the behavior of a high-purity column under one-point control. Their low-order model is somewhat more complicated than the pure first-order transfer-function matrix as given in (2), but the minimal realization of their model contains two time-constants equal to 365 min, while the full model only has one time-constant at 365 min.

The dashed lines in Figure 4 show the response in top composition y_D (y_2) of the Wahl and Harriot low-order model to a step change in feed composition with the composition on plate 4 (y_1) under feedback control. The controller tuning (PI-controller) used here is somewhat different than the one used by Wahl and Harriot, but the responses resemble closely the ones shown in Wahl and Harriot (1970)[1], i.e., a fast response in the composition on plate 4 (y_1) with a slow settling towards steady-state for the uncontrolled top composition (y_2). The slow settling in y_2 was noticed by Wahl and Harriot, but they assumed it to be a property of the process. However, the slow settling to steady-state is simply a result of a modelling error, that is, the model has an excessive slow pole. This is seen from the solid lines in Figure 4 which show the responses obtained using the full linear model. The full model yields a fast response in both compositions.

Also several other authors (e.g., Skogestad et al., 1990; Sandelin et al., 1991) have used inconsistent models for studies of partial feedback control in distillation. This may be seen from their figures by observing the slow settling in the uncontrolled output.

4 Analytical treatment of model with one loop closed.

Consider applying the control law

$$du_1 = -K(dy_1 - dy_{1s}) \tag{5}$$

to the simplified model (2) (here subscript s denotes setpoint). The closed-loop transfer-matrix becomes

$$\begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} = \frac{1}{1+\tau_{CL}s} \begin{pmatrix} \frac{Kk_{11}}{(1+Kk_{11})} & \frac{k_{12}}{(1+Kk_{11})} \\ \frac{Kk_{21}}{(1+Kk_{11})} & \frac{k_{22}(1+\tau_{CL}s)-\frac{Kk_{12}k_{21}}{1+Kk_{11}}}{(1+\tau_1 s)} \end{pmatrix} \begin{pmatrix} dy_{1s} \\ du_2 \end{pmatrix} \tag{6}$$

where

$$\tau_{CL} = \tau_1/(1+Kk_{11}) \tag{7}$$

Thus, three of the elements are first-order with the time-constant, τ_{CL}, whereas the transfer-function $g_{22}(s)$ from u_2 to the uncontrolled output y_2 is second order, as it in addition contains the open-loop dominant time-constant, τ_1. To see how the two time-constants contribute to the overall response in the uncontrolled output y_2, write $g_{22}(s)$ on

[1]Actually Wahl and Harriot have the wrong sign on the change in top composition

the form

$$g_{22}(s) = \frac{X_1}{1+\tau_1 s} + \frac{X_{CL}}{1+\tau_{CL} s} \tag{8}$$

The ratio between the gains X_1 and X_{CL} is given by

$$\frac{X_1}{X_{CL}} = (1+Kk_{11})(\frac{1}{Y}-1) \quad ; Y = \frac{k_{12}k_{21}}{k_{11}k_{22}} \tag{9}$$

Y is the ratio between the off-diagonal and diagonal steady-state gains, and is a well known measure of interactions (e.g., Balchen, 1958, Rijnsdorp, 1965). It is also related to the 1,1–element of the Relative Gain Array (Bristol, 1966) for 2×2 systems

$$\lambda_{11} = \frac{1}{1-Y} \tag{10}$$

The model is ill-conditioned when Y is close to one which corresponds to a large value of λ_{11}.

Consider Y in the range 0 to 1. For cases with $Y = 1$ ($\lambda_{11} = \infty$) we see from (9) that X_1 becomes zero, i.e., there is no gain related to τ_1, and only τ_{CL} remains in $g_{22}(s)$. This is as expected since $Y = 1$ implies that the model is singular at all frequencies and the minimal realization of (2) will only contain one state. On the other hand, if $Y = 0$ ($\lambda_{11} = 1$) we see from (9) that the gain related to τ_{CL} will be zero and only τ_1 will be left in $g_{22}(s)$. This is also as expected since $Y = 0$ implies that the steady-state matrix is triangular or diagonal, in which case it is likely that the identified process actually contains two poles at $-1/\tau_1$ (see discussion below). For values of Y between 0 and 1 ($\lambda_{11} > 1$), both poles will be present in $g_{22}(s)$.

From (9) we see that the ratio X_1/X_{CL} also depends on the gain K used in the controller. The higher the gain is, the larger is the ratio X_1/X_{CL}. This means that the faster the response in the controlled output is, the more marked is the large time-constant τ_1 in the uncontrolled output, y_2.

Example 1, continued. For the heat-exchanger example we have $Y = 0.907$ and $Kk_{11} = 28.1$ which yields $X_1/X_{CL} = 2.98$ for the simplified model (2). That is, a major part of the response in the uncontrolled output y_2 is related to τ_1, which is confirmed by the slow settling for y_2 (dashed line) in Figure 3. For the full model (1) the single time-constant τ_1 is affected by the feedback control, and y_2 has no slow settling (solid line in Figure 3).

Our analysis of equation (9) seems to suggest that it is for weakly interactive processes we get the largest error when an inconsistent model with excessive slow poles is used. However, this conclusion is misleading as it is for ill-conditioned processes we most likely will identify a model with too many slow poles. To see this consider a 2×2 model which is reduced to have two states. The two poles left should be the ones with the largest effect on the input-output behavior of the full model. Each of the two poles will have an input direction related to them, that is, a set of inputs that cancels the other pole. A similarity transformation of the state-space model, so that the A-matrix becomes diagonal, will reveal these directions in the rows of the transformed B-matrix. Changes in one input at the time, i.e., the input vectors $[1 \quad 0]^T$ and $[0 \quad 1]^T$, will span the input space. If one of the poles dominates the responses to both these input perturbations, it means that the gain related to the "hidden" pole must be small compared to the gain related to the dominating pole. This implies that the system has two directions with widely differing gains, i.e., the system is ill-conditioned[2]. From this we conclude that it is only for ill-conditioned systems that the open-loop responses are likely to be well approximated using an inconsistent model with a single time-constant. A diagonal or triangular 2×2 process which has $Y = 0$ ($\lambda_{11} = 1$) and is well described using only one time-constant τ_1 is thus likely to actually contain two poles at $-1/\tau_1$.

Example 1, continued. A similarity transformation of the state-space realization of the full heat exchanger model (1) shows that the input direction cancelling τ_2 is $[1 \quad -1]^T$ and the input direction cancelling τ_1 is $[1 \quad 1]^T$. A singular value decomposition of the model gives a (minimized) condition number of 41 with the high-gain input direction being $[1 \quad -1]^T$ and the low-gain input direction being $[1 \quad 1]^T$. In this case we therefore have a perfect alignment of the singular input vectors and the pole-cancelling vectors, i.e the high-gain input direction has a pole $-1/\tau_1$ and the low-gain input direction a pole $-1/\tau_2$. The gain in the direction of the slow pole $-1/\tau_1$ is consequently 41 times the gain in the direction of the fast pole $-1/\tau_2$, and the fast pole is thus only weakly visible in open-loop simulations with perturbations in single inputs. This explains why a model using only one time-constant yields an excellent fit of the open-loop responses in Figure 2.

5 Implications for low-order models of distillation columns

In this section we discuss the problem of obtaining simple low-order models of high-purity distillation columns (see Figure 5). As mentioned above, high-purity distillation columns may be strongly ill-conditioned and the responses to step changes in inputs are usually well fitted by first-order responses when reflux L and boilup V are used as independent variables.

Example 3. High-purity distillation column. Data for a column with relatively high product purities are given in Table 2. Open-loop responses in product compositions y_D and x_B to step changes in reflux L (keeping boilup V fixed) using a full linear model with 82 states are shown in Figure 6. Note that liquid flow-dynamics are included in the model. Despite the high order of the model, the responses seem to be almost pure first-order with a time-constant of approximately $\tau_1 = 194$ minutes. Fitting each transfer-matrix element to a first-order response with time-constant $\tau_1 = 194$ min yields a model on the same form as (2)[3]

$$\begin{pmatrix} dy_D \\ dx_B \end{pmatrix} = \frac{1}{1+\tau_1 s} \begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix} \begin{pmatrix} dL \\ dV \end{pmatrix} \tag{11}$$

[2]Note that some ill-conditioned systems may have the directions of the poles closely aligned with the input vectors of the perturbations. In this case both poles will show up in the simulations.

[3]Note that we could have fitted four slightly different time-constants instead. However, the results would have been similar.

This model is denoted N1. N means no flow dynamics and 1 indicates one time-constant. This simple model gives an almost perfect fit of the overall open-loop responses of the full model with 82 states as seen from Figure 6; the difference between the responses is hardly visible. However, an analysis of the full model reveals that it has only a single pole at $-1/\tau_1$, and so the model N1 is inconsistent in that it contains two poles at $-1/\tau_1$.

It is clear from the analysis in the previous section that the model N1 will give a poor prediction of the behavior of the full plant under one-point control. In addition, Jacobsen et al. (1991) have shown that this type of model is poor also for the case with both compositions under feedback control ("two-point control"). This is explained by the fact that the directionality of the process, in particular at intermediate and high frequencies, is poorly predicted by a model of the form (11). To see this consider Figure 7 which shows the RGA plotted as a function of frequency for the full model with 82 states and the fitted model N1. Model N1 has $\lambda_{11} = 35$ over all frequencies, that is, strong directional dependence at all frequencies. The RGA for the full model, on the other hand, breaks off at intermediate frequencies and becomes unity at higher frequencies. The process is therefore only weakly directionally dependent at high frequencies. As shown by Skogestad et al. (1990), the low RGA at intermediate and high frequencies is caused by the flow-dynamics, i.e., the lag in reflux from the top to the bottom of the column.

We now want to study how to develop an improved simple low-order model of a distillation column which 1) has correct directions at intermediate and high frequencies and 2) does not contain excessive slow poles. However, let us first consider the low-order models most commonly presented in the distillation literature.

Most low-order models of distillation columns presented in the literature are of the type first-order plus delay. With only output delays and/or input delays the sum of delays would be equal in the diagonal and off-diagonal elements. However, a study of models reported reveals that most of them have a larger sum of delays in the off-diagonal elements than in the diagonal elements (e.g., Wood and Berry, 1973, Hammarström et al., 1982, Waller et al., 1988). This seems reasonable and is probably a result of the flow-dynamics. For example it takes time for the reflux L to affect x_B and $g_{21}(s)$ should contain an additional lag. Most authors use pure dead-times to represent this, while the flow-dynamics physically is a high order lag.

Let us now try to improve the simplified model N1 (11) to get a reasonable low-order model. Because the flow-dynamics reduce the RGA at high frequencies, we will first try to add this to the model. We introduce a lag term $g_L(s)$ of 2.5 minutes in the 2,1 element of model N1 (11), corresponding to the reflux lag of the full model, and obtain the model F1 (F denotes flow dynamics)

$$\begin{pmatrix} dy_D \\ dx_B \end{pmatrix} = \frac{1}{1+\tau_1 s} \begin{pmatrix} k_{11} & k_{12} \\ k_{21} g_L(s) & k_{22} \end{pmatrix} \begin{pmatrix} dL \\ dV \end{pmatrix} \qquad (12)$$

where in our case (time is in minutes)

$$\tau_1 = 194 \quad ; K = \begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix} \quad ; g_L(s) = \frac{1}{(1+0.5s)^5}$$

(the full model actually has $g_L(s)$ with a 39th order lag, but a good approximation is obtained with a 5th order lag). We see from the RGA-plot in Figure 7 that we now obtain a much better fit of the directionality of the process. The model F1 has been studied by Skogestad et al. (1990) and Jacobsen et al. (1991), and they concluded that it was a reasonably good model when both outputs are controlled simultaneously ("two-point" control). However, we find that including the flow-dynamics does not correct the fundamental error of excessive slow poles, and the model will be poor for studies of partially controlled distillation columns. This is seen from Figure 8 which shows the response of the model under one-point control. Top composition y_D is controlled by reflux L using a PI-controller while x_B is left uncontrolled. As seen from curve F1, the model yields an incorrect slow settling towards steady-state for the uncontrolled output x_B. As F1 represents the type of model most commonly presented in the literature, it is likely that most of the models presented contain excessive slow poles (e.g., Wahl and Harriot, 1970, Sandelin et al., 1991).

Skogestad and Morari (1988) suggested to use a two time-constant model with the dominant time-constant τ_1 for the high-gain direction ($dL = -dV$) and a smaller time-constant τ_2 for the low-gain direction ($dL = dV$) (model N2)

$$\begin{pmatrix} dy_D \\ dx_B \end{pmatrix} = \begin{pmatrix} \frac{k_{11}}{1+\tau_1 s} & \left(\frac{k_{11}+k_{12}}{1+\tau_2 s} - \frac{k_{11}}{1+\tau_1 s}\right) \\ \frac{k_{21}}{1+\tau_1 s} & \left(\frac{k_{21}+k_{22}}{1+\tau_2 s} - \frac{k_{21}}{1+\tau_1 s}\right) \end{pmatrix} \begin{pmatrix} dL \\ dV \end{pmatrix} \tag{13}$$

For the column in Example 3, $\tau_1 = 194$ min and $\tau_2 = 15$ min The minimal realization of this model contains two poles (not four as one might expect), one "slow" at $-1/\tau_1$ and one "fast" at $-1/\tau_2$, and gives an excellent fit of a full 41th order model which results from neglecting flow dynamics. The N2 model also agrees well with the 82th order full model with flow dynamics for the case of one-point control. This is seen from curve N2 in Figure 8. However, model N2 (13) does not include flow-dynamics and may therefore be relatively poor for two-point control studies (Jacobsen et al., 1991). To improve the model one may try to add flow-dynamics $g_L(s)$ to (13).

$$\begin{pmatrix} dy_D \\ dx_B \end{pmatrix} = \begin{pmatrix} \frac{k_{11}}{1+\tau_1 s} & \left(\frac{k_{11}+k_{12}}{1+\tau_2 s} - \frac{k_{11}}{1+\tau_1 s}\right) \\ \frac{k_{21}}{1+\tau_1 s} g_L(s) & \left(\frac{k_{21}+k_{22}}{1+\tau_2 s} - \frac{k_{21}}{1+\tau_1 s}\right) \end{pmatrix} \begin{pmatrix} dL \\ dV \end{pmatrix} \tag{14}$$

This model, denoted F2, yields good results for two-point control studies as reported by Jacobsen et al. (1991). However, adding the lag term $g_L(s)$ to the 2,1 element of (13) does again result in a minimal realization with two slow poles at $-1/\tau_1$, and therefore the model is poor for one point control studies. This is seen from curve F2 in Figure 8 where we again get an incorrect slow settling towards the new steady-state for the uncontrolled output.

In fact, we have not been able to obtain a simple low-order model for high-purity columns which includes flow-dynamics and is consistent in terms of the number of slow poles. In particular, it seems difficult to include the effect of disturbances, i.e., inputs not used for control. On the other hand, by mathematical model reduction of the full linear model it is possible for this example to obtain low-order models with only two states which are good for both one- and two-point control studies (Jacobsen et al., 1991). For instance, applying the Optimal Hankel Approximation without balancing (Safonov et al., 1987) to the full 82 order linear model (with flow-dynamics included) yields the model

(Jacobsen et al., 1991)

$$\begin{pmatrix} dy_D \\ dx_B \end{pmatrix} = \frac{1}{(1.61s+1)(194s+1)} \begin{pmatrix} 0.871(2.17s+1) & -0.861(0.721s+1) \\ 1.089(0.45s+1) & -1.101(3.48s+1) \end{pmatrix} \begin{pmatrix} dL \\ dV \end{pmatrix} \quad (15)$$

Note that the small time-constant of 1.61 min is much smaller than the value τ_2= 15 min used in model (13) which applies when flow dynamics are neglected. The model (15) has a minimal realization with only one slow pole at $-1/\tau_1$ and is a good model for both one- and two-point control studies. However, the model structure resulting from mathematical model reduction is not parameterized in terms of a few physical parameters and is therefore difficult to use for fitting experimental data. For instance, in the model (15) the decoupling at intermediate and high frequencies, physically caused by the flow dynamics, is described by letting the zeros in the off-diagonal elements be smaller than in the diagonal elements.

Nevertheless, based on the model (15), one possible model structure for obtaining consistent low-order models is the following

$$G(s) = \frac{1}{(\tau_1 s+1)(\tau_2 s+1)} \begin{pmatrix} k_{11}(z_{11}s+1) & k_{12}(z_{12}s+1) \\ k_{21}(z_{21}s+1) & k_{22}(z_{22}s+1) \end{pmatrix} \quad (16)$$

There are a number of parameters in this model and we discuss below how these may be obtained.

The steady-state gains k_{ij} and the dominant time-constant τ_1 may be obtained from open-loop experiments. However, since it is difficult to observe the low-gain direction of the plant (Skogestad and Morari, 1988, Andersen et al., 1989) one may for high-purity columns with large RGA-values easily get the wrong sign of the RGA (and the determinant) of the steady-state gain matrix, K. The model will then be useless for feedback control purposes. This may be corrected by performing separate experiments for changes in internal flows (e.g., by using the DV-configuration as suggested by Skogestad, 1988, Alsop and Edgar, 1990, Andersen and Kümmel, 1991, Kuong and McGregor, 1991), or by adjusting the steady-state gains to match an estimated steady-state RGA-value as suggested by Jacobsen et al. (1991), or by using a "perturbed model" (Kapoor et al., 1986) based on steady-state where the RGA-elements are smaller. The basis for the two last suggestions is that the steady-state behavior is not of primary importance for feedback control.

Care should be taken to obtain separate data for the flow dynamics so as to get the correct decoupling at intermediate and high frequencies, and the zeros, z_{ij}, and the small time-constant, τ_2, in (16) may be adjusted to match this behavior. Note that only two of the zeros may be adjusted independently as we require a model with only two states, that is, we must have two multivariable zeros at $-1/\tau_1$ and $-1/\tau_2$, respectively.

6 Discussion

The linear process model is often written

$$y(s) = G(s)u(s) + G_d(s)d(s) \quad (17)$$

So far in this paper we have discussed obtaining low-order models, $G(s)$, for the effect of inputs. However, the main purpose of process control is to reject the effect of disturbances, d, entering the process, and so a disturbance model, $G_d(s)$, is usually also required. The common approach is to identify the disturbance model $G_d(s)$ independently of $G(s)$ (e.g., Shunta and Luyben, 1972, Waller et al., 1988), as is also suggested from the model form in (17). However, in reality, these models are often directly coupled and share the same states. For the processes studied in this paper the single dominating pole seen in $G(s)$ will also appear in $G_d(s)$ and will dominate the open-loop responses to disturbances. If the fitted disturbance model $G_d(s)$ is inconsistent with the input model $G(s)$, then these inconsistencies may appear under feedback control. For instance, using the low-order model in example 3 with flow-dynamics included, e.g. (12), we find that the responses to setpoint changes are reasonably correct compared to the full model when using a properly tuned two-point controller. However, with a pure first-order model for disturbances in feed flow rate F, we find that the response to a disturbance in F is erroneous due to inconsistent slow poles left in the closed-loop disturbance model. Thus, care needs to be taken also when identifying the disturbance model of an ill-conditioned process.

The processes studied in this paper contain only one slow pole, and since a rank 2 model requires two states, we need to identify at least one more time-constant. For the heat-exchanger example the correct model is given by (1) and the small time-constant τ_2 may be identified by doing a simultaneous step increase in hot and cold flow of the same magnitude, i.e., applying the input vector $[1 \quad 1]^T$. However, this is impossible in practice, and with a small deviation in the magnitude the low-gain direction disappears. Due to this one should try to explicitly model the "weak" direction of the plant and be careful about only fitting open-loop data.

Although all the process studied in this paper contains only a single slow pole, there are of course many processes that contain several slow poles. When doing open-loop identification it is necessary to know in advance how many slow poles the process actually contains. In a well designed high-purity distillation column there will usually only be one slow pole, while an overdesigned high-purity column with a pinch in the composition profile usually will have two slow poles (one pole related to each of the two column sections). An open-loop black-box identification method will not be able to discriminate between the two cases, and some physical knowledge needs to be added. Closed-loop identification, on the other hand, is likely to reveal the fact that there is indeed a single dominant time-constant in the process. Thus, for ill-conditioned processes where physical knowledge is lacking, one should apply some closed-loop identification method when obtaining low-order models.

The general literature on identification has so far not treated multivariable issues in much detail. Actually, we doubt that there exists identification algorithms that will be able to identify, based on "realistic" (noisy) data generated from the model in Eq. (1), a reasonable model that may be used for control studies. In particular, we believe this is the case if the model structure and order is unknown, and even if it is known we believe it will be very difficult.

7 Conclusions

- The open-loop responses of ill-conditioned processes will often take the form of almost pure first-order dynamics. The responses are dominated by a single slow pole resulting from interactions in the process. The open-loop dynamics of such processes are seamingly well approximated by a low-order model containing only the dominant time-constant. However, the model will contain an excessive number of slow poles and is therefore physically inconsistent.

- It is sufficient to close one feedback loop in order the move the single slow pole of the process, and thus make all the outputs (both the controlled output and the uncontrolled) respond faster. On the other hand, for inconsistent models containing excessive slow poles, only one of the slow poles is moved with one feedback loop closed. Thus, erroneous slow responses may appear for the uncontrolled outputs when an inconsistent model is used.

- We have demonstrated that it is difficult to define a physically motivated low-order model structure for high-purity distillation columns which contains the flow-dynamics and is consistent in terms of the number of slow poles.

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NOMENCLATURE

A - heat transfer area (m^2)
c_P - heat capacity ($kJ/^\circ C kg$)
D - distillate flow (kmol/min)
d - process disturbance vector
F - feed flow (kmol/min)
$G(s)$ - process transfer-matrix for effect of inputs u
$G_d(s)$ - process transfer-matrix for effect of disturbances d
$g_{ij}(s)$ - transfer matrix element i,j
k_{ij} - steady state process gains
I - identity matrix
K - controller gain
L - reflux rate ($kmol/min$)
N - number of theoretical trays
N_F - feed tray
q_C - cold inlet flow (m^3/min)
q_H - hot inlet flow (m^3/min)
T_C - cold outlet temperature ($^\circ C$)
T_H - hot outlet temperature ($^\circ C$)
U - heat transfer coefficient ($kJ/m^2\ ^\circ C min$)
u_i - process input i
V - boilup rate ($kmol/min$)
V_C - liquid volume cold side (m^3)
V_H - liquid volume hot side (m^3)
x_B - bottoms composition
$Y = \frac{k_{12}k_{21}}{k_{11}k_{22}}$ - interaction measure
y_D - distillate composition
y_i - process output i
z_F - feed composition

Greek symbols
α - relative volatility
λ_{11} - 1,1 element of RGA
τ_1 - dominant (largest) process time-constant (min)
τ_2 - smaller process time-constant (min)
τ_{CL} - closed-loop time-constant (min)

Subscripts
s - setpoint change

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APPENDIX. Simple model of heat exchanger.

Consider a simplified heat exchanger with one mixing tank on each side as shown in Figure 1. Assume constant volumes, V, on each side, and constant values of ρ and c_P. A heat balance for the cold and hot side then yields

$$\tau_C \frac{dT_C}{dt} = \frac{q_C}{q_C^*}(T_{Ci} - T_C) + \alpha_C(T_H - T_C) \tag{18}$$

$$\tau_H \frac{dT_H}{dt} = \frac{q_H}{q_H^*}(T_{Hi} - T_H) - \alpha_H(T_H - T_C) \tag{19}$$

where q^* denotes the nominal (steady-state) flow, and

$$\tau_C = \frac{V_C}{q_C^*}; \quad \alpha_C = \frac{UA}{\rho_C q_C^* c_{PC}} \tag{20}$$

$$\tau_H = \frac{V_H}{q_H^*}; \quad \alpha_H = \frac{UA}{\rho_H q_H^* c_{PH}} \tag{21}$$

Linearizing the model assuming UA and thus α constant (independent of flow and temperature), introducing deviation variables, and taking Laplace tranforms yields

$$\tau_C s T_C(s) = T_{Ci}(s) - T_C(s) + (T_{Ci}^* - T_C^*)\frac{q_C(s)}{q_C^*} + \alpha_C(T_H(s) - T_C(s)) \tag{22}$$

$$\tau_H s T_H(s) = T_{Hi}(s) - T_H(s) + (T_{Hi}^* - T_H^*)\frac{q_H(s)}{q_H^*} - \alpha_H(T_H(s) - T_C(s)) \tag{23}$$

where the superscript $*$ denotes steady-state values. In the following we will assume $\tau_C = \tau_H = \tau = 100$ [min], $\alpha_C = \alpha_H = \alpha = 20$ and $q_C^* = q_H^* = q^* = 0.01$ [m^3/min] (see data in Table 1). Rearranging yields

$$\begin{pmatrix} T_C(s) \\ T_H(s) \end{pmatrix} = G(s)\begin{pmatrix} q_C(s) \\ q_H(s) \end{pmatrix} + G_d(s)\begin{pmatrix} T_{Ci}(s) \\ T_{Hi}(s) \end{pmatrix} \tag{24}$$

where

$$G_d(s) = \frac{1}{(\tau s + 1)(\tau s + 1 + 2\alpha)}\begin{pmatrix} \tau s + 1 + \alpha & \alpha \\ \alpha & \tau s + 1 + \alpha \end{pmatrix} \tag{25}$$

and

$$G(s) = G_d(s)\begin{pmatrix} (T_{Ci}^* - T_C^*)/q_C^* & 0 \\ 0 & (T_{Hi}^* - T_H^*)/q_H^* \end{pmatrix} \tag{26}$$

Inserting the numerical values finally yields

$$G_d(s) = \frac{0.02439}{(100s + 1)(2.439s + 1)}\begin{pmatrix} 21(1 + 4.76s) & 20 \\ 20 & 21(1 + 4.76s) \end{pmatrix} \tag{27}$$

$$G(s) = G_d(s)\begin{pmatrix} -3659 & 0 \\ 0 & 3659 \end{pmatrix} \tag{28}$$

Table 1. Steady-state data for heat-exchanger in Example 1 (see also Figure 1).

$V_H = V_C$ $[m^3]$	$q_C = q_H$ $[m^3/min]$	T_{Ci} $[^oC]$	T_{Hi} $[^oC]$	T_C $[^oC]$	T_H $[^oC]$	UA $[kJ/^oC min]$	ρ $[kg/m^3]$	c_P $[kJ/^oC kg]$
1	0.01	25	100	61.59	63.41	300	500	3.0

c_P and ρ are equal for the hot and cold side.

Table 2. Steady-state data for distillation column in Example 3. (see also Figure 5)

z_F	α	N	N_F	$1-y_D$	x_B	D/F	L/F	V/F
0.5	1.5	40	21	0.01	0.01	0.500	2.706	3.206

Feed F is liquid.

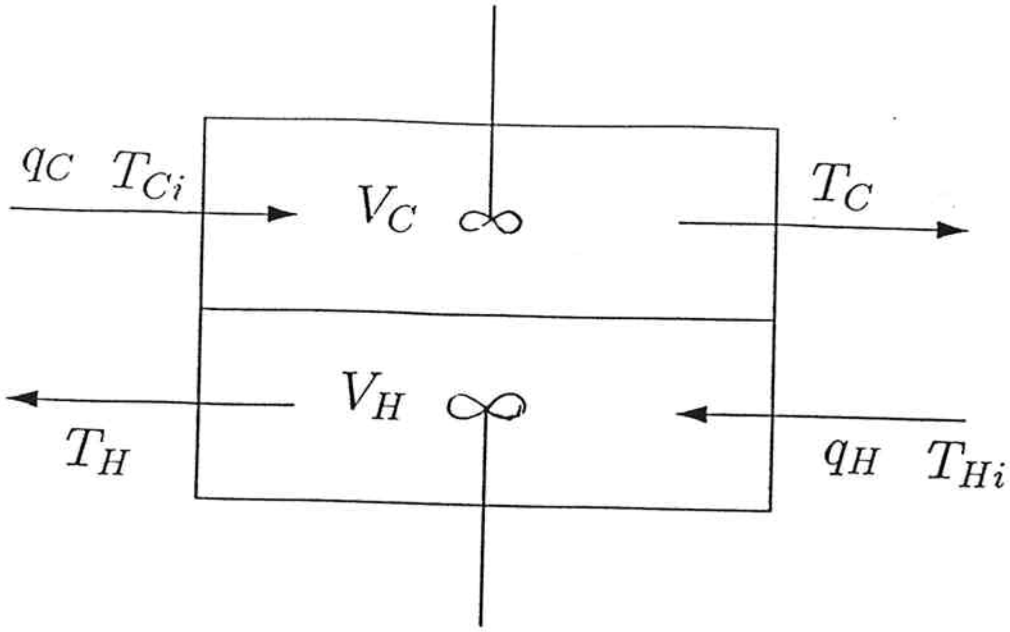

Figure 1. Simplified representation of heat-exchanger with one mixing tank on each side.

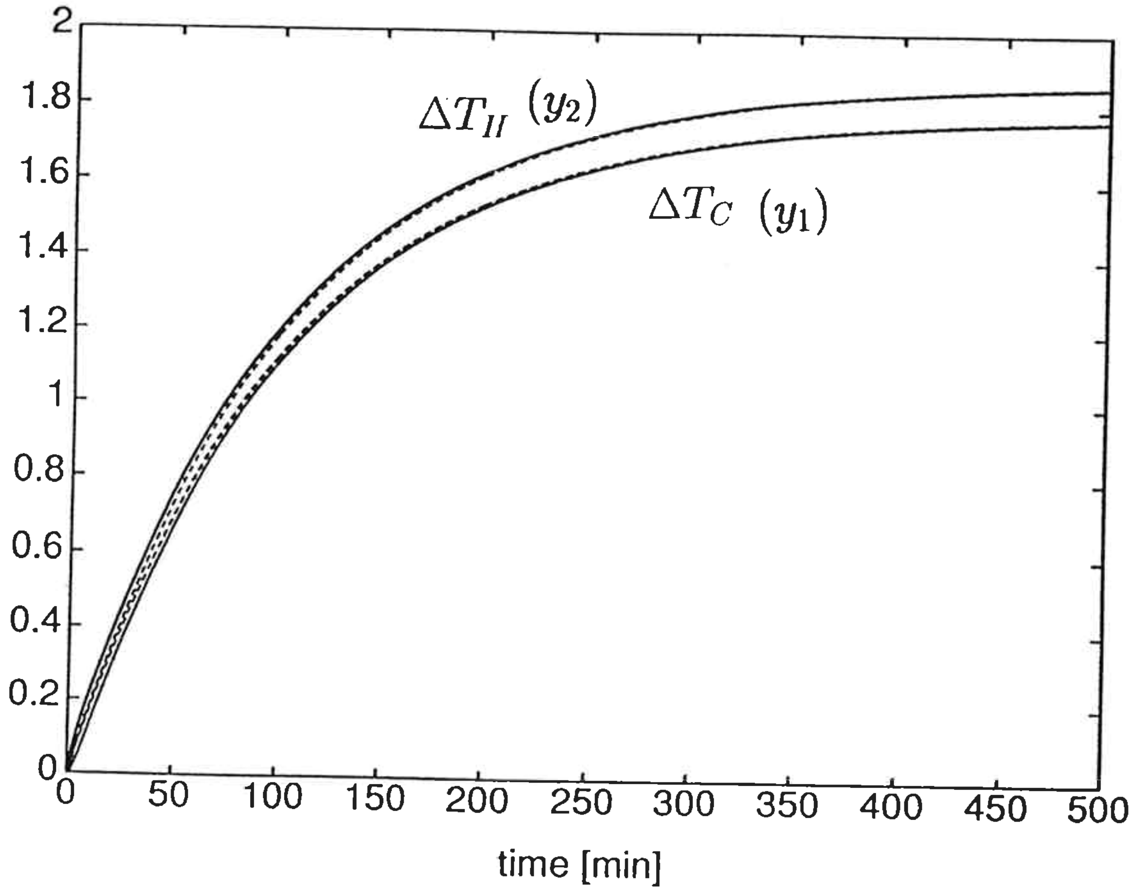

Figure 2. Open-loop dynamic response of heat-exchanger. Responses in outlet temperatures T_C and T_H to a 10 % step increase in hot inlet flow q_H. Solid line: Response of full model (1). Dashed line: Response of fitted model (2).

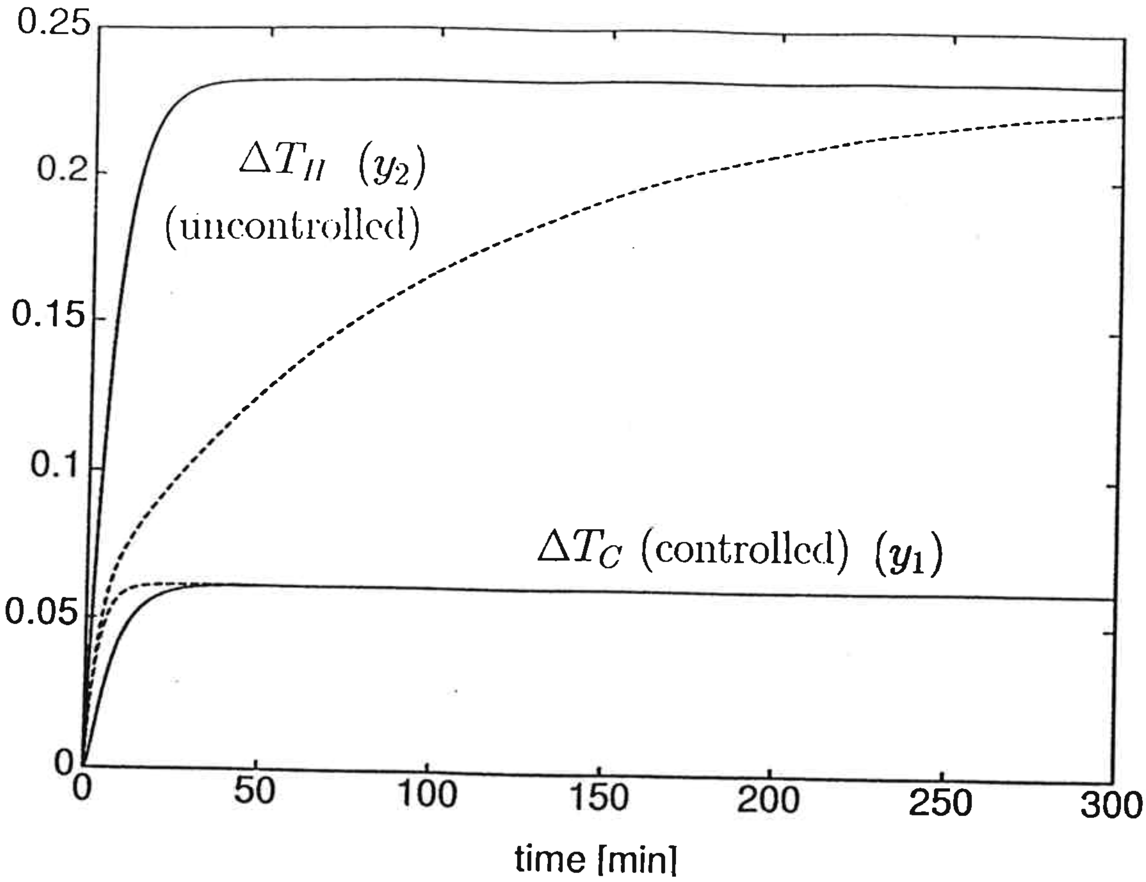

Figure 3. Dynamic response of heat-exchanger with one loop closed. Responses in outlet temperatures to a 10 % step increase in q_H. Cold outlet temperature T_C is controlled by q_C using a pure proportional controller with gain $K = 0.015$. Solid line: Response of full model (1). Dashed line: Response of fitted model (2).

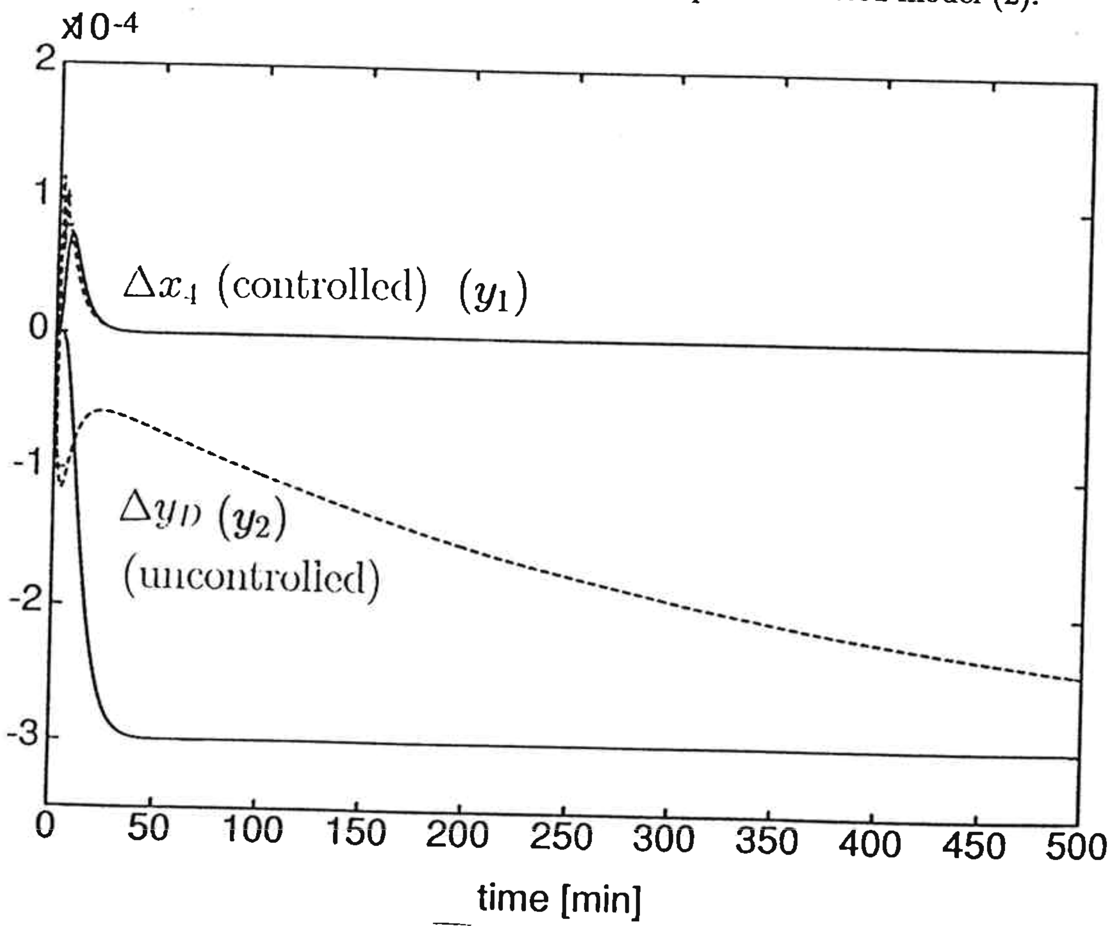

Figure 4. Wahl and Harriot column. Response in top-composition y_D and x_4 to a disturbance in feed composition with x_4 controlled by reflux. Dashed lines: Simulations with low-order model given by Wahl and Harriot (1970). Solid lines: Simulations with full linear model.

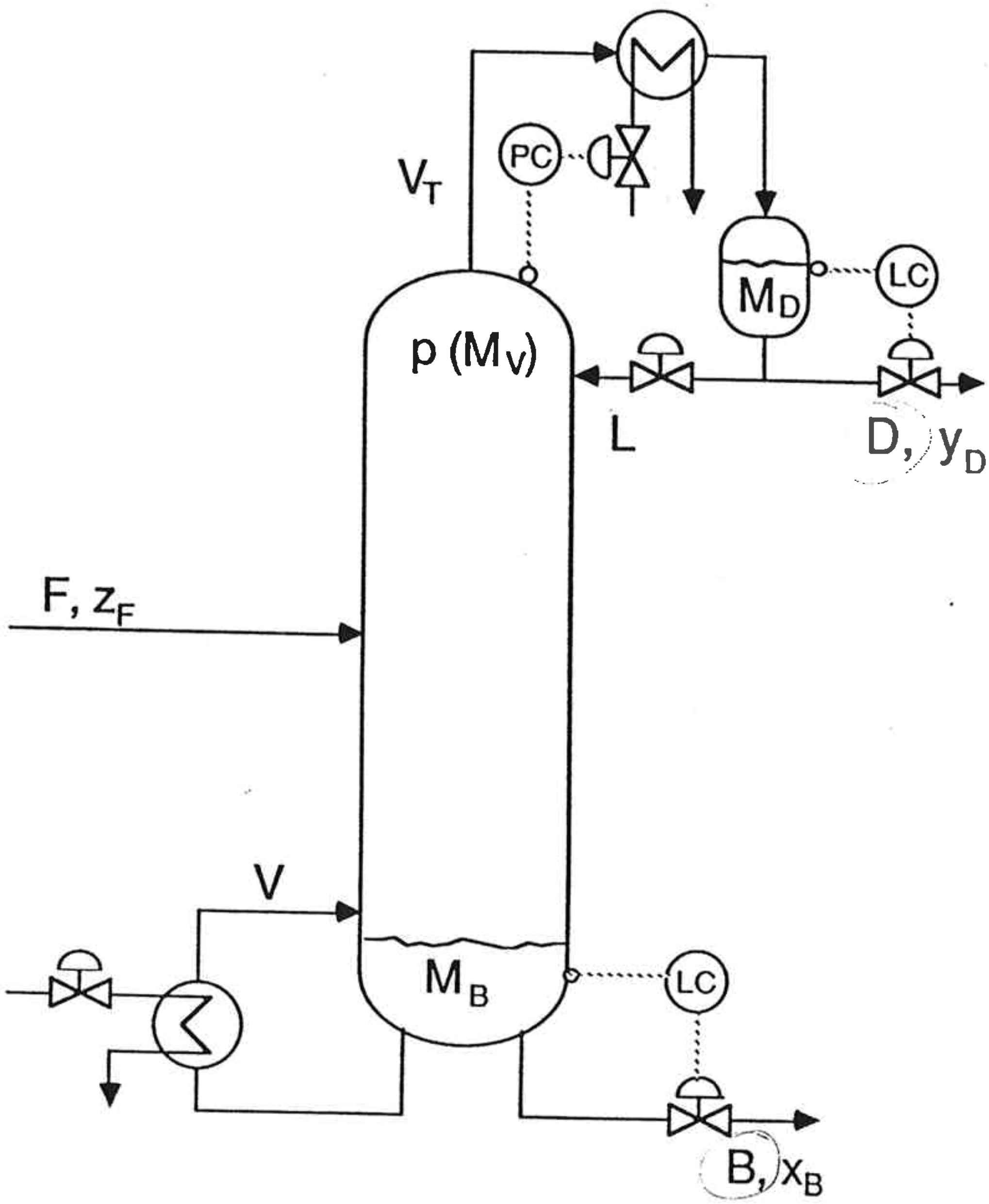

Figure 5. Two-product distillation column with reflux L and boilup V as independent variables.

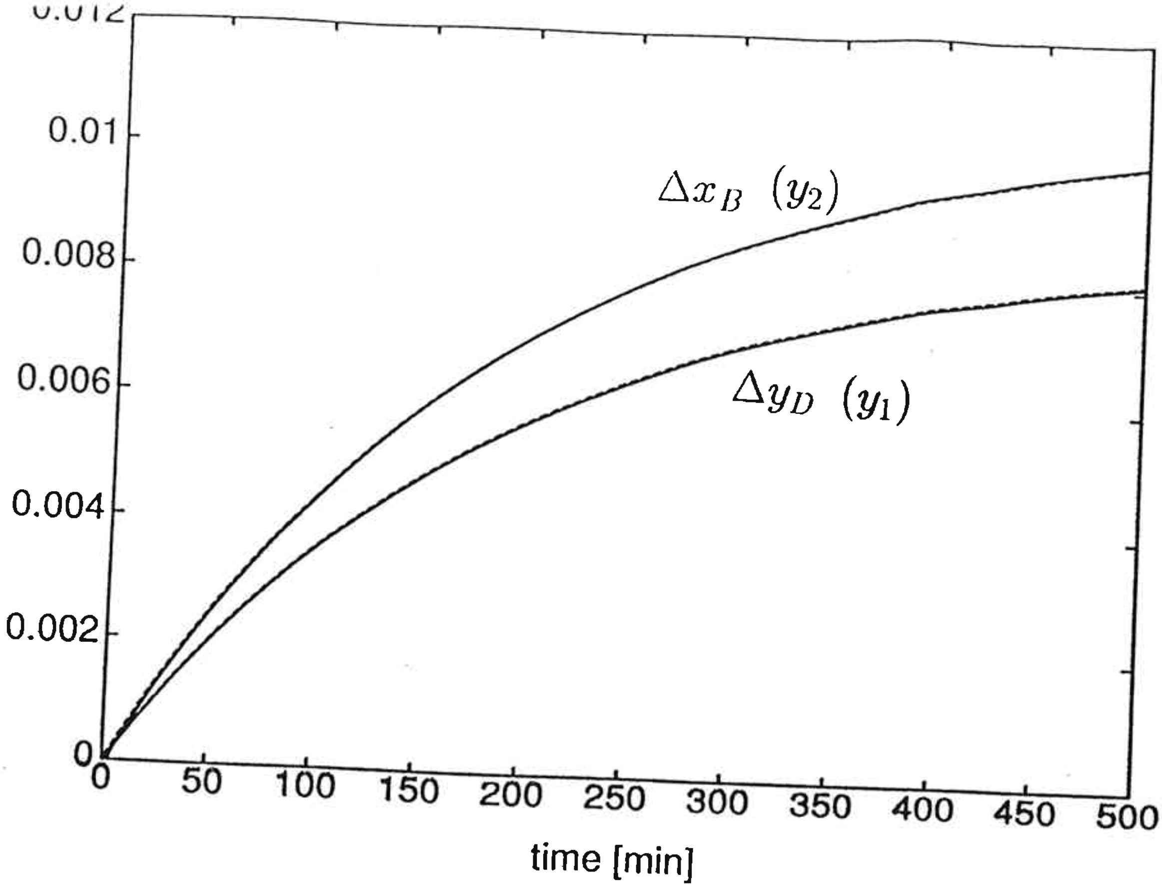

Figure 6. Open-loop responses of distillation column in Example 3 to 1 % step change in reflux L. Solid lines: responses of full linear model. Dashed lines: responses of fitted low-order model N1 (11).

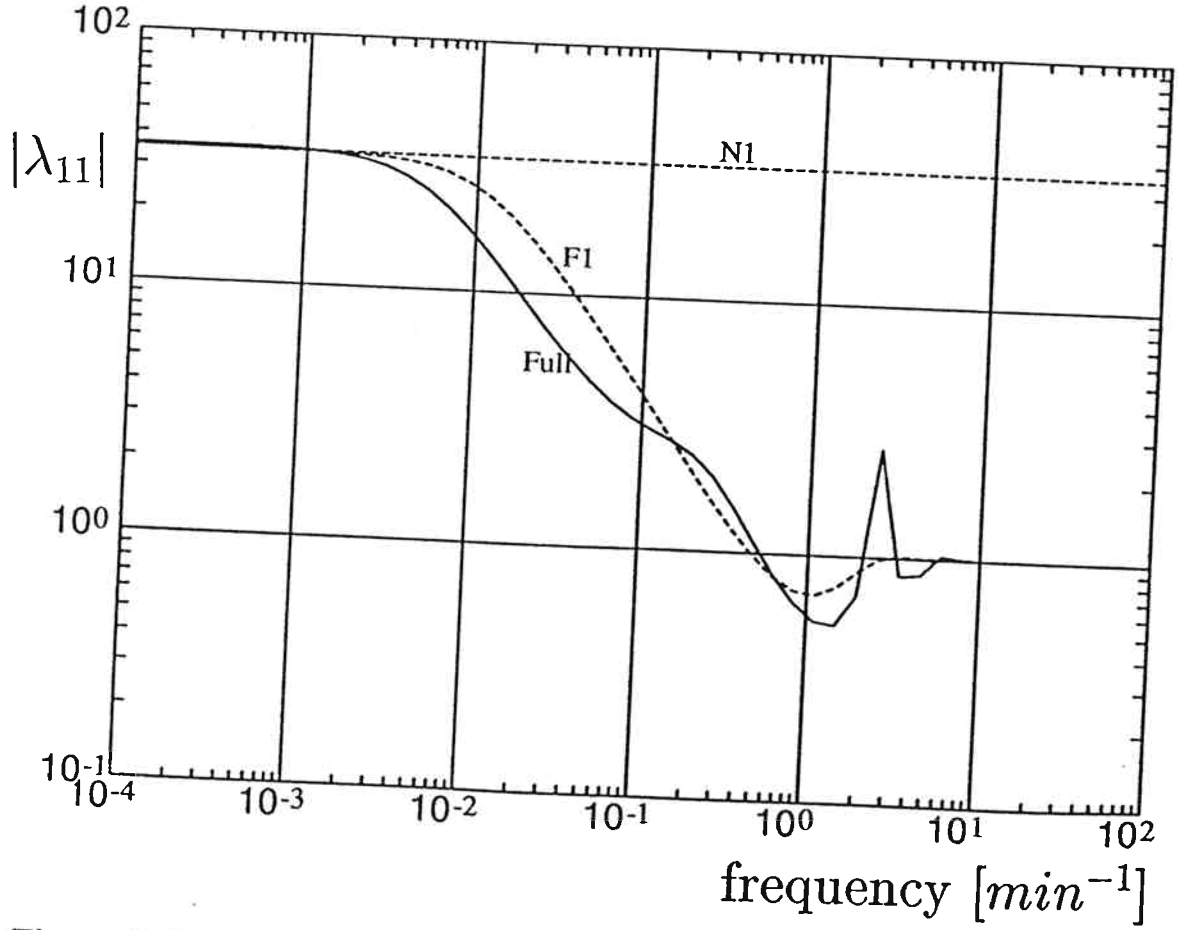

Figure 7. RGA as a function of frequency for distillation column in Example 3. N1: fitted low-order model without flow-dynamics (11). F1: fitted low-order model with flow-dynamics included (12). Full: full 82th order linear model.

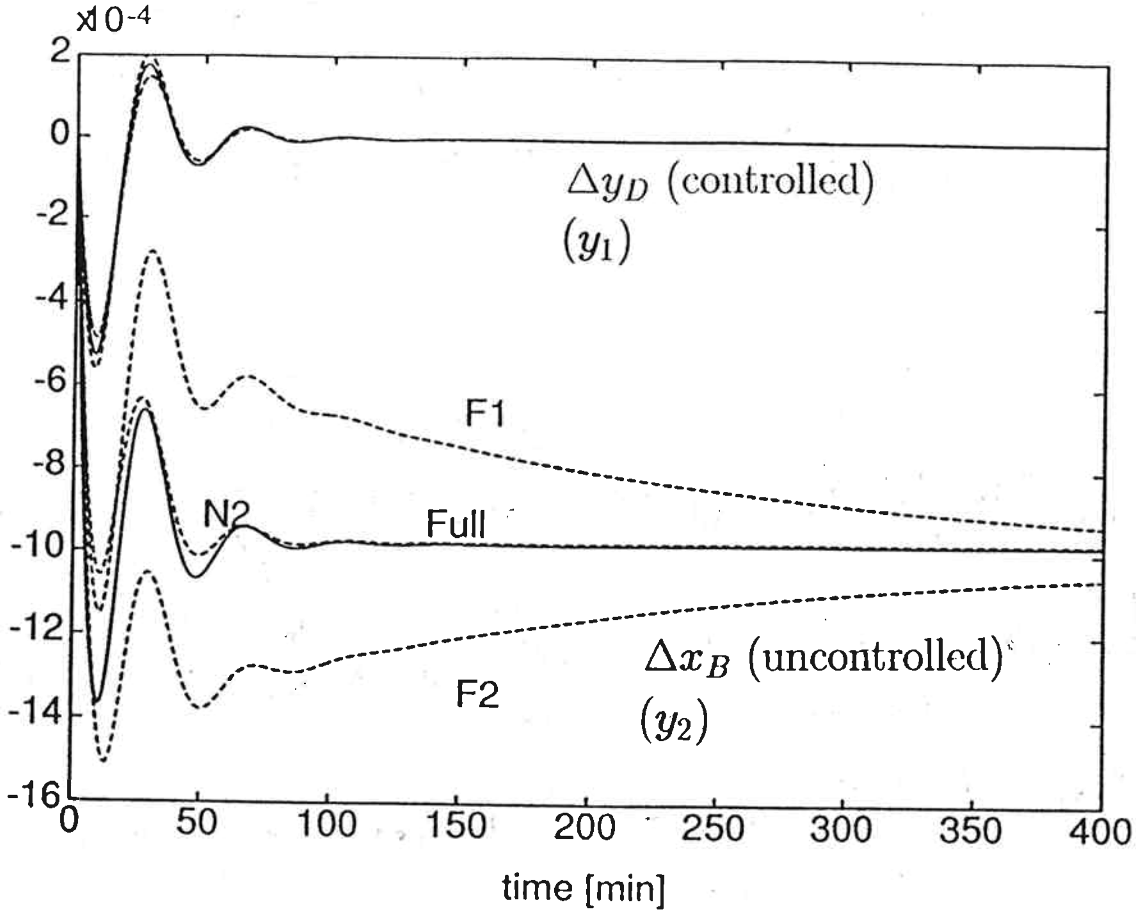

Figure 8. Dynamic response of distillation column in Example 3 with one loop closed. Response to a 1 % increase in boilup. Top composition y_D controlled by reflux L using a PI-controller. Full: full 82th order linear model. F1: fitted low-order model with flow-dynamics included (12). N2: two time-constant model (13) F2: two time-constant model (13) with flow-dynamics included. $F1_{red}$ - model resulting from reducing model F1 down to two states.