

# CONTROL OF IDENTICAL PARALLEL PROCESSES

Sigurd Skogestad \*

Petter Lundström

Morten Hovd

Chemical Engineering  
Norwegian Institute of Technology (NTH)  
N-7034 Trondheim  
Norway

Presented at AIChE Annual Meeting, San Francisco, Nov. 5-10, 1989.

Session: Recent developments in process control. Poster No. 167Ba

UNPUBLISHED.

©Authors

## Abstract

In the paper we study the control of identical parallel processes. Such processes are quite common in industry, for example, in distribution networks or when parallel units (reactors, heat exchangers, etc.) are used. For single-loop control of such processes the interactions between the processes are generally such that DIC (decentralized integral control) is possible. For example, this implies that taking loops out of service is not expected to cause stability problems. When model uncertainty is included and performance is measured in terms of the  $H_\infty$ -norm (that is, the structured singular value  $\mu$  is used as a performance measure), then the optimal single-loop PI- or PID-tunings are not necessarily equal for the individual loops. This is contrary to what one intuitively would expect, and also implies that the optimal solution is non-unique.

---

\*Author to whom correspondence should be addressed.

# 1 Introduction

The paper is concerned with the control of identical processes in parallel which interact with each other. This happens quite often in practice, for example, in distribution networks, or when there are parallel units (reactors, heat exchangers, etc.) in a chemical plant. With  $n$  identical parallel processes the  $n \times n$  transfer matrix of the plant may be written

$$G(s) = \begin{pmatrix} g(s) & i(s) & i(s) & \dots & i(s) \\ i(s) & g(s) & i(s) & \dots & i(s) \\ i(s) & i(s) & g(s) & \dots & i(s) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ i(s) & i(s) & i(s) & \dots & g(s) \end{pmatrix} \quad (1)$$

where the diagonal elements  $g(s)$  denote the transfer function of the process, and the offdiagonal elements  $i(s)$  denote the interactions. At a given frequency this transfer matrix may be written

$$G(j\omega) = g(j\omega) \begin{pmatrix} 1 & a(j\omega) & a(j\omega) & \dots & a(j\omega) \\ a(j\omega) & 1 & a(j\omega) & \dots & a(j\omega) \\ a(j\omega) & a(j\omega) & 1 & \dots & a(j\omega) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a(j\omega) & a(j\omega) & a(j\omega) & \dots & 1 \end{pmatrix} \quad (2)$$

where  $a(j\omega) = i(j\omega)/g(j\omega)$  denotes the degree of interaction at a given frequency.

Our interest in this kind of processes was initiated by results we obtained for a simplified distillation column model (Skogestad et al., 1988). The model used in that paper is

$$G^{LV}(s) = \frac{1}{75s + 1} \begin{pmatrix} 0.878 & -0.864 \\ -1.082 & 1.096 \end{pmatrix} \quad (3)$$

Here the inputs are reflux ( $L$ ) and boilup ( $-V$ ) and the outputs are product compositions. This is of course not an example of identical parallel processes, but its model is quite close (with  $a = -0.986$ ). Skogestad et al. (1988) studied robust control using this model. They considered uncertainty with respect to the actual value of the inputs  $L$  and  $V$ , and used the  $H^\infty$ -norm of the weighted sensitivity as a performance criterion. Note that the uncertainty and performance specifications in this example were identical for the two channels. With the uncertainty and performance weights fixed, the problem of optimizing the worst-case response is solved mathematically by finding the controller  $C$  that minimizes the value of the structured singular value  $\mu_{RP}$  (Doyle, 1982). A value of  $\mu_{RP}$  less than 1 implies that the worst-case response satisfies the robust performance objective. For the process in Eq.3 Skogestad et al. (1988) obtained a multivariable controller, denoted the " $\mu$ -optimal controller", with  $\mu_{RP} = 1.06$ . This means that the  $\mu$ -optimal controller does not quite satisfy the robust performance specification, but it is very close. They also studied use of single-loop PID-controllers and were able to obtain  $\mu_{RP} = 1.34$  by adjusting the six controller parameters ( $k_C, \tau_I, \tau_D$  for each loop). Skogestad et

al. (1988) did not publish the corresponding optimal controller parameters which are:

$$\text{Loop 1 : } K_c = 1.79, \tau_I = 28.9 \text{ min}, \tau_D = 0.31 \text{ min} \quad (4)$$

$$\text{Loop 2 : } K_c = 0.47, \tau_I = 1.38 \text{ min}, \tau_D = 0.27 \text{ min} \quad (5)$$

We note that the integral time for loop 1 is about 20 times larger than that of loop 2, and also the gains are different by a factor of 4. Intuitively, one would expect for single-loop control of identical parallel processes that the optimal tunings would be identical for all loops. However, the results over suggest that this may not be the case. Furthermore, it suggests that the optimal controller is not unique since we may simply interchange the controllers for the two loops and get the same overall performance. Skogestad and Lundström (1990) have obtained results for other distillation column models which support these findings.

Based on these results we decided to study identical parallel processes more systematically and some results are presented in this paper. First we give some examples of processes that may be written on the general form in Eq.1 and 2 above. Note that we so far have only been able to find examples where at steady-state  $-1/(n-1) < a < 1$ . We also study the frequency behavior of the special matrix in Eq.2, and present some general results for these kind of matrices, which we denote a parallel matrix. Finally, we study design of single-loop controllers for such processes and confirm the results obtained for the distillation column example above.

## 2 Examples of parallel processes

We have not performed any extensive search in the literature for examples on parallel processes and for properties of such processes, and would therefore very much welcome possible references. Specifically, we suspect that there might exist literature in the area of distribution networks, and in control of large-scale systems in general. It is quite common in the chemical industries to have units in parallel, either because one single unit would be too large or to add flexibility. Typical examples of parallel units include reactors, heat exchangers and compressors, and some examples may be found in the books of Shinsky.

*Example 1. Flow splitting.* One example is the control of flow in parallel streams from a single source as shown in Fig.1 (Shinsky, 1979, p.201). Opening valve 1 causes  $q_1$  (flow 1) to increase and  $q_2$  (flow 2) to decrease because of the consequent reduction in header pressure  $p$ . If there are two parallel streams the steady-state value of  $a$  is expected to lie between 0 and -1. The value of 0 would be obtained if the source was a large tank such that the header pressure was unaffected by increasing flow 1, and the value of -1 would be obtained if the source was a pump with constant total flow  $q$ . For  $n$  parallel streams from a single source similar arguments yield

$$-1/(n-1) \leq a \leq 0 \quad (6)$$

The lower bound is obtained by considering constant total flow. In this case a change  $\Delta q_1$  in flow 1 would yield  $\Delta q_2 = \Delta q_3 = \dots = \Delta q_n = -\Delta q_1/(n-1)$ . A value less

than the lower bound  $-1/(n-1)$  would imply that the total flow is reduced by opening a valve and does not seem to be possible in a practical situation.

*Example 2. Parallel reactors with combined precooling.* In Fig.2 is shown the cooling system for  $n$  identical mixing tank reactors in parallel. The cooling medium comes from a single source which is split into  $n$  streams and then completely evaporated by heat exchange with the reactors. The streams are then combined and this stream is superheated by precooling the reactor feed. At steady state all temperatures and flows in the parallel streams are assumed equal. Consider the transfer matrix  $G(s)$  between the valve positions (inputs) and the reactor temperatures (outputs). By neglecting the dynamics of the evaporator and the superheater, the model  $G(s)$  can be shown (Appendix) to be on the form

$$G(s) = \frac{k}{\tau s + 1} \begin{pmatrix} 1 & a & a & \dots & a \\ a & 1 & a & \dots & a \\ a & a & 1 & \dots & a \\ \vdots & \vdots & \vdots & & \\ a & a & a & \dots & 1 \end{pmatrix} \quad (7)$$

where  $k$  and  $a$  are real constants and  $\tau$  is the time constant for holdup in each of the reactors. Based on physical arguments we have

$$-1/(n-1) \leq a \leq 1 \quad (8)$$

The lower limit is obtained by considering the case with no precooler ( $UA = 0$  in the model) and assuming constant total flow (recall the parallel flow example above). The upper limit is obtained by considering the case with no evaporator ( $\Delta H_{vap} = 0$  in the model). In this case the cooling streams are split and then recombined without changing their temperatures, and an increase in any cooling stream will affect all reactor temperatures equally and we have  $a = 1$ . Independent control of the reactor temperatures is clearly not possible in this case.

We have looked for other examples of parallel processes with  $a$  outside the bounds given by Eq.8, but have so far not found any. We would be very thankful if someone could direct our attention to such examples.

In the example above the processes were parallel but the control objectives (eg., to keep the individual reactor temperature constant) were otherwise decoupled. There are also interesting examples of control of parallel processes where the control objectives are coupled. For example, consider again the stream split example in Fig.1 and assume the control objective is to keep the total flow  $\sum_i q_i$  at a fixed value  $q_s$  subject to the constraint that the individual flows should be equal, that is,  $q_1 = q_2 = \dots = q_n$ . The requirement of equal flows may be needed, for example, to avoid overheating of tubes in a burner with parallel passes (Shinsky, 1984, p. 104). Since the equal flow objectives yield  $n-1$  objectives of the form  $q_2 - q_1 = 0$ , there are a total of  $n$  control objectives which may be collected in the output vector  $y$ . Subsequently, the control problem may be solved with standard methods from linear control theory, but note that the overall transfer function from inputs (valve positions) to outputs  $y$  is no longer an example of an identical parallel process as defined in Eq.1.

### 3 Results from matrix theory

Consider a complex square  $n \times n$  matrix  $P$  of the form

$$P = k \begin{bmatrix} 1 & a & a & \cdots & a \\ a & 1 & a & \cdots & a \\ a & a & 1 & \cdots & a \\ \vdots & & & \ddots & \vdots \\ a & a & a & \cdots & 1 \end{bmatrix} \quad (9)$$

where the constants  $a$  and  $k$  in general can be functions of frequency. We have not found any name for this matrix in the literature, but we shall refer to it as a *parallel matrix* in the following.

#### 3.1 Classification of the matrix $P$

The parallel matrix  $P$  is a symmetric matrix which belongs to a subclass of both circulant matrices and symmetric Toeplitz matrices. The general form of a circulant matrix  $C$  is:

$$C = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ c_n & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_n & c_1 & \cdots & c_{n-2} \\ \vdots & & & \ddots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_1 \end{bmatrix} \quad (10)$$

The general form of a Toeplitz matrix  $T$  is:

$$T = \begin{bmatrix} t_0 & t_1 & t_2 & \cdots & t_{n-1} \\ t_{-1} & t_0 & t_1 & \cdots & t_{n-2} \\ t_{-2} & t_{-1} & t_0 & \cdots & t_{n-3} \\ \vdots & & & \ddots & \vdots \\ t_{-(n-1)} & t_{-(n-2)} & t_{-(n-3)} & \cdots & t_0 \end{bmatrix} \quad (11)$$

It can easily be seen that if  $c_1 = k$  and  $c_2 = c_3 = \cdots = c_n = ak$  then  $P = C$ . Similarly, if  $t_0 = k$ ,  $t_i = t_{-i}$  and  $t_1 = t_2 = \cdots = t_{n-1} = ak$  then  $P = T$ . The results in this section on circulant and Toeplitz matrices are from Bellman (1970) and Davis (1979).

#### 3.2 Properties of the matrix $P$

**Determinant.**

$$\det P_{n \times n} = k^n [1 + (n-1)a](1-a)^{n-1} \quad (12)$$

*Proof:* The determinant is unaltered by adding  $c$  times one column (row) to another column (row). Subtract the first row from all the other rows. Thereafter, add columns 2 to  $n$  to column 1. The result is a triangular matrix, and the determinant

is the product of the elements on the diagonal.

**Relative Gain Array (RGA) (Bristol, 1966).**

$$\lambda_{ii} = p_{ii} \frac{\det(P_{(n-1) \times (n-1)})}{\det(P_{n \times n})} = \frac{[1 + (n-2)a]}{[1 + (n-1)a](1-a)} \quad (13)$$

$$\lambda_{ij, i \neq j} = \frac{1 - \lambda_{ii}}{(n-1)} = \frac{-a^2}{[1 + (n-1)a](1-a)} \quad (14)$$

Inverse.

$$[P^{-1}]_{ii} = \frac{[1 + (n-2)a]}{[1 + (n-1)a](1-a)} \frac{1}{k} = \frac{\lambda_{ii}}{k} \quad (15)$$

$$[P^{-1}]_{ij, i \neq j} = \frac{-a}{[1 + (n-1)a](1-a)} \frac{1}{k} = \frac{\lambda_{ij}}{ak} \quad (16)$$

**Eigenvalues, eigenvectors and singular value decomposition.**

Let  $r_i$  be a root of the equation  $r^n = 1$ . From the theory of circulant matrices (Davis 1979, Bellman 1970) we then know that the matrix  $P$  has eigenvalues  $\lambda_i$  given by the formula:

$$\lambda_i/k = 1 + a(r_i + r_i^2 + \dots + r_i^{n-1}) \quad (17)$$

However, since

$$1 + r_i + r_i^2 + \dots + r_i^{n-1} = 0 \text{ for } r_i \neq 1 \quad (18)$$

the matrix  $P$  will have at most two distinct eigenvalues. These are given by

$$\lambda_1 = (1 + (n-1)a)k \quad (19)$$

$$\lambda_2 = \lambda_3 = \dots = \lambda_n = (1-a)k \quad (20)$$

This means that the matrix  $P$  will always have eigenvalues in the right half of the complex plane if  $k$  is a positive constant. The eigenvector corresponding to  $\lambda_i$  is:

$$\underline{m}_i = [1 \ r_i \ r_i^2 \ \dots \ r_i^{n-1}]^T \quad (21)$$

Since the equation  $r^n = 1$  has  $n$  distinct roots,  $P$  will always have a complete set of eigenvectors, and will thus always be diagonalizable. In fact, all circulant matrices of the same order have the same eigenvectors, and are therefore diagonalized by the same matrix, the Fourier matrix.

As  $P$  is symmetric, the eigenvectors will be the same as the vectors resulting from singular value decomposition, and the singular values will equal the modulus of the eigenvalues ( $\sigma_i = |\lambda_i|$ ).

### 3.3 Combinations of parallel matrices

If  $A$  and  $B$  are parallel matrices and  $k_i$  a scalar, then  $A^T$ ,  $A^H$ ,  $k_1A + k_2B$ ,  $AB$ ,  $\sum_i k_i A^i$  are parallel matrices and  $A$  and  $B$  commute, that is,  $AB = BA$ . Note that  $A^{-1}$  is also a parallel matrix.

For example, if a process with a parallel transfer function  $G$  is controlled by  $n$  equal single-loop controllers (ie.,  $C = cI$ ), the sensitivity function  $S = (I + GC)^{-1}$  and the complementary sensitivity function  $T = I - S$  are both parallel matrices.

## 4 Limits on interactions for single-loop control

The results in this section are based on the general results from matrix theory presented above.

### 4.1 Conditions for DIC

When single-loop controllers are used it is highly desirable that the system corresponding to the chosen pairing is Decentralized Integral Controllable (DIC). If this is the case then there exist stabilizing single-loop controllers with integral action (no offset) such that the gains of the individual loops can be reduced *independently* without introducing instability. A set of necessary conditions for DIC are presented by Skogestad and Morari (1988). Essentially these conditions are tests for whether the steady-state gain of the system may change sign as the controller gain for other loops are changed. Assume  $G(s)$  is open-loop stable. Two of these conditions in terms of the steady-state gain matrix  $G(0)$  corresponding to the chosen pairing are

1.  $\lambda_{ii}(G(0)) < 0$  for some  $i \Rightarrow$  not DIC
2.  $\min_i \operatorname{Re}\{\lambda_i(G^+(0))\} < 0 \Rightarrow$  not DIC

Here  $G^+(0)$  is the steady-state gain matrix with signs of inputs adjusted such that all diagonal elements are positive. In words the first test tells us to avoid pairings on elements corresponding to negative RGA-elements, and the second test says that DIC is not possible if  $G^+(0)$  has eigenvalues in the left half plane. The system will also not be DIC if any of the two tests fail for the subsystems obtained by deleting a row and corresponding column in  $G(0)$ .

Consider a parallel steady-state gain matrix of the form in Eq.2. At steady state  $a$  and  $k$  are real numbers, and assuming  $k > 0$  yields that the diagonal RGA elements are negative for

$$\frac{-1}{n-2} < a < \frac{-1}{n-1} \quad \text{or} \quad a > 1 \quad (22)$$

One or more eigenvalues of  $G^+(0)$  are in the left half plane for

$$a < \frac{-1}{n-1} \quad \text{or} \quad a > 1 \quad (23)$$

Equations 22 and 23 give regions for  $a$  for which DIC is not possible. Combining the regions for  $a$  excluded by the two conditions gives the region in Eq.23, that is, the eigenvalue condition is the more useful in this case. Applying the eigenvalue test to the submatrices corresponds to choosing  $n$  smaller and this does not exclude more values of  $a$ . In summary, we see that to guarantee DIC we must at least require that

$$-\frac{1}{n-1} < a < 1 \quad (24)$$

This condition is necessary and probably also sufficient for DIC. However, the sufficiency remains to be proven rigorously. It is interesting to note that the region

for which DIC is possible according to Eq.24 is exactly the same region we found that  $a$  should be in for our example processes. This implies that DIC is possible for all the examples of parallel processes we have studied so far. This is of course a relief for the practicing engineer, but from a theoretical point of view we would very much welcome example processes with  $a$  outside this region.

## 4.2 Magnitude of RGA-values at steady state

It is known (eg., Skogestad and Morari, 1987) that large RGA-elements at frequencies corresponding to the desired closed-loop bandwidth imply that the plant is fundamentally difficult to control. For  $a$  in region given by Eq.24 above  $\lambda_{ii}$  at steady state is always larger than 1.  $\lambda_{ii}$  is 1 for  $a = 0$  and it approaches infinity as  $a$  approaches its lower limit  $-1/n - 1$  or upper limit 1. Provided the steady-state behavior is representative for the behavior at high frequency we may conclude that the system is difficult to control (regardless of what controller is used) if  $a$  approaches these limits. In particular, it is known that inverse-based controllers (eg., decouplers) should not be used for plants with large RGA- elements because the system will be extremely sensitive to small input errors.

In this section we have established that single-loop control seems to be a reasonable control strategy for controlling identical processes. Firstly, we established that the requirement of DIC is always satisfied, at least for the examples we have studied. Secondly, we established that the RGA-elements are always larger than 1 in magnitude at steady-state. This implies that multivariable controllers that might be used to reduce the effect of interactions will have robustness problems.

## 5 Design of single-loop controllers for parallel processes

### 5.1 Example process

In this section we consider tuning of single-loop controllers for a  $2 \times 2$  parallel process with transfer function

$$G(s) = \frac{1}{1 + \tau s} \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \quad (25)$$

where  $a$  is a real constant. This plant has a constant value of the RGA given by  $\lambda_{11} = 1/(1 - a^2)$ . An example of such a plant may be cooling of two parallel reactors with combined precooling (Example 2) as shown in Fig.2. We shall consider the following parameter values:

- $a$  : 0, 0.827, 0.949, 0.984, and 0.995 (corresponding to  $\lambda_{11} = 1, 3, 10, 30,$  and 100)
- $\tau$  : 10 and 100 min.



The value of  $\tau$  should be compared to the value of the dead time  $\theta$  of 1 min allowed by the uncertainty weight (see below). Also note that for the special case of  $2 \times 2$  plants the sign of  $a$  does not matter, that is, the same results, for example with respect to optimal tuning parameters, would be obtained with  $a = 0.9$  or  $a = -0.9$ . The reason is that changing the sign of input 1 and output 1 changes the sign of the off-diagonal elements, but keeps the diagonal elements unchanged. This is not the case for  $3 \times 3$  plants or higher.

As controllers we shall consider PI- and PID-controllers on the cascade form with derivative action effective over one decade.

$$C_{PID}(s) = k \frac{1 + \tau_I s}{\tau_I s} \frac{1 + \tau_D s}{1 + 0.1 \tau_D s} \quad (26)$$

The controller parameters  $k, \tau_I, \tau_D$  for each loop are optimized with respect to robust performance of the system in Fig.3 as described in detail below.

## 5.2 Mu-optimal tuning procedure

*Model uncertainty.* One source of uncertainty which always is present and which generally limits achievable closed-loop performance is input uncertainty. Let the relative input error be  $\epsilon$  at steady-state, and assume it increases with frequency such that it reaches 1 (100%) at a frequency of about  $1/\theta$ . For example, this increase of the error with frequency may be caused by neglecting a time delay  $\theta$ . The corresponding input weight is then

$$w_I(s) = \epsilon \frac{\frac{\theta}{2}s + 1}{s + 1} \quad (27)$$

We use the same weight as Skogestad et al. (1988) and Skogestad and Lundström (1989) and choose for both inputs  $\epsilon = 0.2$  (20%) and  $\theta = 1$  min.

*Performance.* Assume that the following performance specifications are given: 1) Steady-state offset less than  $A$ ; 2) Closed-loop bandwidth higher than  $\omega_B$ ; and 3) Amplification of high-frequency noise less than a factor  $M$ . These specifications may be reformulated as a bound on the weighted sensitivity function

$$\bar{\sigma}(w_P S(j\omega)) < 1 \quad \forall \omega \quad (28)$$

(which is equivalent to requiring  $\|w_P S\|_\infty < 1$ ) using the following weight

$$w_P(s) = \frac{1}{M} \frac{\tau_P s + 1}{\tau_P s + A/M}, \quad \text{with } \tau_P = 1/\dot{M}\omega_B \quad (29)$$

We shall for both outputs use  $A = 0$  (no offset) and  $M = 2$  which are the same values as used in previous studies. This gives

$$w_P(s) = 0.5 \frac{\tau_P s + 1}{\tau_P s} \quad (30)$$

Previously (Skogestad et al., 1988)  $\tau_P$  has been fixed at 10 min, but we shall use it as an adjustable parameter.

*Robust Performance (RP)*. This is satisfied if the above-mentioned performance criterion is satisfied for all possible model errors. Mathematically, this is tested by computing  $\mu$  of the matrix  $N_{RP}$  (see Skogestad et al., 1988a).

$$N_{RP} = \begin{pmatrix} w_I C S G & w_I C S \\ w_P S G & w_P S \end{pmatrix} \quad (31)$$

$\mu(N_{RP})$  should be less than one at all frequencies for RP to be satisfied. The peak value of  $\mu(N_{RP})$  will be denoted  $\mu_{RP}$ .

*Controller tuning.* In previous papers (Skogestad et al., 1988) we have obtained the optimal parameters by minimizing  $\mu_{RP}$  with fixed uncertainty and performance weights ( $\tau_P = 10$  min). An optimal  $\mu_{RP}$ -value different from 1, say 0.7, then means that *both* the uncertainty and performance weight may be increased by a factor of  $1/0.7$  and still have robust performance. In most cases it seems more reasonable to fix the uncertainty and find the best achievable performance (that is, adjust the performance weight such that  $\mu_{RP} = 1$ ). This is the approach taken here, and the parameters in the controller  $C(s)$  are obtained by solving the following optimization problem (denoted "Approach 2" in Skogestad and Lundström, 1989):

$$\min_{\tau_P} \left\| \min_C \mu_{RP}(C, \tau_P) - 1 \right\| \quad (32)$$

The optimal parameters are obtained using a general optimization routine and since there are many local minima there is no guarantee that the settings presented really are the true optimal.

### 5.3 Optimal tuning parameters

The results of the parameter optimizations for different combinations of the parameters  $a$  and  $\tau$  are presented in Tables 1-4. The tables give the best achievable performance (as expressed by the value of  $\tau_P$  in the performance weight) for the specific plant as well as the corresponding controller settings. Small values of  $\tau_P$  are good as they imply that fast response may be achieved, even in presence of model uncertainty.

Table 1 present the optimal PI-settings when we require the two single-loop controllers to be identical and Table 2 present the results when the tunings are allowed to be different. For the case with  $\tau=10$  min there is only a very small improvement by allowing the controllers to be different. However, for the case with  $\tau=100$  min we note that the achievable closed-loop time constant  $\tau_P$  may be significantly reduced. This is especially the case for "intermediate" values of  $a$  corresponding to RGA-values between 10 and 30. For example, with  $\tau=100$  min and  $a=0.949$  ( $\lambda_{11}=10$ ) the value of  $\tau_P$  may be reduced from about 26 to 19 min. We also note that for the cases where the improvement is largest, that the integral time for one loop is at about the open-loop time constant  $\tau$  while the integral time for the other loop is much smaller.

Similar results are found when PID-controllers are used as illustrated by Tables 3 and 4.

To understand better what is going on consider the case with  $\tau = 100$  and  $\lambda_{11} = 10$ . The optimal PI-settings when both loops are required to be identical are:

Controller  $c$ :  $k=39.7$ ,  $\tau_I=65.1$

and when the loops are allowed to be different:

Controller  $c_1$ :  $k_1=46.3$ ,  $\tau_{I1}=101$

Controller  $c_2$ :  $k_2=22.8$ ,  $\tau_{I2}=3.15$

Results for the identical controllers ( $c$ ) are shown with solid lines in Figures 4-10, and with dotted lines for the case with different controllers ( $c_1$ ,  $c_2$ ). The  $\mu$ -plots for RP with  $\tau_P$  in the performance weight fixed at 26 min is shown for the two cases in Fig.4. The plots are quite similar at high frequency, at intermediate frequencies the performance with identical controllers is better, but at low frequency the performance with different controllers is best. Next, consider the magnitude of the controllers shown in Fig.5. We see that controller  $c_1$  is quite similar to  $c$ , while controller  $c_2$  has significantly higher gain (the gain is lower at high frequencies, but this is outside the bandwidth when both loops are closed which is at frequency of about  $0.02 \text{ min}^{-1}$ ). However, this high gain does lead to a more oscillating response as is clear from the plot of the nominal (no uncertainty) sensitivity functions  $|S|$  for the individual loops in Fig.6. Note that the sensitivity function should be as small as possible to have tight control and a peak generally signifies an oscillating response. The nominal sensitivity function in the worst direction with both loops closed simultaneously,  $\bar{\sigma}(S)$ , is shown in Fig.7, and we do not from this plot see much difference between identical or different controllers. Comparing Fig.6 and 7 demonstrate how the interactions reduce the closed-loop bandwidth with about one decade.

The findings above are confirmed by simulations of setpoint changes. Note that a dead time of 1 min in each loop has been included in these simulations, but there is otherwise no model error. In Fig.8 is shown the response of the individual loops and we find as expected that loop 2 is more oscillatory, but there is otherwise no improvement from the higher loop gain. In Figures 9 and 10 are shown responses with both loops closed, and with a setpoint change in the "easy" and "difficult" direction, respectively. The easy direction, which corresponds to increasing both outputs simultaneously, is even faster than found for the individual loops. However, for the difficult direction, corresponding to decreasing  $y_1$  and increasing  $y_2$ , we see that the responses are much slower. We also see here that the higher gain in loop 2 does make a difference, as the response with different controllers approach the steady-state significantly faster in this case.

## 6 Discussion

The results above demonstrated that when robust performance is used as a performance measure, the optimal PI- and PID-tunings for single-loop controllers are not necessarily equal even though the problem statement is completely symmetric. In particular, this implies that the solution is not unique as we may simply interchange controller 1 and 2 and get the same optimal  $\mu$ -value.

Physically, the reason for why the optimal controllers are different is related to the fact that plant is ill-conditioned (large RGA-values). The loops then interact such that it is difficult to control both outputs well at the same time. In some way, the optimal controller tries to find a balance between the conflicting objectives of having fast response and avoiding interactions. The optimal PI-controller in the one example above seemed to be able to reduce the interactions at steady-state, but at the expense of a poorer initial response.

It should be noted that although the problem statement is indeed identical as seen from any loop, the model uncertainty does actually allow the parallel processes *not* to be identical. The results with different tunings may be related to this fact.

The fact that the optimal tunings for the loops are not identical is of course a very interesting result from a theoretical point of view, but it has real practical implications only if there is a real improvement by using different controllers. Though we saw in the example that the performance measured in terms of the  $H_\infty$ -norm may be improved quite a bit, the results from the time domain simulations were less convincing (though we clearly see that the response is improved at higher simulation times in that the outputs return to their setpoints sooner). However, one should note that simulations can only be performed for specific choices of setpoint changes and model errors, and one should not necessarily expect a good correlation between a single simulation and the worst-case response for which  $\mu$  is a measure. There will therefore most likely be cases where clear improvements in performance may be seen also in time domain simulations.

For most of the cases where performance may be improved by having different tunings, the optimal  $\mu$ -value is actually quite insensitive to the tuning parameters, that is, we may get very similar  $\mu$ -values for a large range of, for example, integral times. This is of course of some interest from a practical point of view because it implies that exact tuning is not critical. However, it does also make the numerical optimization with respect to optimal controller parameters very difficult.

These results demonstrate that the  $\mu$ -optimal single-loop controllers do not have to be equal for identical parallel processes when we use PI- or PID-controllers. However, it is not at this point obvious if this is caused by the limited number of degrees of freedom in these controllers, or if the "true" optimal single-loop controllers (with no restrictions on the number of adjustable parameters) are indeed different for each loop. Our conjecture at this point is that they will be different, but this still remains to be investigated.

## 7 Conclusion

In the paper we have studied the control of identical parallel processes. For single-loop control of such processes we have found:

1. The interactions between the processes are generally such that DIC (decentralized integral control) is possible (this is a fortunate circumstance which makes life easier for the engineer). For example, this implies that taking loops out of service is not expected to cause stability problems.

2. When model uncertainty is included and performance is measured in terms of the  $H_\infty$ -norm, the optimal single-loop PI- or PID-tunings are not necessarily equal for the individual loops. This is contrary to what one intuitively would expect. It also implies that the optimal solution is non-unique.

### NOMENCLATURE (also see Fig.3)

$C(s)$  - controller

$G(s)$  - linear model of process

RGA - Relative Gain Array, elements are  $\lambda_{ij}$

$S(s) = (I + G(s)C(s))^{-1}$ , sensitivity function

$w_I$  - input uncertainty weight

$w_P$  - performance weight

#### Greek symbols

$\|A\|_\infty = \max_\omega \bar{\sigma}(A(j\omega))$  -  $H_\infty$ -norm of  $A$

$\lambda_i$  - eigenvalue

$\lambda_{11}(j\omega) = (1 - \frac{g_{12}(j\omega)g_{21}(j\omega)}{g_{11}(j\omega)g_{22}(j\omega)})^{-1}$  - 1,1-element in RGA.

$\omega$  - frequency ( $\text{min}^{-1}$ )

$\bar{\sigma}$  - maximum singular value

$\mu$  - structured singular value

### REFERENCES

- Bellman, R., 1970, *Introduction to Matrix Analysis*, McGraw-Hill, New York.
- Bristol, E. H., 1966, "On a New Measure of Interactions for Multivariable Process Control", *IEEE Trans. Automat. Contr.*, **AC-11**, 133-134.
- Davis, P.J., 1979, *Circulant Matrices*, Wiley, New York.
- Doyle, J. C., 1982, "Analysis of Feedback Systems with Structured Uncertainties", *IEEE Proc.*, **129**, D, 242-250.
- Shinskey, F. G., 1979, *Process Control Systems*, 2nd Edition, McGraw-Hill, New York.
- Shinskey, F. G., 1984, *Distillation Control*, 2nd Edition, McGraw-Hill, New York .
- Skogestad, S. and M. Morari, 1987, "Implications of Large RGA-Elements on Control Performance", *Ind. & Eng. Chem. Research*, **26**, 11, 2323-2330.
- Skogestad, S., M. Morari and J.C. Doyle, 1988, "Robust Control of Ill-Conditioned Plants: High-Purity Distillation", *IEEE Automatic Control*, **33**, 12, 1092-1105.
- Skogestad, S. and M. Morari, 1988, "Variable Selection for Decentralized Control", Paper 128c, AIChE Annual Meeting, Washington DC, Nov. 1988.
- Skogestad, S. and P. Lundström, 1990, "Mu-optimal LV-control of Distillation Columns", *Comp. and Chem. Eng.*, In Press.

## APPENDIX

### Model of cooling of parallel reactor system (Example 2)

The flow diagram is shown in Fig.2. Each reactor is cooled by a heat exchanger, which acts as an evaporator for the cooling medium. The evaporated cooling medium is then superheated by heat exchange with the incoming feed.

The following assumptions have been made when developing the dynamic model of the system:

i) The cooling medium enters the evaporators at its bubble point and leaves at its dewpoint  $T_d$ .

ii) The dynamics of the evaporators have been neglected. This is justified by assuming constant holdup of liquid and vapor in the evaporator and neglecting accumulation of heat in the walls.

iii) The dynamics of the superheater are neglected. The arithmetic mean temperature difference  $\Delta T$  is used for the superheater.

iv) All physical properties constant.

v) The reactors have constant volume  $V$ .

vi) The overall heat transfer coefficient  $U$  in the superheater is constant.

Let  $m_i$  (kg/s) denote the flow of the fluid (subscript f) through each reactor and let  $q_i$  (kg/s) denote the flow of coolant (subscript c) through each evaporator. The energy balance for reactor no.  $i$  is:

$$\frac{d}{dt}(\rho_f V C p_f T_i) = m_i C p_f (T_m - T_i) - q_i \Delta H_{vap} \quad (33)$$

The energy balance for the superheater:

$$\sum_i m_i C p_f (T_f - T_m) = \sum_i q_i C p_c (T_h - T_d) = U A \Delta T \quad (34)$$

Fairly straight forward algebra then yields:

$$\frac{d}{dt}(\rho_f V C p_f T_i) = m_i C p_f \left( T_f \frac{2G + F - 1}{2G + F + 1} + T_d \frac{2}{2G + F + 1} - T_i \right) - q_i \Delta H_{vap} \quad (35)$$

Where

$$F = \frac{\sum_i m_i C p_f}{\sum_i q_i C p_c} \quad (36)$$

$$G = \frac{\sum_i m_i C p_f}{U A} \quad (37)$$

This can be linearized to give :

$$\frac{d}{dt} \Delta T_i = -\frac{m_i}{\rho_f V} \Delta T_i - \frac{H(T_f - T_d) + \Delta H_{vap}}{\rho_f V C p_f} \Delta q_i - \frac{H(T_f - T_d)}{\rho_f V C p_f} \sum_{j \neq i} \Delta q_j \quad (38)$$

Where

$$H = \frac{2m_i C p_f}{(2G + F + 1)^2} \frac{\sum_i m_i C p_f}{(\sum_i q_i)^2 C p_c} \quad (39)$$

Note that  $H = 0$  for the special case with no superheater, ie.,  $U A = 0$ . Let  $u$  denote the vector of individual valve positions (inputs) using deviation variables, and let  $\Delta q = P_q u$ , where  $P_q$  is a matrix which describe how the flowrate  $q_j$  in

stream  $j$  changes when the position  $u_i$  of valve  $i$  changes. The equations can then be expressed in matrix notation as:

$$\dot{x} = Ax + BP_q u \quad (40)$$

where  $x$  is a vector of tank temperatures  $\Delta T_i$ ,  $A$  is a diagonal matrix with diagonal elements  $a_{ii}$ , and  $B$  and  $P_q$  are parallel matrices. The product  $BP_q$  is also a parallel matrix. Let the elements of  $P_q$  be denoted  $p_{ij}$ . Then we have

$$a_{ii} = -\frac{m_i}{\rho_f V} \quad (41)$$

$$b_{ii} = -[H(T_f - T_d) + \Delta H_{vap}] \frac{1}{\rho_f V C p_f} \quad (42)$$

$$b_{ij} = -\frac{H(T_f - T_d)}{\rho_f V C p_f} \quad (43)$$

$$[BP_q]_{ii} = b_{ii} p_{ii} + (n-1) b_{ij} p_{ij} \quad (44)$$

$$[BP_q]_{ij} = b_{ii} p_{ij} + b_{ij} p_{ii} + (n-2) b_{ij} p_{ij} \quad (45)$$

Consequently, the process transfer function  $G(s)$  will be a parallel matrix given by:

$$G(s) = \frac{\tau}{\tau s + 1} BP_q \quad \text{where} \quad \tau = \frac{\rho_f V}{m_i} \quad (46)$$

Table 1. Identical PI-controllers.

$\tau$ min.	$a$	$\tau_P$ min.	$k$	$\tau_I$ min.
10	0.000	1.30	5.57	11.5
10	0.827	6.20	3.86	5.39
10	0.949	13.3	2.91	3.03
10	0.984	43.0	2.95	3.19
10	0.995	133	2.93	3.13
100	0.000	1.33	55.6	111
100	0.827	8.37	41.7	58.9
100	0.949	26.0	39.7	65.1
100	0.984	61.5	38.5	39.8
100	0.995	104	37.3	19.3

Table 2. Different PI-controllers.

$\tau$ min.	$a$	$\tau_P$ min.	$k_1$	$k_2$	$\tau_{I1}$ min.	$\tau_{I2}$ min.
10	0.000	1.30	5.57	5.57	11.5	11.5
10	0.827	6.15	3.95	3.81	5.89	4.92
10	0.949	13.2	2.98	2.91	3.21	2.91
10	0.984	42.2	2.94	2.90	3.15	3.05
10	0.995	133	2.93	2.93	3.13	3.13
100	0.000	1.33	55.6	55.6	111	111
100	0.827	7.86	35.3	36.2	90.1	3.56
100	0.949	18.9	46.3	22.8	101	3.15
100	0.984	51.2	51.8	13.1	86.5	3.34
100	0.995	103	38.2	36.0	19.6	18.1



**Table 3.** Identical PID-controllers.

$\tau$ min.	$a$	$\tau_P$ min.	$k$	$\tau_I$ min.	$\tau_D$ min.
10	0.000	0.822	8.40	11.0	0.58
10	0.827	4.47	5.85	5.52	0.40
10	0.949	9.42	4.47	2.13	0.53
10	0.984	18.6	4.10	1.90	0.52
10	0.995	52.5	3.85	1.60	0.60
100	0.000	0.824	83.1	99.1	0.59
100	0.827	5.92	59.6	83.6	0.37
100	0.949	19.0	56.8	65.0	0.34
100	0.984	48.5	55.7	50.8	0.35
100	0.995	91.1	54.1	26.5	0.39

**Table 4.** Different PID-controllers.

$\tau$ min.	$a$	$\tau_P$ min.	$k_1$	$k_2$	$\tau_{I1}$ min.	$\tau_{I2}$ min.	$\tau_{D1}$ min.	$\tau_{D2}$ min.
100	0.000	0.824	83.1	83.1	99.1	99.1	0.59	0.59
100	0.827	5.50	56.6	51.4	121	3.63	0.18	0.56
100	0.949	13.2	67.4	30.3	95.7	1.90	0.30	0.55
100	0.984	43.1	62.9	25.1	88.0	4.90	0.18	0.11
100	0.995	88.4	75.8	26.3	51.3	8.09	0.34	0.37