Weight selection for H-infinity and mu-control methods - Insights and examples from process control

Petter Lundström, Sigurd Skogestad *and Zi-Qin Wang
Chemical Engineering
University of Trondheim, NTH
N-7034 Trondheim, Norway

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Abstract

The objective of the paper is to give some insight into the practical use of H-infinity (H_∞) and mu methods. Mu (or μ) is the structured singular value (SSV) introduced by Doyle (1982). Controller synthesis software using H_∞-methods is now readily available. Mu is a very powerful tool for *analysis* of control systems. However, to effectively use this tool one must be able to choose reasonable performance and uncertainty weights. One of the main topics in this paper is performance weight selection. Mu may also be used for controller synthesis. However, mu-synthesis is generally not a convex optimization problem and is presently not straightforward. We will discuss some of the problems we have encountered.

1 Introduction

This paper is based on the paper on ill-conditioned plants by Skogestad, Morari and Doyle (1988), and the reader is referred to that paper for notation and background material. We shall use the same simplified distillation column (the LV-configuration) as our example. In the previous paper the effect of various uncertainty descriptions was studied, but here we mainly discuss alternative choices for the performance weight.

Performance is defined using the H_∞-norm. For a stable transfer function $M(s)$ the H_∞-norm is given by

$$\|M\|_\infty = \sup_\omega \bar{\sigma}(M(j\omega)) \tag{1}$$

where $\sup_\omega$ denotes the peak value over all frequencies. The H_∞-norm may be viewed as a direct generalization of the frequency domain performance specifications used in classical control for single-input single-output (SISO) systems. There are several physical interpretations (Doyle, 1987) of this norm which give rise to different procedures for selecting performance weights. This is discussed in detail below.

In many cases it would be more natural to define performance in the time domain (eg., maximum deviation, maximum overshoot for step response, etc.). However, analysis and design techniques are

*Author to whom correspondence should be addressed. E-mail: skoge@kjemi.unit.no, Phone: 47-7-594154, Fax: 47-7-591410

not readily available. Another option is the traditional H_2-norm where good analysis and synthesis techniques exist (eg., LQG control through solution of Riccati equations). Recently, Doyle et al. (1989) have shown that also the analysis and design using the H_∞-norm for performance may be solved using similar methods. This has made the H_∞-approach more reliable and simpler from a computational point of view.

It is clear from the discussion above that the main reason for using the H_∞-norm is usually *not* because it is the most natural framework to define performance. However, other design specifications, specifically model uncertainty, may be formulated readily within this framework. In particular, this applies to uncertain or neglected high-frequency dynamics which is always present. Because the uncertainty usually is structured (stems from specific physical sources), one must use multiple perturbations, Δ_i. Using weights, each of these may be bounded such that the H_∞-norm, $\|\Delta_i\|_\infty$, is less than 1. To test for robust stability the system is rearranged such that N_{RS} (which includes the uncertainty weights) represents the interconnection matrix from the outputs to the inputs of the uncertainty-blocks, Δ. In the following we assume that N_{RS} is stable. Using the small gain theorem, we know that robust stability will be satisfied if $\|N_{RS}\|_\infty < 1$, or equivalently

$$\text{RS} \quad \text{if} \quad \bar{\sigma}(N_{RS}) < 1; \quad \forall\omega \tag{2}$$

However, this bound is generally conservative. First, the issue of stability should be independent of scaling. Thus, an improved robust stability condition is

$$\text{RS} \quad \text{if} \quad \min_{D(\omega)} \bar{\sigma}(D N_{RS} D^{-1}) < 1; \quad \forall\omega \tag{3}$$

where D is a real block-diagonal scaling matrix with structure corresponding to that of Δ, such that $\Delta D = D\Delta$. A further refinement of this idea led to the introduction of the structured singular value (Doyle, 1982, 1987). We have (essentially, this is the defintion of μ)

$$\text{RS} \quad \text{iff} \quad \mu(N_{RS}) < 1; \quad \forall\omega \tag{4}$$

Thus $\min_D \bar{\sigma}(DMD^{-1})$ is an upper bound on $\mu(M)$. It is usually very close in magnitude. The largest deviation reported so far is about 10-15% (Doyle, 1982, 1987). Computationally tractable lower bounds for μ also exist and are in common use.

An additional bonus of using the H_∞-norm both for performance and uncertainty is that the robust performance problem may be recast as a robust stability problem (Doyle, 1982), with the performance specification represented as a fake uncertainty block. To test for robust performance one considers the interconnection matrix N_{RP} from the outputs to the inputs of *all* the Δ-blocks, including the "full" Δ_P-block for performance. N_{RP} depends on the plant G, the controller C and on the weights used to define uncertainty and performance. The condition for robust performance within the H_∞-framework is (for stable N_{RP})

$$\text{RP} \quad \text{iff} \quad \mu(N_{RP}) < 1; \quad \forall\omega \tag{5}$$

Analysis of robust performance for a given controller using μ is straightforward, but controller design using μ-synthesis is still rather involved. The present "DK-iteration" uses the upper bound on μ, and involves solving a number of "scaled" H_∞-problems. We will discuss this further in section 6.

One alternative is to use another method (eg., H_2 or H_∞) for controller synthesis and iterate on the tuning parameters ("weights") in this method until acceptable μ-values are obtained. For H_∞-synthesis one may minimize a "mixed" H_∞-norm for nominal performance and robust stability by considering, for example

$$T = \begin{bmatrix} W_1(I+GC)^{-1} \\ W_2GC(I+GC)^{-1} \end{bmatrix} \tag{6}$$

This is the problem discussed by Chiang and Safonov (1988, 1990). A similar mixed objective is studied by Yue and Postlethwaite (1988). Minimizing the H_∞-norm of (6) corresponds to defining nominal performance in terms of weighted sensitivity and robust stability in terms of unstructured output uncertainty. For this particular problem minimizing the H_∞-norm of T is actually equivalent to minimizing $\mu(T)$. However, for many plants, in particular ill-conditioned plants, uncertainty at the inputs (which is always present) is much more restrictive in terms of performance. The "mixed" problem with input uncertainty is obtained by using $W_2CG(I+CG)^{-1}$ rather than $W_2GC(I+GC)^{-1}$ in the expression for T. In this case minimizing the H_∞-norm of T yields nominal performance and robust stability, but the overall system may not satisfy robust performance. This *may* be the case if the condition number of the plant *and* the controller is large (see eq. 32 in Skogestad et al., 1988). For example, this applies to the example column studied by Skogestad et al. (1988, Fig. 11). In such cases one would have to use non-physical weights, W_1 and W_2, to get a "good" final controller (with robust performance) by minimizing the H_∞-norm of T.

Software to synthesize H_∞-controllers has been available for some time, for example, through the Robust Control toolbox in MATLAB (Chiang and Safonov, 1988). Recently, a μ-toolbox for MATLAB has become available (Balas et al., 1990). This toolbox includes alternative H_∞-software, and μ-analysis and synthesis is included as outlined above. All computations presented in this paper have been done employing this toolbox.

2 Uncertainty weights

Since uncertainty modelling using the H_∞-framework is a worst-case approach, one should generally not include too many sources of uncertainty, since it otherwise becomes very unlikely for the worst case to occur in practice. One should therefore lump various sources of uncertainty into a single perturbation whenever this may be done in a non-conservative manner. On the other hand, one should be careful about excluding physically meaningful uncertainty that limit achievable performance. From this it follows that selecting appropriate uncertainty weights is very problem-dependent, and it is important that guidelines for specific classes of problems be developed.

Sometimes one might use a smaller uncertainty set for robust performance than for robust stability. The idea is to guarantee stability for a large set of possible plants, but require performance only for a subset. This is to avoid very conservative designs with poor nominal performance.

In this paper we only consider input uncertainty. The effect of output uncertainty, time constant uncertainty and correlated gain uncertainty was studied by Skogestad et al. (1988). They found that these sources of uncertainty were less important than the input uncertainty for this particular ill-conditioned plant.

Combining gain uncertainty and time delay uncertainty. Consider input uncertainty in each separate channel. We want to show how gain uncertainty and time delay uncertainty may be combined in a single perturbation. Let ϵ denote the allowed steady-state relative gain error on each input, and let θ be the allowed time delay error (may also represent neglected dynamics corresponding in phase lag to a time delay of θ). If a time delay $e^{-\theta s}$ is negected, that is, modelled as 1, then a close approximation on the the magnitude of the relative error $(e^{-\theta s}-1)/1$ is given by $2\frac{\theta}{2}s/(1+\frac{\theta}{2}s)$ (using a first-order Pade approximation). Adding this to the constant term ϵ and assuming $\epsilon << 2$ yields the following weight for the overall multiplicative (relative) input uncertainty

$$w_I(s)=\epsilon\frac{\frac{\theta}{\epsilon}s+1}{\frac{\theta}{2}s+1} \tag{7}$$

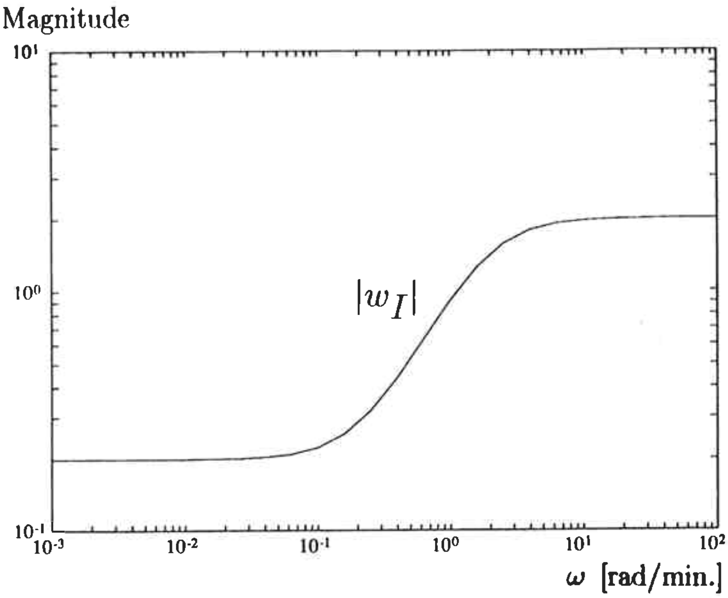

Figure 1: Uncertainty weight, $|w_I(j\omega)|$ (Eq.7), with ϵ=0.2 and θ=1 min.

To obtain the specific weight used in the example choose $\epsilon = 0.2$ (20%) and θ=1 min. This weight is shown graphically as a function of frequency in Fig.1. The asymptote of the weight reaches one (corresponding to 100% uncertainty) approximately at a frequency $1/\theta$, and then levels off at two (200% uncertainty). In some cases it may be better to let the weight continue increasing at higher frequencies, that is, use a weight $w_I(s) = \epsilon(\frac{\theta}{\epsilon}s + 1)/(\frac{\theta}{p}s + 1)$ where p is a large number. This may be advantageous for μ-synthesis to avoid that a lot of additional states in the controller are needed to "take advantage" of the fact that the uncertainty levels off at 200% at high frequencies (of course, since the uncertainty is over 100% the overall response will only be marginally improved).

3 Performance weights

The are several different physical interpretations of the H_∞-norm (Doyle, 1987, Zhou et al., 1990), and this gives rise to different methods for weight selection.

A. Since $||M||_\infty = \sup_\omega \bar{\sigma}(M(j\omega)$ (provided $M(s)$ is stable), the H_∞-norm may be viewed as a direct generalization of classical frequency-domain bounds on transfer functions (loop-shaping) to the multivariable case.

B. Alternatively, we may consider $M(j\omega)$ as the frequency-by-frequency sinusoidal response. That is, for a unit sinusoidal input to channel j with frequency ω, the steady-state ouput in channel i is equal to $m_{ij}(j\omega)$. To consider all the channels combined, we use the maximum singular value, $\bar{\sigma}(M(j\omega))$, which gives the worst-case (with respect to choice of direction) amplification of a unit sinusoidal input of frequency ω through the system.

C. The induced norm from bounded power spectrum inputs to bounded power spectrum outputs in the time domain is equal to the H_∞-norm.

There are also other interpretations of the H_∞-norm: It is equal to the induced 2-norm (energy) in the time domain. It is equal to the induced power norm. It is also equal to induced norm in the time domain from signals of bounded magnitude to outputs of bounded power.

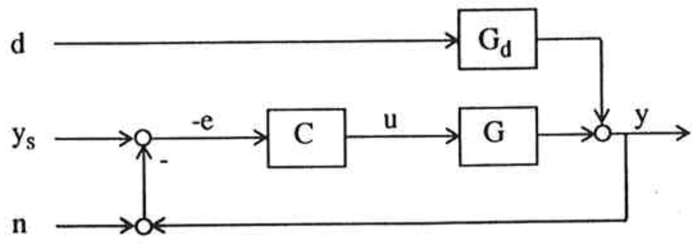

Figure 2: Block diagram of conventional feedback system.

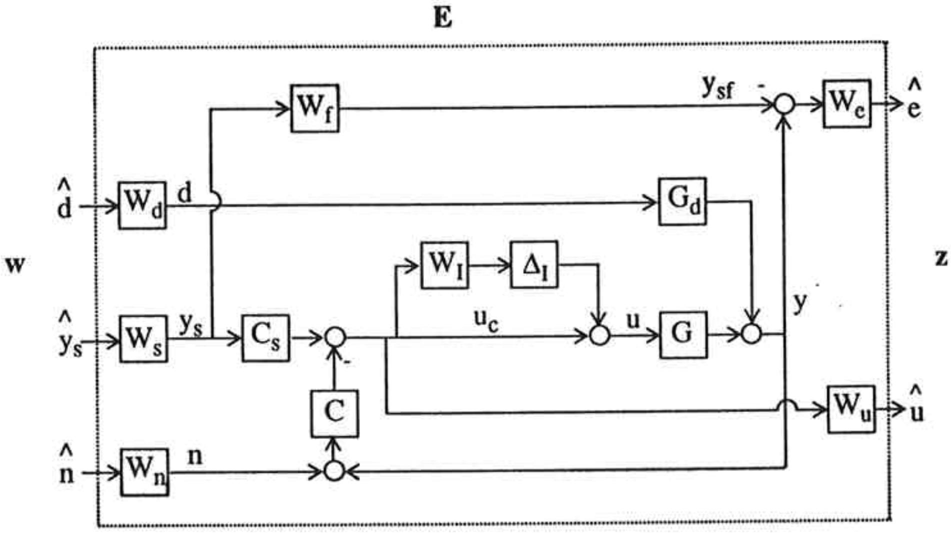

Figure 3: General feedback system with weights, a two-degree-of-freedom controller and input uncertainty. It is assumed that the outputs are measured directly. The transfer function E (with $\Delta_I = 0$) is used for general H_∞-performance.

The following discussion is mostly relevant to approaches B and C. A block diagram of a conventional feedback system with disturbances d, setpoints y_s and noise n is shown in Fig.2. The most general way to define performance within the H_∞-framework is to consider the H_∞-norm of the closed-loop transfer function E between the external weighted input vector w (disturbances, setpoints, noise) and the weighted output vector z (may include $y - y_s$, manipulated inputs u which should be kept small, etc.). Weights are chosen such that the magnitude (in terms of the 2-norm) of the normalized external input vector is less than one at all frequencies, ie. $\|w(j\omega)\|_\infty \leq 1$, and such that for acceptable performance the normalized output vector is less than 1 at all frequencies, ie. $\|z(j\omega)\|_\infty < 1$. With $z = Ew$ the performance requirement becomes

$$\|E\|_\infty = \sup_\omega \bar{\sigma}(E(j\omega)) < 1 \tag{8}$$

Introducing the weights W_d, W_s, W_n, W_e and W_u into Fig.2 yields the block diagram in Fig.3 where E is given as shown by the dotted box. We also have introduced an "ideal response" from y_s to y (W_f), and use a "two degree of freedom controller" (C, C_s). For the conventional case in Fig. 2, $C_s = C$ or $C_s = CC_f$, where C_f is a filter for the setpoints.

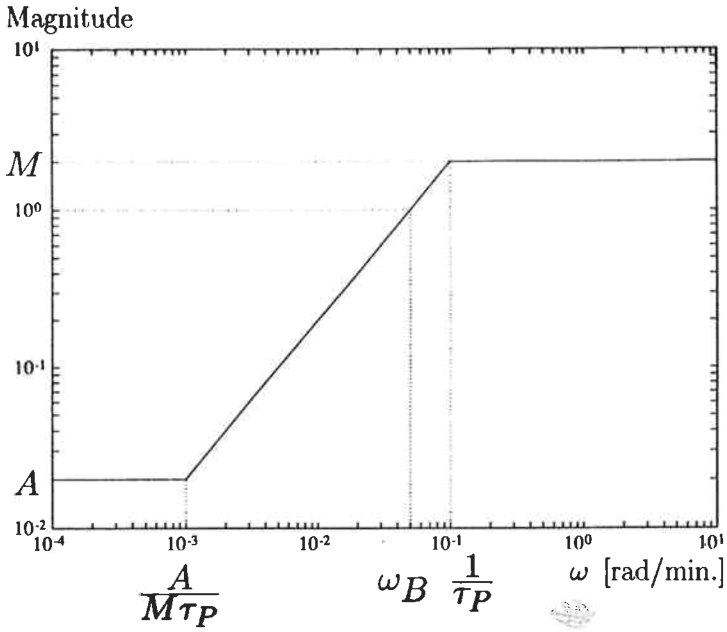

Figure 4: Asymptotic plot of $1/w_P = M\frac{\tau_P s+A/M}{\tau_P s+1}$ where $\tau_P = 1/M\omega_B$. $|S(j\omega)|$ should lie below $1/|w_P|$ to satisfy classical frequency-domain specifications in terms of A, M and ω_B.

3.1 Performance approach A. Weights on transfer functions (Loop shaping)

In many cases it is simple and instructive to translate the desired performance specifications into an upper bound $1/|w_P|$ on the frequency plot of the magnitude of the sensitivity function $S = (I+GC)^{-1}$.

$$\bar{\sigma}(S(j\omega)) < 1/|w_P(j\omega)|, \quad \forall\omega \tag{9}$$

This is equivalent to (8) with $E = w_P S$ (weighted sensitivity). The concept of bandwidth, which is here defined as the frequency ω_B where the asymptote of $\bar{\sigma}(S)$ first crosses one, is closely related to this kind of performance specification, and most classical frequency domain specifications may be captured by this approach.

Classical frequency domain specifications. For example, assume that the following specifications are given in the frequency domain

1. Steady-state offset less than A.
2. Closed-loop bandwidth higher than ω_B.
3. Amplification of high-frequency noise less than a factor M.

These specifications may be reformulated in terms of Eq.9 using

$$w_P(s) = \frac{1}{M}\frac{\tau_P s+1}{\tau_P s + A/M}, \qquad \text{with } \tau_P = 1/\ (M\omega_B) \tag{10}$$

and the resulting bound $1/|w_P(j\omega)|$ is shown graphically in Fig.4.

General case. In the multivariable case the generalized weighted sensitivity is $W_{P1}SW_{P2}$. For example, one may use different bounds on the sensitivity function for various outputs. Assume we want the response in channel 1 to be about 10 times faster than that in channel 2. Then we might use the performance specification

$$\|W_P S\|_\infty < 1; \qquad W_P = \begin{pmatrix} w_{P11} & 0 \\ 0 & w_{P22} \end{pmatrix} \tag{11}$$

with $\omega_{B11} = 10\omega_{B22}$. We shall study the effect of introducing similar weights for the distillation example later.

Introducing matrix valued weights on the inputs or outputs is necessary in some cases, in particular, if the disturbances have strong directionality. However, the direct implications for the shape of the sensitivity function then become less clear, and it is probably better to shift to the more general signal-oriented approach discussed next.

3.2 Performance approach B. Frequency by frequency sinusoidal signals

This approach is used, for example, in the space shuttle application study by Doyle et al. (1987).

In this approach we consider the effect of persistent sinusoidal input signals of a given frequency. The weights W_d, W_s and W_n will be diagonal matrices which give the expected magnitude of each input signal at each frequency. Typically, the disturbance weight, W_d, and the setpoint weight, W_s, do not vary very much with frequency [1], while the noise weight, W_n, usually has its peak value at high frequency. W_e is a diagonal matrix which at each frequency specifies the *inverse* of the allowed magnitude of a specific output error. If we want no steady-state offset the weight should include an integrator such that its magnitude is infinite at steady-state (we require offset-free response to slow-varying sinusoids) [2]. Typically, we let the weight level off at high frequencies at a value $1/M$, where M is approximately equal to maximum allowed error (overshoot) in the time response. Often the value of M is about 2 times the allowed magnitude of the noise at high frequency (or of the setpoints if no measurement noise is included). The corner frequency for the weight (where it levels off) should be approximately $1/\tau_{Pe}$ where τ_{Pe}/M is the maximum allowed closed-loop time constant for that output. The actuator penalty weight, W_u, is usually small or zero at steady-state [3]. W_u may be close to a pure differentiator (s) if we want to penalize fast changes in the inputs. In many cases the weights W_u and W_I (for input uncertainty weight) have similar effects on the resulting design and only one of them is used.

It is important to check that the various performance requirements are consistent. This may be done by evaluating their influence of the required loop shapes (approach A), in particular, at low and high frequencies. Alternatively, one may test if it is possible to get $\mu < 1$ for NP by performing a H_∞-synthesis with no uncertainty.

Combining performance weights. It is preferable to have as few and simple weights as possible. To illustrate how specifications on setpoints and disturbance rejection (approach B) may be reformulated as bounds on the weighted sensitivity (approach A) consider Fig.3 and evaluate the transfer function

[1] That is, in this approach B we should *not* add a integrator ($1/s$) to the weight even if step changes in disturbances or setpoints are expected (however, in approach C below this is correct). The reason is that in approach B we consider the response frequency-by-frequency and a step change cannot really be modelled very well, and certainly not as a slow-varying sinusoid of infinite magnitude. A more reasonable approach is to consider a range of sinusoids and use a nearly constant weight with the same magnitude as of the step.

[2] Note that we may not require offset-free response for $y - y_s$ if the measurement noise is nonzero at steady-state ($\omega = 0$). Therefore, to get a controller with integral action we may select W_n to be zero at $\omega = 0$. Alternatively, we may require no offset for $y_m - y_s$, where $y_m = y + n$ is the measurement.

[3] The use of actuators inputs of a certain magnitude is often unavoidable (independent of the controller) in order to reject slow-varying disturbances, and penalizing the inputs at low frequencies makes little sense in such cases.

from normalized disturbances and setpoints to normalized errors. We have

$$z = \hat{e} = E \begin{pmatrix} \hat{d} \\ \hat{y}_s \end{pmatrix} = Ew \tag{12}$$

With conventional feedback control with no setpoint filtering ($C_s = C; W_f = I$) and with no uncertainty ($\Delta_I = 0$) we have

$$E = (W_e S G_d W_d \quad W_e S W_s). \tag{13}$$

The performance specification is $\|E\|_\infty < 1$ and we want to find a weight $w_P(j\omega)$ such that at each frequncy $\bar{\sigma}(w_P S) = \bar{\sigma}(E)$. For the SISO (scalar) case we get $\bar{\sigma}(w_P S) = |w_P S|$ and $\bar{\sigma}(E) = |W_e S| \sqrt{|G_d W_d|^2 + |W_s|^2}$, and we have at each frequency

$$|w_P| = |W_e| \sqrt{|G_d W_d|^2 + |W_s|^2} \tag{14}$$

Consider the following special SISO case where we assume: i) G_d has been scaled such that disturbances $\hat{d}$ are less than 1 in magnitude, and $W_d = 1$; ii) Disturbance model $G_d = k_d/(1 + \tau_d s)$; iii) G has been scaled such that for setpoints $W_s = 1$; iv) The errors, $\hat{e}$, should be less than M in magnitude at high frequencies, and we want integral action and require a response time better than about τ_{Pe}/M. Then $W_e = (\tau_{Pe} s + 1)/M\tau_{Pe} s$.

With the exception of at most a factor $\sqrt{2}$ (at frequencies where $|G_d| \approx 1$) we may then use the following approximation for Eq.(14):

$$w_P(s) \approx W_e(s) \left(\frac{|k_d|}{1 + \tau_d s} + 1 \right) = \frac{s + 1/\tau_{Pe}}{Ms} \frac{s + \frac{|k_d|+1}{\tau_d}}{s + 1/\tau_d} \tag{15}$$

Obviously, if the scaled disturbance gain, $|k_d|$, is small compared to 1, then $w_P(s) \approx W_e(s)$, and the disturbance does not affect the bound on $S(j\omega)$. However, in general the requirement of disturbance rejection may require a faster response than the response time, τ_{Pe}, required by the weight W_e. The most important feature of the performance weight, $w_P(s)$, is it's bandwidth requirement, ω_B^*, which we define as the frequency where the asymptote of $w_P(s)$ crosses 1. Introduce the performance time constant imposed by disturbances

$$\tau_{Pd} = \frac{\tau_d}{|k_d| + 1} \tag{16}$$

A closer analysis of (15) shows that $M\omega_B^* = \max\{1/\tau_{Pe}, 1/\tau_{Pd}\}$. That is, for $\tau_{Pd} < \tau_{Pe}$ the bandwidth requirement is determined by disturbance rejection. For $\tau_{Pe} < \tau_d$ ("slow disturbances") the weight in (15) has a region at low frequencies where $|w_P(j\omega)|$ has a slope of -2 on a log$|w_P|$-logω plot. This is illustrated in Fig.5 where we show the bound $1/|w_P|$ on $|S|$ as a function of frequency for the three cases: 1) $\tau_{Pd} < \tau_{Pe}$, 2) $\tau_{Pd} = \tau_{Pe}$, 3) $\tau_{Pd} > \tau_{Pe}$. It is important to notice that any significant disturbance ($|k_d| > 1$) will require a tighter bound on $S(j\omega)$ at low frequencies.

In the multivariable case we must use matrix-valued weights, and it is not possible to transform approach B into a scalar bound on S. Specifically, $\bar{\sigma}(SG_d(j\omega))$ may be significantly smaller than $\bar{\sigma}(S(j\omega))\bar{\sigma}(G_d(j\omega))$ when G_d is in the "good" direction corresponding to the large plant gains (see Skogestad et al., 1988).

3.3 Performance approach C. Power signals - Power spectrum weights

This is not a frequency-by-frequency approach. Rather one must consider the entire frequency spectrum. One may think of the weights W_d, W_n and W_s as upper bounds on the power spectral density

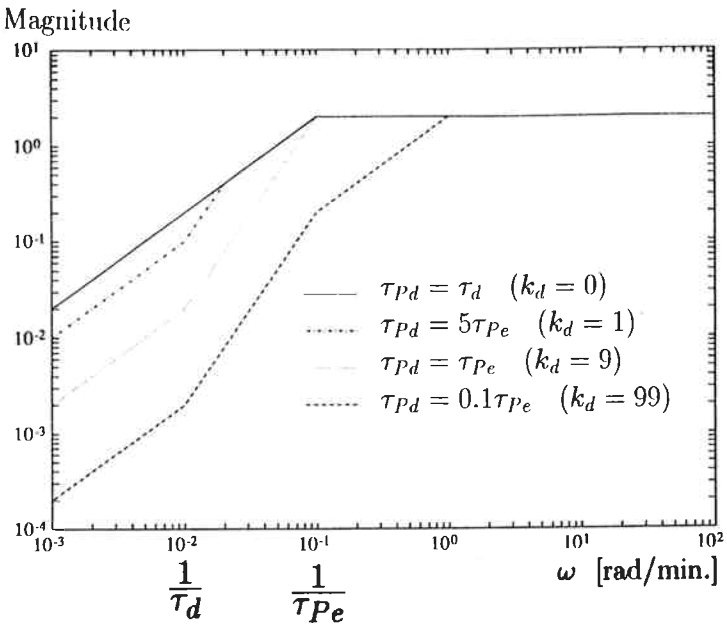

Figure 5: Asymptotic plot of $1/w_P$ (Eq.15) for cases where $\tau_{Pe} < \tau_d$ ($\tau_{Pe} = 10$ and $\tau_d = 100$ is used in the plot).

of the input signals, whereas W_e and W_u are equal to the inverse of the upper bounds on the power spectral density of the output signals. For example, if we allow for step changes of the setpoints, we may choose a weight $W_s = 1/s$ (but we will also allow a lot of other signals bounded by this spectral density). We will not discuss this approach any further, but just note that it compared to approach B in many cases corresponds to shifting integrators from the output weights to the input weights.

4 Fixed or adjustable weights

One advantage with H_∞ or μ-optimal control is that it is relatively well-defined what an objective function with a value close to 1 means: The worst-case response will satisfy our performance objective. If μ at a given frequency is different from 1 then the interpretation is that at this frequency we can tolerate $1/\mu$ times more uncertainty and still satisfy our performance objective with a margin of $1/\mu$.

Controller synthesis almost always consists of a series of steps where the designer iterates between mathematical formulation of the control problem, synthesis and analysis. In μ-synthesis the designer will usually redefine the control problem by adjusting some performance or uncertainty weight until the final optimal μ-value is reasonably close to 1. In most cases this is done in an more or less *ad hoc* fashion, but it may also be done systematically. One attractive option is to keep the uncertainty weight fixed (of course, it must be possible to satisfy RS) and evaluate the achievable performance with this level of uncertainty, that is, adjust some performance weight to make $\mu(N_{RP}) = 1$. There are two obvious options to adjust the performance weight:

1) Scale the performance frequency-by-frequency such that $\mu(N_{RP}) = 1$ at all frequencies, that is, at each frequency find a $k(\omega)$ which solves

$$\mu \begin{pmatrix} N_{RP_{11}} & N_{RP_{12}} \\ kN_{RP_{21}} & kN_{RP_{22}} \end{pmatrix} = 1 \tag{17}$$

This option is most attractive for analysis with a given controller. The numerical search for k is straightforward since μ increases monotonically with k, and since a solution always exists provided we have RS.

2) Adjust some parameter in the performance weight such that the peak value of $\mu(N_{RP})$, denoted $\|N_{RP}\|_\mu$, is 1. This option is most reasonable for μ-synthesis, that is, if the controller is not given. For example, with the performance weight (10) we may adjust the time constant τ_P such that the optimization problem becomes

$$\min_{\tau_P} | \min_C \|N_{RP}(C, \tau_P)\|_\mu - 1| \tag{18}$$

Different plants may then be compared based on their maximum achievable bandwidth. Two disadvantages with this approach are: 1) It introduces a rather time-consuming outer loop in the μ-synthesis. 2) It may be impossible to achieve $\mu(N_{RP}) = 1$ by adjusting τ_P in the performance weight if, for example, the high-frequency specification (value of M) is limiting. Skogestad and Lundström (1990) have used this approach to compare alternative control structures for a distillation column example. One might consider keeping τ_P and M in the weight (10) fixed, and rather adjust the weight at *all* frequencies with the same constant. This would avoid problem 2, but generally does not make sense from a physical point of view since we cannot adjust the weight very much at high frequencies (since $S \approx I$ at high frequencies). This approach would therefore in most cases be very similar to adjusting M.

In this paper we do not employ these approaches, but use fixed weights only.

5 Skogestad et al. (1988) example revisited

We shall use the same plant as studied previously by Skogestad et al. (1988). The plant model is

$$G(s) = \frac{1}{75s+1} \begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix} \tag{19}$$

The unit for time is minutes. This is a very crude model of a distillation column, but it is an excellent example for demonstrating the problems with ill-conditioned plants. Freudenberg (1989) and Yaniv and Barlev (1990) used this model to demonstrate design methods for robust control of ill-conditioned plants.

In Skogestad et al. (1988) the following specifications were used: 1) The relative magnitude of the uncertainty in each of the two input channels is given by $w_I(s) = 0.2(5s+1)/(0.5s+1)$. Thus the uncertainty is 20% at low frequencies and reaches 1 at a frequency of approximately 1 rad/min. Note that the corresponding uncertainty matrix, Δ_I, is a diagonal matrix since we assume that uncertainty does not "spread" from one channel to another (for example, we assume that a large input signal in channel 1 does not affect the signal in channel 2).

2) RP-specification (using performance approach A): The worst case (in terms of uncertainty) H_∞-norm of $w_P S$ should be less than 1. Here $w_P(s) = 0.5(10s+1)/10s$. This requires integral action, a bandwidth of approximately 0.05 rad/min and a maximum peak for $\bar{\sigma}(S)$ of 2.

In the following we shall keep the uncertainty description fixed, but consider alternative performance specifications.

In the time domain this problem specification may be formulated *approximately* as follows: Let the plant be

$$G_p(s) = G(s) \begin{pmatrix} k_1 e^{-\theta_1 s} & 0 \\ 0 & k_2 e^{-\theta_2 s} \end{pmatrix} \tag{20}$$

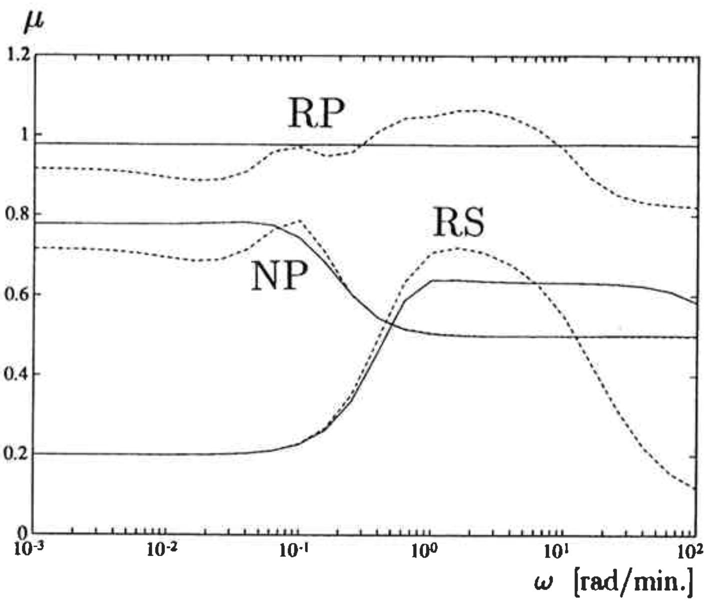

Figure 6: μ-plots for $C_{\mu new}$ (solid curves) and $C_{\mu old}$ (dashed curves).

where $G(s)$ is given in (19). Let $0.8 \leq k_1 \leq 1.2$, $0.8 \leq k_2 \leq 1.2$, $0 \leq \theta_1 \leq 1$, and $0 \leq \theta_2 \leq 1$[4] The response to a step change in setpoint should have a closed-loop time constant less than about 20 minutes. Specifically, the error of each output to a unit setpoint change should be less than 0.37 after 20 minutes, less than 0.13 after 40 minutes, and less than 0.02 after 80 minutes, and with no large overshoot or oscillations in the response.

5.1 Original problem formulation (Performance approach A)

Skogestad et al. (1988) used a software package based on the H_∞-minimization in Doyle (1985) (denoted "the 1984-approch" in Doyle et al., 1989) to design a "μ-optimal" controller. Their controller has six states and gives $\mu_{RP} = \|N_{RP}\|_\mu$=1.067 for both structured and unstructured Δ_I. We will denote this controller $C_{\mu old}$. Freudenberg (1989) used another design method and achieved a controller with five states giving μ_{RP}=1.054 for unstructured Δ_I. Yaniv and Barlev (1990) do not present a μ value for their design, but show some time responses[5].

New optimal design. With the new H_∞-software (Balas et al., 1990) based on the state-space solution of Doyle et al. (1989), also the μ-synthesis performs better than with the 1984-approach. We were able to design a controller which, compared to $C_{\mu old}$, lowered μ_{RP} from 1.067 to 0.978. The new controller will be denoted $C_{\mu new}$. It has 22 states and a state space representation is given in Appendix 1.

Fig.6 shows μ for RP, NP and RS as a function of frequency for $C_{\mu new}$ (solid curves) and $C_{\mu old}$ (dashed curves). $\mu(N_{RP})$ for the new controller is extremely flat and the peak value, μ_{RP}, is substantially lower than for the old controller. The *nominal* performance is generally worse for the new controller, while robust stability is improved for some frequencies.

Fig.7 shows the time response to a setpoint change in top composition for controller $C_{\mu new}$ (solid

[4]The uncertainty given in terms if $w_I(s)$ does not quite allow this uncertainty at high frequency. This follows since we do not really have $\epsilon = 0.2$ much less than $M = 2$ as required when deriving the weight (7)). On the other hand, $w_I(s)$ allows for a lot of other possible uncertainties which are not included in the gain-timedelay-uncertainty described above.

[5]Based on the data in Yaniv and Barlev (1990) we obtained $\mu_{RP} = 1.97$ for their design. However, our time responses did not quite match those presented in their paper.

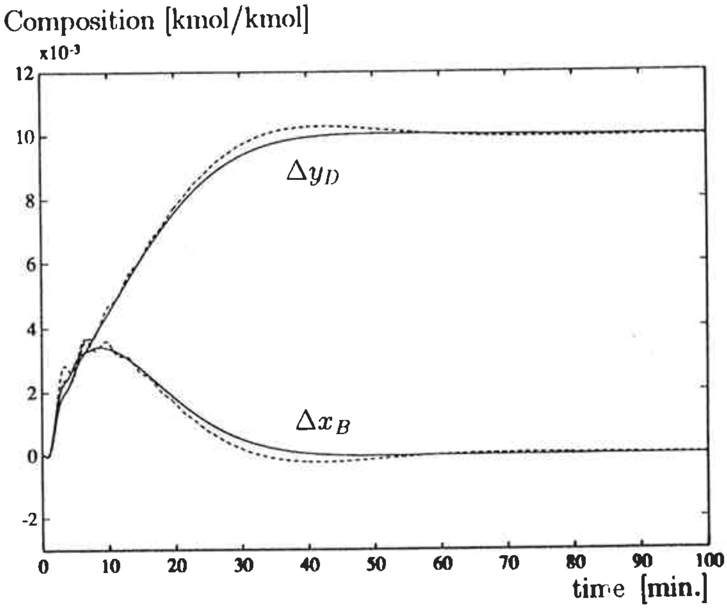

Figure 7: Simulation of setpoint change in y_D using controller $C_{\mu new}$ (solid curves) and $C_{\mu old}$ (dashed curves). The setpoint change is a step at $t = 0$ acting through the filter $1/(5s+1)$. Input uncertainty (Eq.21) is used in the simulations.

curves) and $C_{\mu old}$ (dashed curves). The setpoint change is a step going through the filter $\frac{1}{5s+1}$. In the simulations we use in each channel +20% gain error and a time delay $\theta = 1$ minute (using a second order Pade approximation), that is for each input

$$u(s) = 1.2\frac{\theta^2 s^2 - 6\theta s + 12}{\theta^2 s^2 + 6\theta s + 12}u_c(s) \tag{21}$$

(actually, as discussed below this is not quite covered by the uncertainty description, $w_I(s)$). We see from the time response that the new design has better robustness properties, but otherwise the response with this specific uncertainty is only marginally improved.

5.2 Uncertainty sets and worst perturbations

As mentioned earlier, there is not a one-to-one map between the parametric (time-domain) uncertainty in Eq.20 and the frequency-domain set $G(s)(I + \Delta_I w_I(s))$. It is interesting to compare these two sets frequency-by-frequency. Since the uncertainty is structured (scalar) we can use the complex plane. Fig.8 shows the two sets

$$ke^{-\theta j\omega};\ 0.8 \le k \le 1.2;\ 0 \le \theta \le 1.0 \tag{22}$$

and

$$1 + \Delta(j\omega)|0.2\frac{5j\omega + 1}{0.5j\omega + 1}|;\ |\Delta(jw)| \le 1 \tag{23}$$

at $\omega = 1$ rad/min and at $\omega \to \infty$. Note that the allowed set defined by w_I (Eq.23) is larger than what is required from Eq.22, but still there are elements of $ke^{-\theta j\omega}$ that are not quite covered by the uncertainty set. Also note that +20% gain error *and* -180^o phase error is not covered by the allowed uncertainy set at any frequency.

Controller $C_{\mu new}$ is used in the following. It is interesting to study the worst perturbation $\Delta_I^* w_I(j\omega)$ [6]

[6] We obtained Δ_I^* by selecting the appropriate elements (upper left corner) of the unitary perturbation matrix, U, which is obtained from the lower bound $\rho(N_{RP}U)$ on $\mu(N_{RP})$.

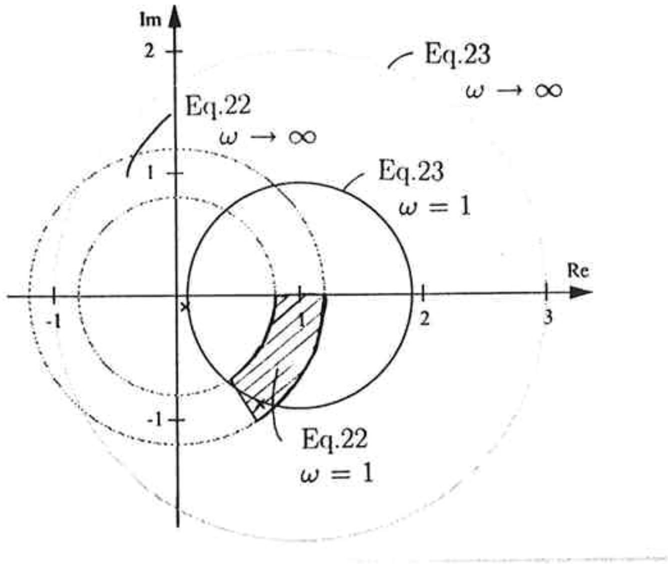

Figure 8: **Graphic representation of the uncertainty sets in Eq.22 and Eq.23 for** $\omega = 1$ rad/min (solid curves) **and** $\omega \to \infty$ **(dotted curves).**

, which at each frequency gives the worst case amplication through the system. That is, subsitituting Δ_I^* into the block diagram gives at each frequency $\bar{\sigma}(w_P S_p) = \mu(N_{RP})$ where S_p denotes the perturbed sensitivity function with the uncertainty Δ_I^*. The worst perturbation in each channel at $\omega = 1$ rad/min is marked with an "x" in Fig.8. The distance from (1,0) to each "x" is equal to $1/\mu(N_{RP})$. Note that at this frequency the worst case frequency domain uncertainty in one of the two channels *is* included in the time domain set, while the uncertainty in the other channel *is not.*

$\bar{\sigma}(S_p)$ for the four corner values for the gain in Eq.20 with a constant delay of 1 minute, is shown as function of frequency in Fig.9. The worst perturbation at low frequencies is if both gains are reduced to 0.8, and at high frequencies with both gains equal to 1.2. It is clear also from this figure that the controller will not quite achieve RP for +20% gain uncertainty and 1 minute delay. The curves also demonstrate that none of these four perturbations is very severe at $\omega = 0.5$ rad/min. However, since $\mu(N_{RP}) = 0.978$ at $\omega = 0.5$ there is some perturbation allowed by $\Delta_I w_I$ that gives $\bar{\sigma}(w_P S_p(0.5)) = 0.978$. Written on the gain-timedelay form (Eq.20) one such perturbation is $1.017e^{-1.07s}$ in channel 1 and $0.636e^{0.99s}$ in channel 2. Note that the phase-error in channel 2 corresponds to a prediction. This is allowed by the frequency-set (w_I) but is not included in time-domain set in Eq.20.

5.3 Other performance weights (Approach A)

Here we use the same problem formulation as in section 5.1, except for using different performance weights in each output channel.

$$W_P(s) = \begin{pmatrix} w_{P_1} & 0 \\ 0 & w_{P_2} \end{pmatrix}; \quad w_{P_i}(s) = \frac{1}{M}\frac{\tau_{P_i}s + 1}{\tau_{P_i}s} \tag{24}$$

Intuitively, we may reduce the "interactions" (this is a term which is relevant for single-loop control) in the system by having one channel with a fast response, and one channel with a slow response. Optimal μ-values for different choices of τ_{P_1} and τ_{P_2} are shown in Table 1. We keep the "avarage" response time constant by holding $\tau_{P_1}\tau_{P_2}$ constant. We see that the interactions are less and the μ-values somewhat lower when we allow different response times in the two channels. Of course, this is only true to a

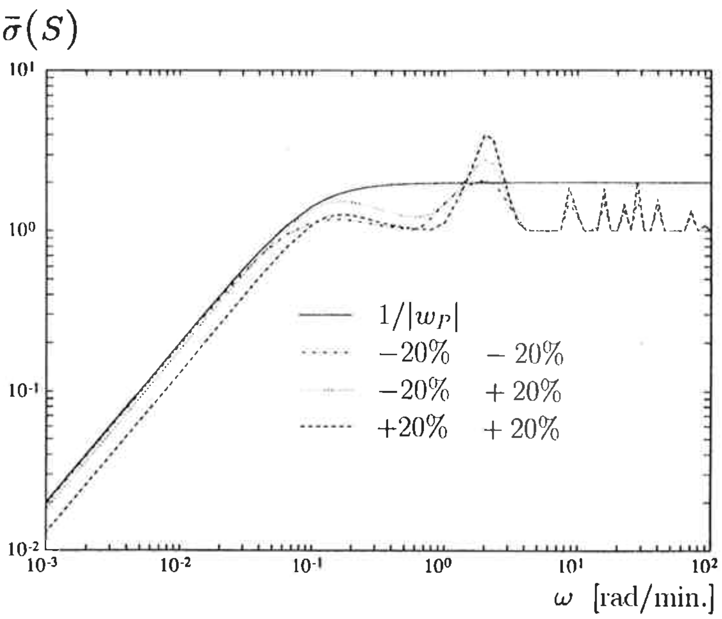

Figure 9: The maximum singular value of S for four specific perturbations (Eq.20) compared to the performance requirement ($1/w_P$, solid curve). All perturbations include a 1 minute delay.

Table 1: Optimal μ-values obtained by μ-synthesis for different performance weights, Eq.24.

τ_{P1} min	τ_{P2} min	μ_{RP}
10	10	0.9777
20	5	0.9703
40	2.5	0.9372
10	**5**	**1.0982**

Table 2: Optimal PID tuning parameters obtained by minimizing μ_{RP} for different performance weights, Eq.24.

τ_{P1} min	τ_{P2} min	μ_{RP}	k_1	k_2	τ_{I1} min	τ_{I2} min	τ_{D1} min	τ_{D2} min
10	10	1.32	162	−39.1	41.2	0.836	0.382	0.286
20	5	1.21	93.9	−52.2	53.3	2.36	0.333	0.195
40	2.5	1.16	62.6	−60.8	82.8	2.66	0.527	0.273
10	**5**	**1.33**	**164**	**−37.4**	**39.4**	**0.661**	**0.381**	**0.285**

limited extent, since the response time of the fast channel is limited by the allowed time delay of about 1 minute. The last entry in the table does not have the same "average" response time, but is included to illlustrate that the μ_{RP}-value increases markedly if we require that only one loop is made faster.

As expected, the reduced interaction becomes even clearer if we study single loop (decentralized) control using two PID-controllers of the form below.

$$C(s) = \begin{pmatrix} c_{PID_1}(s) & 0 \\ 0 & c_{PID_2}(s) \end{pmatrix}; \quad c_{PID_i}(s) = k_i \frac{1+\tau_{I_i}s}{\tau_{I_i}s} \frac{1+\tau_{D_i}s}{1+0.1\tau_{D_i}s} \tag{25}$$

The tuning parameters and μ_{RP} for different choices of performance weights are given in Table 2. The PID-controllers were obtained using a general-purpose optimization algorithm to minimize μ_{RP} with respect to the six parameters. The last entry in the table shows that for the PID-controller we can increase the speed of one channel at almost no cost in terms of μ_{RP}.

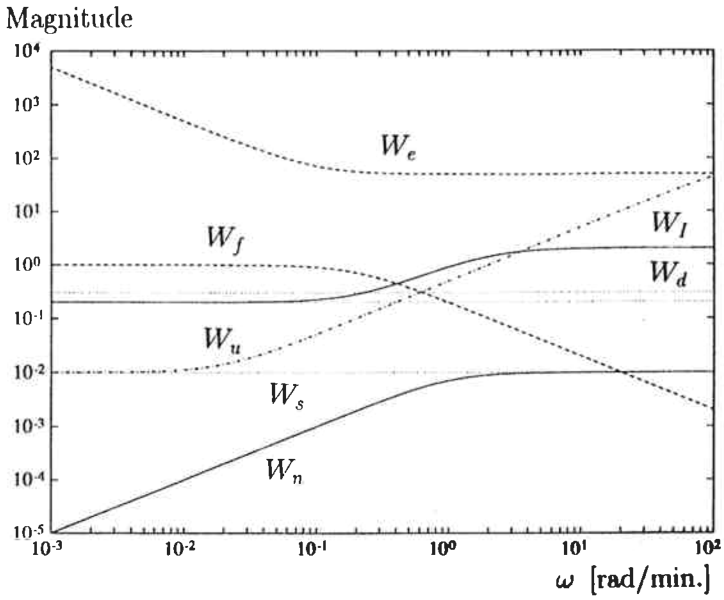

Figure 10: Frequency plot of the weights used in Approach B (Eq.27-29).

5.4 Performance approach B

Consider the block diagram in Fig. 3. We shall design a two degree-of-freedom controller using approach B. The plant G is given in Eq.19. G_d describes the effect of disturbances (feed flow, F, and feed composition, z_F) on the two controlled variables (top and bottom composition, y_D and x_B).

$$G_d(s) = \frac{1}{75s+1}\begin{pmatrix} 0.394 & 0.881 \\ 0.586 & 1.119 \end{pmatrix} \tag{26}$$

We use the following weights to define the problem.

$$W_d(s) = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.2 \end{pmatrix} \quad ; \quad W_s(s) = 0.01 I_{2x2} \quad ; \quad W_n(s) = 0.01\frac{s}{s+1}I_{2x2} \tag{27}$$

$$W_f(s) = \frac{1}{5s+1}I_{2x2} \quad ; \quad W_I(s) = 0.2\frac{5s+1}{0.5s+1}I_{2x2} \tag{28}$$

$$W_e(s) = \frac{100}{2}\frac{10s+1}{10s}I_{2x2} \quad ; \quad W_u(s) = 0.01\frac{50s+1}{0.005s+1}I_{2x2} \tag{29}$$

These weights are plotted in Fig.10.

G and G_d were found by linearizing a non-linear model at an operating point where $y_D = 0.99, x_B = 0.01, F = 1.0$ and $z_F = 0.5$ (Skogestad and Morari, 1987). W_d shows that we are expecting up to 0.3/1=30% variation in F and 0.2/0.5=40% in z_F. Similarly, W_s specifies the setpoint variations, $0.98 \leq y_{Dsp} \leq 1.00$ and $0.00 \leq x_{Bsp} \leq 0.02$. These weights reflect the relative importance of the external inputs, i.e. we consider 30% variation in F to be comparable to a setpoint variation of 0.01 kmol/kmol. The noise at high frequency is allowed to be of magnitude 0.01. The factors 0.01 and 100 in the weights for W_s, W_n and W_e correspond to an output scaling, and could alternatively have been accomplished by multiplying the elements in G and G_d by 100.

The optimal controller, $C_{\mu B}$, gives a μ_{RP} value for this problem definition of 1.05, whereas the controller $C_{\mu new}$ (with $C_s = C$) gives a peak value of about 1800 at high frequencies. The reason to this extremely high value is that $C_{\mu new}$ is tuned without any direct penalty on the manipulated inputs, while in the new

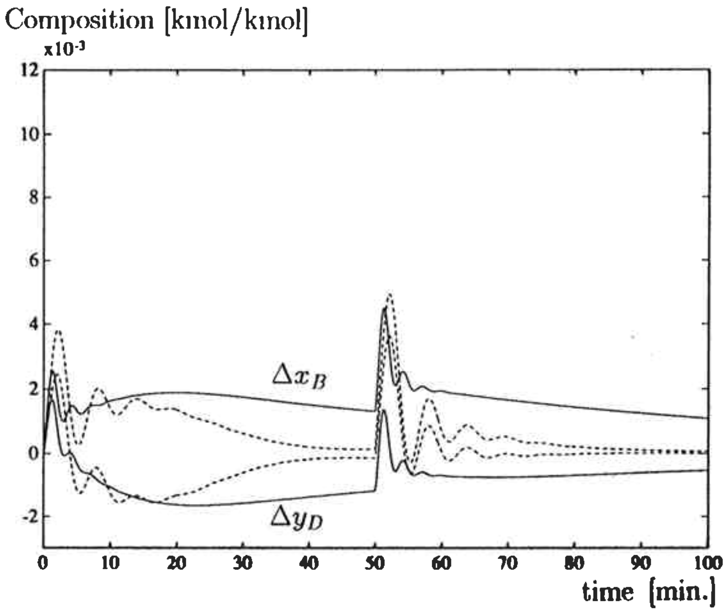

Figure 11: Simulation of a disturbance in F (+30%) at time $t = 0$ and in z_F (+40%) at $t = 50$ min using controller $C_{\mu new}$ (solid curves) and $C_{\mu B}$ (dashed curves). The input error in Eq.21 is used in the simulations.

formultion such a penalty (W_u) is included. At low frequencies $C_{\mu new}$ gives $\mu_{RP} = 1.75$. Conversely, when applied to the original problem definition, $C_{\mu B}$, gives $\mu_{RP} = 1.19$, whereas $C_{\mu new}$ gives 0.978.

Recall the analysis of Eq.14 where we analyzed the relative importance of disturbance and setpoint tracking on performance. If we in this example look at the disturbance rejection from a scalar point of view, the performance time constants, τ_{Pd} in Eq.16, for the effect of the two disturbances in F and z_F on x_B are about $75/(0.3 \cdot 58.6 + 1) = 4.0$ min and $75/(0.2 \cdot 111.9 + 1) = 3.2$ min, respectively, whereas τ_{Pe} for setpoints is 10 min. However, this does not take into account the *direction* of the disturbances. In our cases the disturbance condition number (Skogestad et al., 1988) for the two disturbances are 11.5 and 1.8, respectively, whereas the "disturbance" condition number for the two setpoints are 111 and 89 (Skogestad and Morari, 1987). Thus, the disturbances are in the "good" directions of the plant, and the bandwidth requirements imposed by the disturbances are not as hard as computed above. However, the disturbances do put tighter restrictions at lower frequencies (the "slope two" requirement) than the setpoint requirement. This is also clear from the simulations discussed next.

Fig.11 shows the response to a disturbance in F (+30%) at time $t = 0$ and in z_F (+40%) at $t = 50$ min. Solid curves show the response for controller $C_{\mu new}$ and dashed curves are for controller $C_{\mu B}$. We note that controller $C_{\mu new}$ gives a rather sluggish return to the setpoint. This dominant (low-frequency) part of the response is significantly improved with the controller $C_{\mu B}$. The controller, $C_{\mu new}$ for approach A, could have been improved by using a performance weight, w_P, with slope two at intermediate frequencies. Also, note that the disturbance in z_F is simpler to reject because it is almost exclusively in the "good" direction.

6 Mu-synthesis

The μ-synthesis procedure employed today makes use of the upper bound of μ, trying to "solve"

$$\min_{C,D} ||DN_{RP}(C)D^{-1}||_\infty \tag{30}$$

The algorithm, often called "D-K iteration", is as follows:

1 Scale the problem with a stable and minimum-phase transfer matrix D with appropriate structure.

2 Find a controller C by minimizing the H_∞-norm of $DN_{RP}(C)D^{-1}$.

3 Compute $\mu(N_{RP}(C))$ and obtain at each frequency the optimal "D-scales" from $\min_D \bar{\sigma}(DN_{RP}D^{-1})$.

4 Fit the magnitude of each element of $D(\omega)$ to a stable and minimum phase transfer function.

5 Test a stop criterion. Stop or go to 1.

The major problem with μ-synthesis is that the D-K iteration is not guaranteed to find the global optimum of Eq.30 (Doyle and Chu, 1985). A second problem is the difficulty to define a stop criterion for the optimization.

Good initial D-scales in step 1 of the algorithm, reduces the number of iterations, and may even, because of local minimas, affect the final minimal μ-value. For our example problem with the original problem definition, we observed that a natural physical scaling of the problem (using "logarithmic compositions" as discussed by Skogestad and Morari, 1988), that corresponds to multiplying all elements in $G(s)$ by a factor 100, gave very good initial D-scales. With this simple scaling the μ-value after the first iteration was reduced from 14.9 to 1.2.

The D-K iteration depends heavily on optimal D-scales. If the D-scales are not optimal, then the controller C is not optimal either. The μ software in the toolbox do not seem to compute a sufficiently tight upper bound of μ. Thereby the D-scales are not optimal, and the D-K iteration suffers. We have experienced cases where, for some frequencies, the computed μ-value has been larger than the maximum singular value. When this occurs the D-K iteration often starts diverging.

An other critical factor is the fitting of the D-scales. It is important to get a good fit, preferably by a transfer function of low order. The software for D-scale fitting in the μ-toolbox requires that the user specifies the order of the transfer function and decides if the fit is good enought. The optimal order of the tranfer function D varies as the D-K iteration progress. It is sometimes better to *increase* the order, and sometimes the order should be *decreased.*

The final problem is to determine when to stop the iteration. Two reasonable candidates for criterion for terminating the iteration are: 1) An iteration criterion

$$\mu_{\mathbf{k-1}} - \mu_{\mathbf{k}} < \epsilon_1 \tag{31}$$

and, 2) A "flatness" criterion

$$\max_\omega(\mu_{\mathbf{peak}} - \mu(\omega)) < \epsilon_2 \tag{32}$$

1) In Eq.31 the subscript denotes the $\mathrm{k}-1^{\mathrm{th}}$ and the k^{th} iteration respectively. This is a standard criterion, the iteration terminates if the objective function (μ) does not improve. There are two problems with this criterion. First, we may have found a local minimum, which means it is possible to improve μ by using a different D. Second, this criterion would terminate the iteration if μ increases. That may sound reasonable, but we have experienced situations where μ increases for a number of iterations and then start to decrease again. 2) Eq.32 relies on the optimal controller giving a flat μ versus frequency plot. However, this is not always true. The optimal solution to the problem in Skogestad et al. (1988) does not give a flat μ-plot, instead μ always goes to 0.5 at high frequencies (since S goes to I, and w_P goes to 0.5 I). As the number of iterations are increased one is able to extend the frequency where μ starts dropping down to 0.5, but the curve never becomes flat at all frequencies.

Acknowledgements. Support from NTNF is gratefully acknowledged.

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APPENDIX 1:

State space description of $C_{\mu new}(s) = C(sI - A)^{-1}B + D$. The controller has 22 states, 2 inputs and 2 outputs. The A matrix is given in tridiagonal form with the complex conjugate roots in real two by two form, i.e. A is a bandmatrix with all non-zero elements on the main diagonal and the two adjacent diagonals. The D matrix is a zero 2 by 2 matrix.

A:

row number	diagonal below main	main diagonal	diagonal above main
1		$-1.0000e-07$	0
2	0	$-1.0000e-07$	0
3	0	$-5.3681e-04$	0
4	0	$-6.8364e-04$	0
5	0	$-3.4883e-03$	0
6	0	$-5.5976e-02$	0
7	0	$-5.7017e-02$	0
8	0	$-2.0050e-01$	0
9	0	$-2.6267e-01$	$-1.1744e-01$
10	$1.1744e-01$	$-2.6267e-01$	0
11	0	$-4.8527e-01$	0
12	0	$-3.1117e+00$	$-6.9774e-01$
13	$6.9774e-01$	$-3.1117e+00$	0
14	0	$-1.9255e+01$	0
15	0	$-4.1007e+01$	0
16	0	$-1.1341e+02$	0
17	0	$-1.2966e+02$	$-8.7070e+01$
18	$8.7070e+01$	$-1.2966e+02$	0
19	0	$-1.3042e+02$	$-8.6556e+01$
20	$8.6556e+01$	$-1.3042e+02$	0
21	0	$-1.8112e+02$	0
22	0	$-6.3929e+05$	

B and C^T:

B column one	B column two	C row one	C row two
$8.5088e-01$	$1.0625e+00$	$9.6138e-01$	$-9.6366e-01$
$1.6792e+00$	$-1.3426e+00$	$1.5210e+00$	$1.5194e+00$
$2.6054e-02$	$-2.0838e-02$	$-2.3158e-02$	$-2.3122e-02$
$1.1099e-01$	$1.3877e-01$	$1.1712e-01$	$-1.1731e-01$
$-7.4077e-02$	$-9.2619e-02$	$7.4475e-02$	$-7.4592e-02$
$1.1255e+00$	$1.4072e+00$	$1.2294e+00$	$-1.2313e+00$
$6.1913e-01$	$-4.9518e-01$	$5.1127e-01$	$5.1046e-01$
$-1.4905e+00$	$-1.8635e+00$	$-1.6054e+00$	$1.6079e+00$
$-6.5002e+00$	$5.1989e+00$	$-5.6709e+00$	$-5.6620e+00$
$7.4959e+00$	$-5.9952e+00$	$-2.4034e+00$	$-2.3997e+00$
$-1.0500e+00$	$-1.3128e+00$	$-7.4675e-01$	$7.4792e-01$
$8.4252e-01$	$1.0534e+00$	$-6.4611e-01$	$6.4712e-01$
$-3.0556e+00$	$-3.8204e+00$	$-1.9699e-01$	$1.9730e-01$
$-8.0530e+01$	$6.4408e+01$	$5.4282e+01$	$5.4197e+01$
$4.1453e+01$	$-3.3154e+01$	$-3.3348e+01$	$-3.3296e+01$
$7.7061e+02$	$9.6349e+02$	$-1.7348e+02$	$1.7375e+02$
$-2.4073e+01$	$1.9254e+01$	$-1.8973e+02$	$-1.8944e+02$
$3.3649e+02$	$-2.6913e+02$	$2.4277e+01$	$2.4239e+01$
$-5.1808e+02$	$-6.4776e+02$	$-5.7626e+01$	$5.7717e+01$
$-3.8541e+02$	$-4.8188e+02$	$3.3056e+02$	$-3.3108e+02$
$1.3625e+03$	$1.7036e+03$	$2.8059e+02$	$-2.8103e+02$
$-1.2773e+04$	$-1.5970e+04$	$-1.4449e+04$	$1.4471e+04$