Automatica, Vol. 25, No. 1, pp. 119–125, 1989
Printed in Great Britain.

0005–1098/89 $3.00 + 0.00
Pergamon Press plc
© 1989 International Federation of Automatic Control

Brief Paper

Robust Performance of Decentralized Control Systems by Independent Designs*

SIGURD SKOGESTAD† and MANFRED MORARI†

Key Words—Decentralized control; robustness; large-scale systems; (structured singular value).

Abstract—Decentralized control systems have fewer tuning parameters, are easier to understand and retune, and are more easily made failure tolerant than general multivariable control systems. In this paper the decentralized control problem is formulated as a series of independent designs. Simple bounds on these individual designs are derived, which when satisfied, guarantee robust performance of the overall system. The results provide a generalization of the μ-interaction measure introduced by Grosdidier and Morari (*Automatica*, **22**, 309–319 (1986)).

1. *Introduction*

Robust performance. The goal of any controller design is that the overall system is stable and satisfies some minimum performance requirements. These requirements should be satisfied at least when the controller is applied to the *nominal* plant (G), that is, we require nominal stability (NS) and nominal performance (NP). In addition, when a decentralized controller is used, it is desirable that the system be failure tolerant. This means that the system should remain stable as individual loops are opened or closed.

In practice the real (or "perturbed") plant G_p is not equal to the model G. The term "robust" is used to indicate that some property holds for a set Π of possible plants G_p as defined by the uncertainty description. In particular, by robust performance (RP) we mean that the performance requirements are satisfied for all $G_p \in \Pi$. Mainly for mathematical convenience, we choose to define performance using the H_∞-norm. Define

$$\mathrm{NP} \Leftrightarrow \bar{\sigma}(\Sigma) \le 1, \quad \forall\omega \tag{1a}$$

$$\mathrm{RP} \Leftrightarrow \bar{\sigma}(\Sigma_p) \le 1, \quad \forall\omega, \quad \forall G_p \in \Pi. \tag{1b}$$

In most cases Σ is the weighted sensitivity operator

$$\Sigma = W_1 S W_2, \quad S = (\mathrm{I} + GC)^{-1} \tag{2a}$$

$$\Sigma_p = W_1 S_p W_2, \quad S_p = (\mathrm{I} + G_p C)^{-1}. \tag{2b}$$

The input weight W_2 is often equal to the disturbance model. The output weight W_1 is used to specify the frequency range over which the sensitivity function should be small and to weight each output according to its importance, and C is defined in the next section.

The definition of Robust Performance is of no value without simple methods to test if conditions like (1b) are satisfied for all G_p in the set Π of possible plants. Doyle *et al.* (1982) have derived a computationally useful condition for (1b) involving the Structured Singular Value μ (μ is defined in the Appendix). To use μ we must model the uncertainty (the set Π of possible plants G_p) as norm bounded perturbations (Δ_i) on the nominal system. Through weights each perturbation is normalized to be of size one:

$$\bar{\sigma}(\Delta_i) \le 1, \quad \forall\omega. \tag{3}$$

The perturbations, which may occur at different locations in the system, are collected in the diagonal matrix Δ_U (the subscript U denotes uncertainty)

$$\Delta_U = \mathrm{diag}\,\{\Delta_1, \ldots, \Delta_n\} \tag{4}$$

and the system is rearranged to match the structure in Fig. 1. The interconnection matrix M in Fig. 1 is determined by the nominal model (G), the size and nature of the uncertainty, the performance specifications and the controller. For Fig. 1 the robust performance condition (1b) becomes (Doyle *et al.*, 1982)

$$\mathrm{RP} \Leftrightarrow \mu(M) < 1, \quad \forall\omega \tag{5}$$

where $\mu(M)$ depends on both the elements in the matrix M and the *structure* of the perturbation matrix $\Delta = \mathrm{diag}\,\{\Delta_U, \Delta_P\}$. Sometimes this is shown explicity by using the notation $\mu(M) = \mu_\Delta(M)$. Δ_P is a full square matrix with dimension equal to the number of outputs (the subscript P denotes performance). In addition to satisfying (5), the system must be nominally stable (i.e. M is stable). Also note that within this framework, the issue of robust stability (RS) is simply a special case of robust performance.

Decentralized control. Decentralized control involves using a diagonal or block-diagonal controller (see Fig. 2)

$$C = \mathrm{diag}\,\{c_i\}.$$

Some reasons for using a decentralized controller are

- tuning and retuning is simple
- they are easy to understand
- they are easy to make failure tolerant.

The design of a decentralized control system involves two steps.

(A) Choice of pairings (control structure).
(B) Design of each SISO-controller c_i (or block).

The best way to proceed for each of these steps is still an active area of research. The RGA (Bristol, 1966) has proven to be an efficient tool for eliminating undesirable pairings in

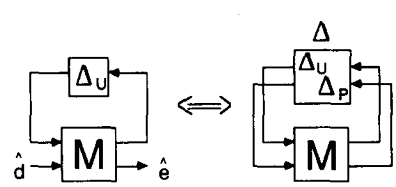

FIG. 1. General structure for studying effect of uncertainty (Δ_U) on stability or performance. M is a function of the plant model (G) and the controller. $\hat{d}$: external inputs (disturbances, reference signals), $\hat{e}$: external outputs (weighted errors $y - r$), $\hat{e} = \Sigma_P \hat{d}$.

* Received 29 January 1987; revised 9 March 1988; received in final form 23 June 1988. The original version of this paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor A. Mahalanabis under the direction of Editor H. Kwakernaak.

† Chemical Engineering, 206–41, California Institute of Technology, Pasadena, CA 91125, U.S.A.

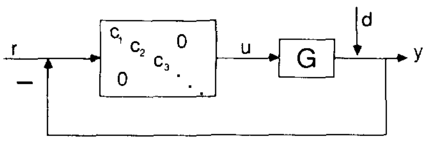

FIG. 2. Decentralized control structure.

Step A. This paper deals with Step B. Two design methods which may be applied for this step are (1) sequential loop-closing and (2) independent design of each loop.

(1) *Sequential loop-closing.* This design approach [e.g. Mayne (1973)] involves designing each element (or block) in C sequentially. Usually the controller corresponding to a fast loop is designed first. This loop is then *closed* before the design proceeds with the next controller. This means that the information about the "lower-level" controllers is directly used as more loops are closed. The final step in the design procedure is to test if the overall system satisfies the RP-condition (5). The main disadvantages of this design method are as follows.

- Failure tolerance is not guaranteed when "lower-level" loops fail.
- The method depends strongly on which loop is designed first and how this controller is designed.
- There are no guidelines on how (and in which order) to design the controllers for each loop in order to guarantee robust performance of the overall system. Therefore the design proceeds by "trial-and-error".

(2) *Independent design of each loop.* This is the design approach used in this paper. In this case each controller element (or block) is designed independently of the others. We present a procedure for these designs which guarantees robust performance of the overall system. The proposed method has the following advantage.

- Failure tolerance: nominal stability (of the remaining system) is guaranteed if any loop fails.
- Each controller is designed directly with no need for trial-and-error.

The main limitation of the approach is the assumption of independent designs, which means that we do not exploit information about the controllers used in the other loops. Therefore the derived bounds are sufficient and not necessary for robust performance.

Problem definition. This paper addresses the following problem: let $\tilde{G}$ denote the diagonal (or block-diagonal) version of the plant corresponding to the chosen structure of C (i.e. $\tilde{G}$ is found from G by deleting the off-diagonal elements). Assume that uncertainty and "interactions" are neglected when designing the controller C, that is, design each element (or block) of C independently based on the information contained in $\tilde{G}$ only. What constraints have to be placed on individual designs in order to guarantee robust performance of the overall system (which can be any plant G_p from the set Π)?

The constraints on the individual designs are chosen to be in terms of bounds on $|\tilde{h}_i|$ and $|\tilde{s}_i|$ where $\tilde{h}_i$ and $\tilde{s}_i$ are the closed-loop transfer functions for loop i:

$$\tilde{h}_i = g_{ii}c_i(1 + g_{ii}c_i)^{-1}, \quad \tilde{H} = \operatorname{diag}\{\tilde{h}_i\} \tag{6a}$$

$$\tilde{s}_i = (1 + g_{ii}c_i)^{-1}, \quad \tilde{S} = \operatorname{diag}\{\tilde{s}_i\}. \tag{6b}$$

[In general, if C is *block*-diagonal, $\tilde{h}_i$, $\tilde{s}_i$ and g_{ii} are matrices corresponding to the block-structure of C, and $|\tilde{h}_i|$ and $|\tilde{s}_i|$ are replaced by $\bar{\sigma}(\tilde{H}_i)$ and $\bar{\sigma}(\tilde{S}_i)$.]

We solve the decentralized problem as defined above, by deriving the tightest possible bounds on

$$\bar{\sigma}(\tilde{H}) = \max_i |\tilde{h}_i| \quad \text{and} \quad \bar{\sigma}(\tilde{S}) = \max_i |\tilde{s}_i|$$

which guarantee robust performance:

$$\text{RP} \Leftarrow \bar{\sigma}(\tilde{H}) < \bar{c}_H \quad \text{or} \quad \bar{\sigma}(\tilde{S}) < \bar{c}_S, \quad \forall \omega. \tag{7}$$

In addition to satisfying (7) the system has to be nominally stable. The μ-interaction measure, introduced by Grosdidier and Morari (1986), gives a sufficient condition for nominal stability:

$$\text{NS} \Leftarrow \bar{\sigma}(\tilde{H}) \le \mu_C(E_H), \quad \forall \omega, \quad E_H = (G - \tilde{G})\tilde{G}^{-1} \tag{8}$$

(μ is computed with respect to the structure of C which is equal to the structure of $\tilde{G}$, $\tilde{H}$ and $\tilde{S}$). This paper provides a generalization of the μ-interaction measure from the case of nominal stability (NS) to the case of robust performance (RP). The results derived here also apply to robust stability (RS) or nominal performance (NP) if the μ-condition (5) is an RS- or NP-condition rather than an RP-condition.

Notation

The most important notation is summarized below.

G – model of the plant.

$\tilde{G} = \operatorname{diag}\{g_{ii}\}$ (corresponding to structure of C).

$G_p = f(G, \Delta_U)$, Δ_U: uncertainty, $G_p = G$ when $\Delta_U = 0$.

$$\tilde{S} = (I + \tilde{G}C)^{-1}, \qquad \tilde{H} = I - \tilde{S}$$

$$S = (I + GC)^{-1}, \qquad H = I - S$$

$$S_p = (I + G_pC)^{-1}, \qquad H_p = I - S_p.$$

Stability of individual loops $\Leftrightarrow \tilde{H}$ (and $\tilde{S}$) is stable.

NS $\Leftrightarrow H$ (and S) is stable (overall system stable with no uncertainty).

RS $\Leftrightarrow H_p$ (and S_p) is stable (for all $G_p \in \Pi$).

NP $\Leftrightarrow S$ satisfies the performance specification.

RP $\Leftrightarrow S_p$ satisfies the performance specification (for all $G_p \in \Pi$).

2. *Nominal stability (of H and S)*

To apply the general robust performance condition $\mu(M) < 1$ in equation (5) we must require that the system is nominally stable, that is, that the interconnection matrix M is stable. Nominal stability is satisfied if H (and S) is stable. However, note that nominal stability (i.e. stability of H and S) is *not* necessarily implied by the stability of the individual loops (i.e. stability of $\tilde{H}$ and $\tilde{S}$). The "interactions" (difference between G and $\tilde{G}$) may cause stability problems as discussed by Grosdidier and Morari (1986). If either one of the following conditions on $\bar{\sigma}(\tilde{H})$ and $\bar{\sigma}(\tilde{S})$ is satisfied, then the stability of $\tilde{H}$ (or $\tilde{S}$) implies nominal stability.

Condition 1 for NS (Grosdidier and Morari, 1986). Assume $\tilde{H}$ is stable (each loop is stable by itself), and that G and $\tilde{G}$ have the same number of RHP (unstable) poles. Then H is stable (the system is stable when all loops are closed) if

$$\bar{\sigma}(\tilde{H}) \le \mu_C^{-1}(E_H), \quad \forall \omega \tag{10}$$

where

$$E_H = (G - \tilde{G})\tilde{G}^{-1}. \tag{11}$$

$\mu_C(E_H)$ is the μ-interaction measure and μ is computed with respect to the structure of the decentralized controller C. Note that the condition that G and $\tilde{G}$ have the same number of RHP poles, is generally satisfied only when G and $\tilde{G}$ are stable. In order to allow integral action ($\tilde{H}(0) = I$), we have to require that $\mu(E_H) < 1$ at $\omega = 0$, that is, we need diagonal dominance at low frequencies. If this is not the case the following alternative condition may be used.

Condition 2 for NS (Postlethwaite and Foo, 1985; Grosdidier, personal communication, 1985). Assume $\tilde{S}$ is stable, and that G and $\tilde{G}$ have the same number of RHP-zeros. Then S (and H) is stable if

$$\bar{\sigma}(\tilde{S}) \le \mu_C^{-1}(E_S), \quad \forall \omega \tag{12}$$

where

$$E_S = (G - \tilde{G})G^{-1}. \tag{13}$$

Since we have to require $\tilde{S} = I$ as $\omega \to \infty$ for any real system, we have to require $\mu(E_S) < 1$ as $\omega \to \infty$, in order to be able to satisfy (12), that is, we must have diagonal dominance at high frequencies.

Conditions 1 and 2 are conditions for nominal stability (i.e. stability of H and S). These conditions *cannot* be combined over different frequency ranges as is sometimes possible for true uncertainties (Postlethwaite and Foo, 1985). The reason is that our "uncertainties" $\bar{H}$ and $\bar{S}$ do not necessarily cover the same uncertainty set; for this to be the case we would at least have to allow $\bar{\sigma}(\bar{S}) \geq 1$ and $\bar{\sigma}(\bar{H}) \geq 1$ in order to include the nominal case with no "uncertainty" (i.e. $\bar{H} = 0$ and $\bar{S} = 0$).

What to do when both conditions fail. In some cases it may be impossible to satisfy either (10) or (12). For example, in order to satisfy (10) and to have integral action ($\tilde{H}(0) = \mathrm{I}$) we must require at least

$$\rho(E_H(0)) < 1 \quad (14)$$

(ρ is the spectral radius of E_H). (14) is derived from (10) by assuming $\tilde{H} = \tilde{h}\mathrm{I}$ (all loops identical) which yields the least restrictive bound $\bar{\sigma}(\tilde{H}) \leq \rho^{-1}(E_H)$ in (10). In general (14) is conservative. For example, it is easily shown (Skogestad and Morari, 1987b), that it is always possible to find a diagonal controller which yields NS if the less restrictive condition

$$\mathrm{Re}\{\lambda_i(E_H(0))\} > -1, \quad \forall i \quad (15)$$

[λ_i is the ith eigenvalue] is satisfied. One example for which (15) is satisfied, but not (14) is the following 2×2 plant:

$$G(0) = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}, \qquad \tilde{G}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\lambda_i(E_H(0)) = \pm i2, \quad \rho(E_H(0)) = 2.$$

For 2×2 plants, (15) is always satisfied when $\mathrm{RGA}_{11} > 0$ [RGA_{11} is the 1,1-element of the RGA (Bristol, 1966)], while (14) is only satisfied when $\mathrm{RGA}_{11} > 0.5$.

Similarly, condition (12) may be impossible to satisfy because (i) G and $\tilde{G}$ do not have the same number of RHP-zeros, or (ii) $\mu(E_S(j\infty)) \geq 1$.

In cases when neither conditions (10) or (12) can be satisfied we may try to redefine the nominal model (G and $\tilde{G}$) such that either condition 1 or 2 is satisfied.

However, since the set Π of possible plants (G_p) still has to be the same, this generally means that we have to increase the magnitude of the model uncertainty. The three following "tricks" may be used (the last two of these are probably easiest to apply since uncertainty always dominates at high frequency).

- To satisfy (10). The plant is made diagonal dominant at low frequencies [$\mu(E_H(0)) < 1$], by reducing the magnitude of the nominal off-diagonal elements and replacing it by element uncertainty (at low frequency) [see Skogestad and Morari (1987a) on how to treat element uncertainty within the μ-framework].
- To satisfy (12). The plant is made diagonal dominant at high frequencies [$\mu(E_S(j\infty)) < 1$], by reducing the magnitude of the nominal off-diagonal elements and replacing it by element uncertainty (at high frequency).
- To have the same number of RHP-zeros in G and $\tilde{G}$: RHP-zeros (or time delays) are "removed" by treating them as uncertainty.

One extreme is obviously to treat the off-diagonal elements entirely as additive element uncertainty. In this case $\mu(E_H) = 0$ at all frequencies, and nominal stability (stability of $\tilde{H}$) is obviously satisfied if each loop $\tilde{h}$ is stable (since $G = \tilde{G}$ and $H = \tilde{H}$ in this case). This approach is generally more conservative, however, since the off-diagonal elements in G (which nominally are equal to g_{ij}) for the case of element uncertainty are allowed to be *any* transfer function of magnitude $|g_{ij}|$ (in particular, both g_{ij} and $-g_{ij}$ are allowed). This additional uncertainty makes it more difficult to satisfy the *robust* stability and performance conditions.

3. *Robust performance*

Having derived conditions for nominal stability, we can now proceed to the case of robust performance. The objective of this section is to derive bounds on the individual designs ($\tilde{H}$ and $\tilde{S}$), which when satisfied guarantee robust performance of the overall system (that is, $\mu(M) < 1$). This is accomplished in two steps.

(1) Sufficient conditions for RP in terms of bounds on $\bar{\sigma}(H)$ and $\bar{\sigma}(S)$ are derived by writing M as a linear fractional transformation (LFT) of H and S.

(2) These bounds are used to derive sufficient conditions for RP in terms of bounds on $\bar{\sigma}(\tilde{H})$ and $\bar{\sigma}(\tilde{S})$.

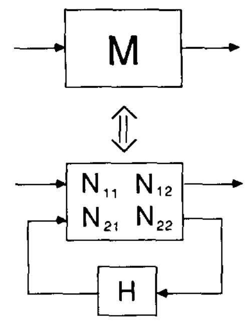

FIG. 3. M written as an LFT of H. N is independent of the controller.

3.1. *Robust performance conditions in terms of H and S.* The robust performance condition [equation (5)]

$$\mathrm{RP} \Leftrightarrow \mu_\Delta(M) \leq 1, \quad \forall \omega$$

may be used to derive sufficient conditions for RP in terms of bounds on $\bar{\sigma}(H)$ and $\bar{\sigma}(S)$ (Skogestad and Morari, 1988). To this end write M as an LFT of H (Fig. 3)

$$M = N_{11}^H + N_{12}^H H(\mathrm{I} - N_{22}^H H)^{-1} N_{21}^H. \quad (16)$$

The matrix N^H, which is independent of C, can be obtained from M by inspection in many cases. Otherwise, the procedure given by Skogestad and Morari (1988) can be used. They also point out that in general M is affine in H, that is, $N_{22}^H = 0$. Applying Theorem 1 of Skogestad and Morari (1988) (the theorem is reproduced in the Appendix) the following sufficient condition for (5) is derived.

RP-condition in terms of H. Assume M is given as an LFT of H (Eq. 16). Then at any given frequency

$$\mu_\Delta(M) \leq 1 \quad \text{if } \bar{\sigma}(H) \leq c_H \quad (17a)$$

where at this frequency c_H solves

$$\mu_{\hat{\Delta}}\begin{pmatrix} N_{11}^H & N_{12}^H \\ c_H N_{21}^H & c_H N_{22}^H \end{pmatrix} = 1 \quad (17b)$$

and μ in (17b) is computed with respect to the structure $\hat{\Delta} = \mathrm{diag}\{\Delta, H\}$.

Note that H is generally a "full" matrix if the controller is diagonal. A similar bound in terms of S is derived by replacing H by S in equations (16) and (17). (17) applies on a frequency-by-frequency basis. This implies that $\mu(M) \leq 1$ at a given frequency is guaranteed if either $\bar{\sigma}(H) < c_H$ *or* $\bar{\sigma}(S) < c_S$ at this frequency. Consequently, the bounds on $\bar{\sigma}(H)$ and $\bar{\sigma}(S)$ can be combined over different frequency ranges. In particular, the following holds

$$\mathrm{RP} \Leftarrow \bar{\sigma}(H) \leq c_H \quad \text{or} \quad \bar{\sigma}(S) \leq c_S, \quad \forall \omega. \quad (18)$$

Example. Robust performance with input uncertainty (Fig.

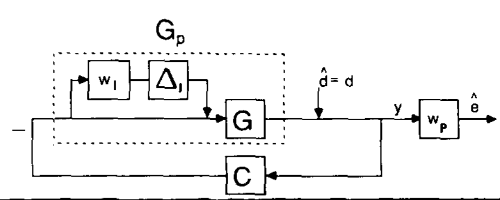

FIG. 4. Plant with input uncertainty (Δ_I) of magnitude $w_I(s)$. Robust performance is satisfied if $\bar{\sigma}(w_P(\mathrm{I} + G_p C)^{-1}) \leq 1$, for all $\Delta_I (\bar{\sigma}(\Delta_I) \leq 1)$.

4). Let the set Π of possible plants be given by

$$G_p = G(I + w_I\Delta_I), \quad \bar{\sigma}(\Delta_I) \le 1, \quad \forall\omega. \tag{19}$$

Here w_I is the magnitude of the relative (multiplicative) uncertainty at the plant inputs. For robust performance we require that the magnitude of the sensitivity operator is bounded by $|w_P|^{-1}$

$$\begin{aligned} \mathrm{RP} &\Leftrightarrow \bar{\sigma}(w_P S_p) \\ &= \bar{\sigma}(w_P(I + G_p C)^{-1}) \le 1, \quad \forall\omega, \quad \forall G_p \in \Pi. \end{aligned} \tag{20}$$

This condition is most easily checked using μ [equation (5)]:

$$\mathrm{RP} \Leftrightarrow \mu_\Delta(M) \le 1, \quad \forall\omega \tag{21a}$$

where the interconnection matrix M is (Skogestad *et al.*, 1988):

$$M = \begin{pmatrix} -w_I CSG & -w_I CS \\ w_P SG & w_P S \end{pmatrix} \tag{21b}$$

and $\mu(M)$ is computed with respect to the structure $\Delta = \mathrm{diag}\{\Delta_I, \Delta_P\}$. Δ_P is always a "full" matrix of the same dimensions as S. Δ_I is often a diagonal matrix (if the inputs do not affect each other). Rewrite M in terms of S and H such that C does not appear

$$M = \begin{pmatrix} -w_I G^{-1}HG & -w_I G^{-1}H \\ w_P SG & w_P S \end{pmatrix}. \tag{22}$$

By inspection M may be written as an LFT (16) of H (recall $S = I - H$)

$$\begin{aligned} M &= N_{11}^H + N_{12}^H H N_{21}^H \\ &= \begin{pmatrix} 0 & 0 \\ w_P G & w_P I \end{pmatrix} + \begin{pmatrix} -w_I G^{-1} \\ -w_P I \end{pmatrix} H (G \quad I). \end{aligned} \tag{23}$$

We derive from (23) and (17)

$$\mathrm{RP} \quad \text{if} \quad \bar{\sigma}(H) \le c_H, \quad \forall\omega \tag{24a}$$

where at each frequency c_H solves

$$\mu\begin{pmatrix} 0 & 0 & -w_I G^{-1} \\ w_P G & w_P I & -w_P I \\ c_H G & c_H I & 0 \end{pmatrix} = 1. \tag{24b}$$

Similarly, a bound on $\bar{\sigma}(S)$ is derived by writing M as an LFT of S

$$\begin{aligned} M = N_{11}^S + N_{12}^S S N_{21}^S &= \begin{pmatrix} -w_I I & -w_I G^{-1} \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} w_I G^{-1} \\ w_P I \end{pmatrix} S (G \quad I) \end{aligned} \tag{25}$$

equations (25) and (17) (with H replaced with S) yield

$$\mathrm{RP} \quad \text{if} \quad \bar{\sigma}(S) \le c_S, \quad \forall\omega \tag{26a}$$

where at each frequency c_S solves

$$\mu\begin{pmatrix} -w_I I & -w_I G^{-1} & w_I G^{-1} \\ 0 & 0 & w_P I \\ c_S G & c_S I & 0 \end{pmatrix} = 1. \tag{26b}$$

In both equations (24b) and (26b) μ is computed with respect to the structure $\mathrm{diag}\{\Delta_I, \Delta_P, H\}$. The bounds (24) and (26) may be combined over different frequency ranges, and RP is guaranteed if either one is satisfied at any frequency [equation (18)]. In practice, (24) is most easily satisfied at high frequencies and (26) at low frequencies.

3.2. *Robust performance condition in terms of $\tilde{H}$ and $\tilde{S}$.* Sufficient conditions for robust performance in terms of $\bar{\sigma}(\tilde{H})$ and $\bar{\sigma}(\tilde{S})$ may now be derived using the identities (Grosdidier, personal communication, 1985)

$$H = G\tilde{G}^{-1}\tilde{H}(I + E_H\tilde{H})^{-1} \tag{27}$$

$$S = \tilde{S}(I - E_S\tilde{S})^{-1}\tilde{G}G^{-1}. \tag{28}$$

Note that (27) and (28) both are LFTs of H (and S) in terms of $\tilde{H}$ (and $\tilde{S}$). In Section 3.1 we pointed out that in general M can be written as an LFT of H with $N_{22}^H = 0$:

$$M = N_{11}^H + N_{12}^H H N_{21}^H. \tag{29}$$

Substituting (27) into (29) yields

$$M = N_{11}^H + N_{12}^H G\tilde{G}^{-1}\tilde{H}(I + E_H\tilde{H})^{-1}N_{21}^H \tag{30}$$

which is an LFT of M in terms of $\tilde{H}$. Using Theorem 1 (Appendix) and (30) we derive the following.

RP-condition in terms of $\tilde{H}$. Let $M = N_{11}^H + N_{12}^H H N_{21}^H$. Then at any frequency

$$\mu_\Delta(M) \le 1 \quad \text{if} \quad \bar{\sigma}(\tilde{H}) \le \bar{c}_H \tag{31a}$$

where at this frequency $\bar{c}_H$ solves

$$\mu_{\hat{\Delta}}\begin{pmatrix} N_{11}^H & N_{12}^H G\tilde{G}^{-1} \\ \bar{c}_H N_{21}^H & -\bar{c}_H E_H \end{pmatrix} = 1 \tag{31b}$$

and μ is computed with respect to the structure $\hat{\Delta} = \mathrm{diag}\{\Delta, C\}$.

Note that the structure of C is *block-diagonal* and equal to that of $\tilde{H}$. An entirely equivalent condition may be derived in terms of $\bar{\sigma}(\tilde{S})$.

RP-condition in terms of $\tilde{S}$. Let $M = N_{11}^S + N_{12}^S S N_{21}^S$. Then at any frequency

$$\mu_\Delta(M) \le 1 \quad \text{if} \quad \bar{\sigma}(\tilde{S}) \le \bar{c}_S \tag{32a}$$

where $\bar{c}_S$ solves

$$\mu_{\hat{\Delta}}\begin{pmatrix} N_{11}^S & N_{12}^S \\ \bar{c}_S\tilde{G}G^{-1}N_{21}^S & \bar{c}_S E_S \end{pmatrix} = 1 \tag{32b}$$

and μ is computed with respect to the structure $\hat{\Delta} = \mathrm{diag}\{\Delta, C\}$.

Again, the bounds (31) and (32) may be combined over different frequency ranges.

Combined RP-condition.

$$\mathrm{RP} \quad \text{if} \quad \bar{\sigma}(\tilde{H}) \le \bar{c}_H \quad \text{or} \quad \bar{\sigma}(\tilde{S}) < \bar{c}_S, \quad \forall\omega \tag{33}$$

Example. Robust performance with input uncertainty (continued). Consider the same example as above (Fig. 4). However, in this case we will derive bounds in terms of $\bar{\sigma}(\tilde{H})$ and $\bar{\sigma}(\tilde{S})$. An RP-condition in terms of $\bar{\sigma}(\tilde{H}) = \max_i |\tilde{h}_i|$ is derived by combining (31) and (23):

$$\mathrm{RP} \quad \text{if} \quad \bar{\sigma}(\tilde{H}) \le \bar{c}_H, \quad \forall\omega \tag{34a}$$

where at each frequency $\bar{c}_H$ solves

$$\mu_{\hat{\Delta}}\begin{pmatrix} 0 & 0 & -w_I\tilde{G}^{-1} \\ w_P G & w_P I & -w_P G\tilde{G}^{-1} \\ \bar{c}_H G & \bar{c}_H I & -\bar{c}_H E_H \end{pmatrix} = 1. \tag{34b}$$

Similarly, the RP-condition in terms of $\bar{\sigma}(\tilde{S}) = |\tilde{s}_i|$ derived from (32) and (25) is

$$\mathrm{RP} \quad \text{if} \quad \bar{\sigma}(\tilde{S}) \le \bar{c}_S, \quad \forall\omega \tag{35a}$$

where at each frequency $\bar{c}_S$ solves

$$\mu_{\hat{\Delta}}\begin{pmatrix} -w_I I & -w_I G^{-1} & w_I G^{-1} \\ 0 & 0 & w_P I \\ \bar{c}_S\tilde{G} & \bar{c}_S G\tilde{G}^{-1} & \bar{c}_S E_S \end{pmatrix} = 1. \tag{35b}$$

In both (34b) and (35b) μ is computed with respect to the structure $\hat{\Delta} = \mathrm{diag}\{\Delta_I, \Delta_P, C\}$. Conditions (34) and (35) can be combined as shown in (33).

4. *Design procedure*

The following design procedure for decentralized control systems based on the "independent designs"-assumption is proposed. Find a decentralized controller which yields individual loops ($\tilde{H}$ and $\tilde{S}$) which are stable and in addition satisfy the following.

(1) Nominal stability: satisfy $\bar{\sigma}(\tilde{H}) \le \mu^{-1}(E_H)$ (10) at all frequencies or satisfy $\bar{\sigma}(\tilde{S}) \le \mu^{-1}(E_S)$ (12) at all frequencies. It is *not* allowed to combine (10) and (12).

(2) Robust performance: at each frequency satisfy either $\bar{\sigma}(\tilde{H}) \le \bar{c}_H$ (31) or $\bar{\sigma}(\tilde{S}) \le \bar{c}_S$ (32). Combining (31) and (32) over different frequency ranges is allowed.

Consequently, two separate conditions must be satisfied by the individual designs: one for nominal stability and one for robust performance.

Remarks

(1) The nominal stability condition $\bar{\sigma}(\tilde{H}) \le \mu^{-1}(E_H)$ (10) is automatically satisfied at any *frequency* where the robust performance condition (31) is satisfied. This follows from the inequality

$$\mu_{\mathrm{diag}\{\Delta_1,\Delta_2\}}\begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \ge \max\{\mu_{\Delta_1}(N_{11}), \mu_{\Delta_2}(N_{22})\} \quad (36)$$

applied to (31b). We find $\bar{c}_H \le \mu^{-1}(E_H)$ and therefore the the RP-condition puts a tighter bound on $\bar{\sigma}(\tilde{H})$ than the NS-condition (10). A similar relationship exists between the RP-condition (32) on $\bar{\sigma}(\tilde{S})$ and the NS-condition (12).

(2) This may seem to imply that NS is automatically guaranteed if RP is satisfied. This is *not* the case, however, since the NS-condition (10) [or (12)] must be satisfied at *all* frequencies. This is not necessarily implied by the combined RP-condition (33) since neither $\bar{\sigma}(\tilde{H}) \le \bar{c}_H$ or $\bar{\sigma}(\tilde{S}) \le c_S$ have to be satisfied at *all* frequencies to satisfy (33). In the following two cases RP does imply NS.

(3) If it happens that the RP-bound (31) on $\bar{\sigma}(\tilde{H})$ is satisfied at *all* frequencies, and if $\tilde{H}$ is stable, then RP and NS are *both* guaranteed using a single condition. However, to be able to satisfy (31) at all frequencies we must require that there exists a $c_H > 0$ which solves (31b). This is equivalent to requiring $\mu(N_{11}^H) \le 1$, which from (30) is equivalent to $\mu(M(\tilde{H}=0)) \le 1$. Consequently, to be able to satisfy (31) we must require that at each frequency the performance requirements are such that $\tilde{H}=0$ is a possible solution. This may be the case, for example, if we are interested in *robust stability* only.

(4) If it happens that the RP-bound (32) on $\bar{\sigma}(\tilde{S})$ is satisfied at *all* frequencies, and if $\tilde{S}$ is stable, then RP and NS are *both* guaranteed. However, to be able to satisfy (32) at all frequencies we must require that at each frequency $\tilde{S}=0$ is a possible solution. This may be the case if there is no uncertainty, that is, if we are interested in *nominal performance* only.

5. *Numerical example*

In this section we continue the previous example of RP with diagonal input uncertainty (Fig. 4). Consider the following plant (time is in minutes):

$$\hat{G} = \frac{1}{1+75s}\begin{bmatrix} -0.878\dfrac{1-0.2s}{1+0.2s} & 0.014 \\ -1.082\dfrac{1-0.2s}{1+0.2s} & -0.014\dfrac{1-0.2s}{1+0.2s} \end{bmatrix}. \quad (37)$$

Physically, this may correspond to a high-purity distillation column using distillate (D) and boilup (V) as manipulated inputs to control top and bottom composition (Skogestad *et al.*, 1988). We want to design a decentralized (diagonal) controller for this plant such that robust performance is guaranteed when there is 10% uncertainty on each manipulated input. The uncertainty and performance weights are

$$\hat{w}_I(s) = 0.1 \quad (38a)$$

$$w_P(s) = 0.25\frac{7s+1}{7s}. \quad (38b)$$

The robust performance condition is

$$\bar{\sigma}(S_p) < 1/|w_P|, \quad \forall G_p \in \Pi \quad (39)$$

(38b) implies that we require integral action ($w_P(0) = \infty$) and allow an amplification of disturbances at high frequencies of at most a factor of four ($w_P(j\infty) = 0.25$). A particular sensitivity function which matches the performance bound (39) exactly at low frequencies and satisfies it easily at high frequency is $S = \dfrac{28s}{28s+1}\mathrm{I}$. This corresponds to a first order response with time constant 28 min.

Nominal stability (*NS*). The nominal model has $RGA_{11} = 0.45$ and we find $\mu(E_H(0)) = 1.11$. Consequently, it is impossible to satisfy the NS-condition (10).

The NS-condition (12) for $\bar{\sigma}(\tilde{S})$ cannot be satisfied either. Firstly, $\hat{G}$ has one RHP-zero, while the diagonal plant has two. Secondly, the plant is clearly not diagonal dominant at high frequencies, and $\mu(E_S(j\omega))$ is larger than one for $\omega > 4\,\mathrm{min}^{-1}$. The simplest way to get around this problem is to treat the RHP-zeros as uncertainty. (This is actually not very conservative, since RHP-zeros limit the achievable performance anyway.) To this end define the following "new" nominal model:

$$G = \frac{1}{1+75s}\begin{bmatrix} -0.878 & 0.014 \\ -1.082 & -0.014 \end{bmatrix} \quad (40)$$

and include the RHP-zeros in the input uncertainty by using the following new uncertainty weight:

$$w_I(s) = 0.1\frac{5s+1}{0.25s+1} \quad (41)$$

$|w_I(j\omega)|$ reaches a value of one at about $\omega = 2\,\mathrm{min}^{-1}$. This includes the neglected RHP-zeros since the relative uncertainty introduced by replacing $\dfrac{1-0.2s}{1+0.2s}$ by 1 is $\left|1 - \dfrac{1-0.2j\omega}{1+0.2j\omega}\right|$, which reaches a value of one at about $\omega = 3\,\mathrm{min}^{-1}$.

With the new model (40) we still cannot satisfy the NS-condition (10) for $\bar{\sigma}(\tilde{H})$. However, the NS-condition (12) on $\bar{\sigma}(\tilde{S})$ is easily satisfied since G and $\tilde{G}$ have the same number of RHP-zeros (none), and $\mu(E_S) = 0.743$ at all frequencies. The only restriction this imposes on $\tilde{S}$ is that the maximum peaks of $|\tilde{s}_1|$ and $|\tilde{s}_2|$ must be less than $1/0.743 = 1.35$. This is easily satisfied since both $\tilde{g}_{11} = \dfrac{-0.878}{1+75s}$ and $\tilde{g}_{22} = \dfrac{-0.014}{1+75s}$ are minimum phase.

In the remainder of this section the model of the plant (G) is assumed to be given by (40) and the uncertainty weight (w_I) by (41).

Nominal performance (*NP*). The NP-requirement is

$$\mathrm{NP} \Leftrightarrow \bar{\sigma}(S) \le |w_P|^{-1}, \quad \forall\omega. \quad (42)$$

How should the individual loops ($\tilde{S} = \mathrm{diag}\{\tilde{s}_1, \tilde{s}_2\}$) be designed in order to satisfy this requirement? Intuitively, we might expect that we have to require *at least* that the individual loops satisfy (42), that is, $\bar{\sigma}(\tilde{S}) \le |w_P|^{-1}$. However, this is not necessarily the case, as illustrated by the example: (42) is equivalent to $\mu_\Delta(M) \le 1$ with $M = w_P S$ and $\Delta = \Delta_P$ (Δ_P is a full matrix). (32) then yields the following sufficient condition for NP in terms of $\tilde{S}$:

$$\mathrm{NP} \Leftarrow \bar{\sigma}(\tilde{S}) \le c_{NP}, \quad \forall\omega \quad (43a)$$

where c_{NP} at each frequency solves

$$\mu_{\hat{\Delta}}\begin{pmatrix} 0 & w_P\mathrm{I} \\ c_{NP}\tilde{G}G^{-1} & c_{NP}E_S \end{pmatrix} = \mathrm{I} \quad (43b)$$

and $\hat{\Delta} = \mathrm{diag}\{\Delta_P, C\}$. In our example Δ_P is a "full" 2×2 matrix, and C is a diagonal 2×2 matrix. c_{NP} is shown graphically in Fig. 5 and it is seen to be *larger* than $|w_P|^{-1}$ at low frequency. Consequently, the performance of the overall system (S) may be *better* than that of the individual loops ($\tilde{s}_1$ and $\tilde{s}_2$), that is, the interactions may improve the performance.

Robust performance (*RP*). *Bound on* $\bar{\sigma}(\tilde{H})$. The bound $\bar{c}_H$ on $\bar{\sigma}(\tilde{H})$ is given by equation (34) and is shown graphically in Fig. 6. [μ of the matrix in (34b) is computed with respect to the structure $\hat{\Delta} = \mathrm{diag}\{\Delta_I, \Delta_P, C\}$, where Δ_I is a diagonal 2×2 matrix, Δ_P is a full 2×2 matrix and C is a diagonal 2×2 matrix.] It is clearly *not* possible to satisfy the bound $\bar{\sigma}(\tilde{H}) < \bar{c}_H$ at all frequencies. In particular, we find $\bar{c}_H \le 0$ for $\omega < 0.03\,\mathrm{min}^{-1}$. The reason is that the

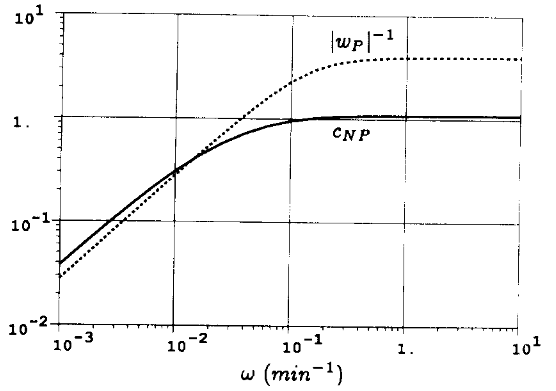

FIG. 5. NP is satisfied iff $\bar{\sigma}(S) \leq |w_P|^{-1}$ which is satisfied if $\bar{\sigma}(\tilde{S}) \leq c_{NP}$.

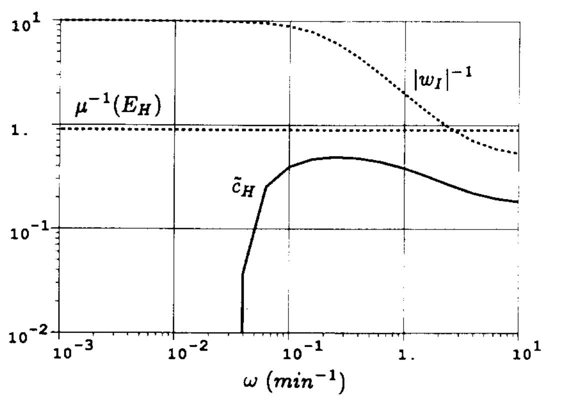

FIG. 6. Bounds on $\bar{\sigma}(\tilde{H})$.

performance weight $|w_P| > 1$ in this frequency range, which means that feedback is required (i.e. $\tilde{H} = 0$ is not possible, see Remark 3 in Section 4).

Bound on $\bar{\sigma}(\tilde{S})$. The bound $\tilde{c}_S$ on $\bar{\sigma}(\tilde{S})$ is given by equation (35) and is shown graphically in Fig. 7 (μ is computed with respect to the same structure as above). Again it is not possible to satisfy this bound at all frequencies. In particular, we find $\tilde{c}_S \leq 0$ for $\omega > 2\,\text{min}^{-1}$. The reason is that the uncertainty weight $|w_I| > 1$ in this frequency range, which means that perfect control ($\tilde{S} = 0$) is not allowed.

Combining bounds on $\bar{\sigma}(\tilde{H})$ *and* $\bar{\sigma}(\tilde{S})$. The bound on $\bar{\sigma}(\tilde{S})$ is easily satisfied at low frequencies, and the bound on $\bar{\sigma}(\tilde{H})$ is easily satisfied at high frequencies. The difficulty is to find an $\tilde{S} = I - \tilde{H}$ which satisfies either one of the conditions in the frequency range from 0.1 to 1 min^{-1}. The following design is

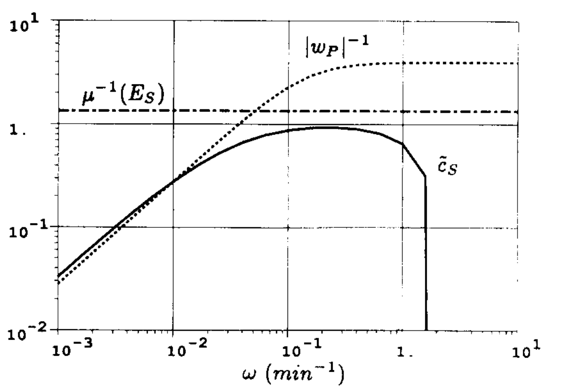

FIG. 7. Bounds on $\bar{\sigma}(\tilde{S})$.

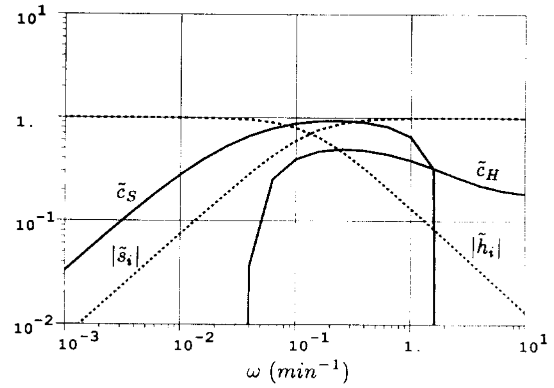

FIG. 8. RP is guaranteed since $|\tilde{s}_i| < \tilde{c}_S$ for $\omega < 0.3\,\text{min}^{-1}$ and $|\tilde{h}_i| < \tilde{c}_H$ for $\omega > 0.23\,\text{min}^{-1}$. $\tilde{h}_i = 1/(1 + 7.5s)$.

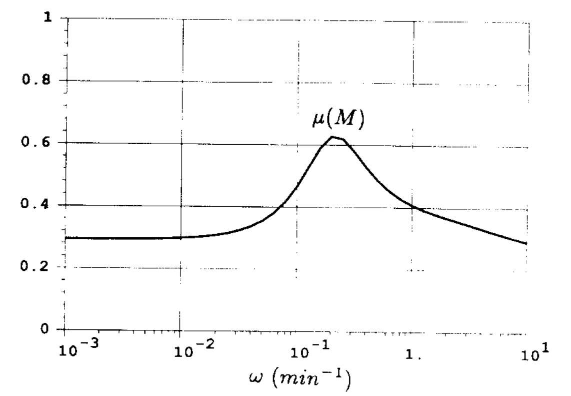

FIG. 9. $\mu(M)$ as a function of frequency. RP is guaranteed since $\mu(M) < 1$ at all frequencies.

seen to do the job (Fig. 8):

$$\tilde{h}_1 = \tilde{h}_2 = \frac{1}{7.5s + 1}, \quad \tilde{s}_1 = \tilde{s}_2 = \frac{7.5s}{7.5s + 1}. \tag{44}$$

The bound on $|\tilde{s}_i|$ is satisfied for $\omega < 0.3\,\text{min}^{-1}$, and the bound on $|\tilde{h}_i|$ is satisfied for $\omega > 0.23\,\text{min}^{-1}$. Equation (44) corresponds to the following controller:

$$C = k\frac{(1 + 75s)}{s}\begin{pmatrix} \frac{-1}{0.878} & 0 \\ 0 & \frac{-1}{0.014} \end{pmatrix}, \quad k = 0.133. \tag{45}$$

Because the bounds $\tilde{c}_H$ and $\tilde{c}_S$ are almost flat in the cross-over region, the result is fairly insensitive to the particular choice of controller gain; it turns out that $0.06 < k < 0.25$ yields a design which satisfies at each frequency $\bar{\sigma}(\tilde{S}) < \tilde{c}_S$ or $\bar{\sigma}(\tilde{H}) < \tilde{c}_H$ and thus has RP. The controller (45) obviously yields an overall system which satisfies the robust performance condition, that is, $\mu(M)$ is less than one. This is also seen from Fig. 9 which shows $\mu(M)$ [M is given by (21b)] as a function of frequency. We find $\mu_{RP} = \sup_{\omega} \mu(M) = 0.63 < 1$ and RP is guaranteed. The fact that μ_{RP} is so much smaller than one, demonstrates some of the conservativeness of conditions (34) and (35) (which are only sufficient for RP).

6. *Conclusion*

This paper solves the problem of robust performance using independent designs as introduced in the Introduction. The example illustrates that this design approach may be useful for designing decentralized controllers.

The main limitation of the approach stems from the initial assumption regarding independent designs. Since each loop is designed separately, we cannot make use of information about the controllers used in the other loops. The consequence is that the bounds on $\bar{\sigma}(\tilde{S})$ and $\bar{\sigma}(\tilde{H})$ are only sufficient for robust performance; there will exist decentralized controllers which violate the bounds on $\bar{\sigma}(\tilde{S})$ and $\bar{\sigma}(\tilde{H})$, but which satisfy the robust performance condition. However, the derived bounds on $\bar{\sigma}(\tilde{S})$ and $\bar{\sigma}(\tilde{H})$ are the tightest norm bounds *possible,* in the sense that in such cases there will exist another controller with the same values of $\bar{\sigma}(\tilde{H})$ and $\bar{\sigma}(\tilde{S})$ which does *not* yield robust performance.

The bounds on $\bar{\sigma}(\tilde{H})$ and $\bar{\sigma}(\tilde{S})$ tend to be most conservative in the frequency range around crossover where $\bar{\sigma}(\tilde{H})$ and $\bar{\sigma}(\tilde{S})$ are both close to one. If, for a particular case, it is not possible to satisfy either $\bar{\sigma}(\tilde{H}) < \tilde{c}_H$ or $\bar{\sigma}(\tilde{S}) < \tilde{c}_S$ in this frequency range, then try the following. Design a controller for which the frequency range where both bounds are violated is as small as possible. Since the bounds are only sufficient for RP, this may still yield an acceptable design with robust performance. This may be checked using the tight RP-condition $\mu(M) < 1$ in equation (5).

Another potential source of conservativeness is the inherent assumption of similar or equal bandwidths in all loops which is made when the same bounds on $|\tilde{h}_i|$ and $|\tilde{s}_i|$ are used for all loops. This limitation may be partially eliminated by including matrix valued weights on $\tilde{h}$ and $\tilde{S}$ [see Grosdidier and Morari (1986)]. However, it is not obvious how these weights should be chosen *a priori.*

Acknowledgements—Partial support from the National Science Foundation and Norsk Hydro is gratefully acknowledged. We are thankful to Professor John C. Doyle and Dr Pierre Grosdidier for numerous useful discussions and remarks.

References

Bristol, E. H. (1966). On a new measure of interaction for multivariable process control. *IEEE Trans. Aut. Control,* **AC-11,** 133–134.

Doyle, J. C., J. E. Wall and G. Stein (1982). Performance and robustness analysis for structured uncertainty. *Proc. IEEE Conf. on Decision and Control,* Orlando, FL.

Grosdidier, P. and M. Morari (1986). Interaction measures for systems under decentralized control. *Automatica,* **22,** 309–319.

Mayne, D. Q. (1973). The design of linear multivariable systems. *Automatica,* **9,** 201–227.

Postlethwaite, J. and V. K. Foo (1985). Robustness with simultaneous pole and zero movement across the $j\omega$-axis. *Automatica,* **21,** 433–443.

Skogestad, S. and M. Morari (1987a). Effect of model uncertainty on dynamic resilience. *Chem. Engng Sci.,* **42,** 1765–1780.

Skogestad, S. and M. Morari (1987b). Letter to the Editor concerning paper by Mijares *et al. A.I.Ch.E. J.,* **33,** 701–702.

Skogestad, S. and M. Morari (1988). Some new properties of the structured singular value. *IEEE Trans. Aut. Control,* **33**(12), 1151–1154.

Skogestad, S., M. Morari and J. C. Doyle (1988). Robust control of ill-conditioned plants: high purity distillation. *IEEE Trans. Aut. Control,* **33**(12), 1092–1105.

Appendix

Definition of the structured singular value μ. Let M be a square complex matrix. $\mu(M)$ is defined such that $\mu^{-1}(M)$ is equal to the smallest $\bar{\sigma}(\Delta)$ needed to make $(\mathrm{I} + \Delta M)$ singular, i.e.

$$\mu^{-1}(M) = \min_{\delta} \{\delta \mid \det(\mathrm{I} + \Delta M) = 0 \text{ for some } \Delta,\ \bar{\sigma}(\Delta) < \delta\}. \tag{A1}$$

(If M is a transfer matrix this definition applies frequency-by-frequency.) Δ is a block-diagonal perturbation matrix with a given *structure*

$$\Delta = \begin{bmatrix} \Delta_1 & & \\ & \ddots & \\ & & \Delta_n \end{bmatrix}, \quad \bar{\sigma}(\Delta_i) < \delta, \quad \forall i$$

where Δ is allowed to be *any* complex matrix satisfying $\bar{\sigma}(\Delta) < \delta$. [It turns out that Δ may be restricted to being unitary without changing $\mu(M)$ (Doyle *et al.,* 1982).] $\mu(M)$ depends on both the matrix M *and* the *structure* of the perturbations Δ. This is sometimes shown explicitly by using the notation $\mu(M) = \mu_\Delta(M)$. An equivalent statement of (A1) which is more useful for our purposes is the following:

$$\begin{aligned} &\det(\mathrm{I} + \Delta M) \neq 0, \quad \forall \Delta(\bar{\sigma}(\Delta) < \delta) \\ &\Leftrightarrow \rho(\Delta M) \leq 1, \quad \forall \Delta(\bar{\sigma}(\Delta) < \delta) \\ &\Leftrightarrow \mu_\Delta(M) \leq 1/\delta. \end{aligned} \tag{A2}$$

The reader is referred to Doyle *et al.* (1982) for further properties and computational aspects of μ.

Theorem 1. Let M be written as an LFT of T:

$$M = N_{11} + N_{12}T(\mathrm{I} - N_{22}T)^{-1}N_{21} \tag{A3}$$

and let k be a given constant. Assume $\mu_\Delta(N_{11}) < k$ and $\det(\mathrm{I} - N_{22}T) \neq 0$. Then

$$\mu_\Delta(M) \leq k \tag{A4}$$

if

$$\bar{\sigma}(T) \leq c_T \tag{A5}$$

where c_T solves

$$\mu_{\hat{\Delta}} \begin{bmatrix} N_{11} & N_{12} \\ kc_T N_{21} & kc_T N_{22} \end{bmatrix} = k \tag{A6}$$

and $\hat{\Delta} = \mathrm{diag}\{\Delta, T\}$.

Proof. The theorem follows directly from the definition of μ (A2) after some algebra: assume that T is defined such that $\bar{\sigma}(T) < c_T$. Then at each frequency the following holds

$$\mu_\Delta(M) \leq k(\omega), \quad \forall T(\bar{\sigma}(T) < c_T)$$

$$\Leftrightarrow \det(\mathrm{I} + \Delta M) \neq 0, \quad \forall \Delta(\bar{\sigma}(\Delta) < 1/k), \quad \forall T \tag{A7}$$

$$\Leftrightarrow \det \begin{bmatrix} \mathrm{I} + \Delta N_{11} & \Delta N_{12} \\ -TN_{21} & \mathrm{I} - TN_{22} \end{bmatrix} \neq 0, \quad \forall \Delta, \quad \forall T \tag{A8}$$

$$\Leftrightarrow \det\left(\mathrm{I} + \begin{bmatrix} k\Delta & 0 \\ 0 & -\frac{1}{c_T}T \end{bmatrix} \begin{bmatrix} \frac{1}{k}N_{11} & \frac{1}{k}N_{12} \\ c_T N_{21} & c_T N_{22} \end{bmatrix}\right) \neq 0, \quad \forall \Delta, \quad \forall T$$

$$\Leftrightarrow \mu_{\hat{\Delta}} \begin{bmatrix} \frac{1}{k}N_{11} & \frac{1}{k}N_{12} \\ c_T N_{21} & c_T N_{22} \end{bmatrix} \leq 1$$

$$\Leftrightarrow \mu_{\hat{\Delta}} \begin{bmatrix} N_{11} & N_{12} \\ kc_T N_{21} & kc_T N_{22} \end{bmatrix} \leq k(\omega).$$

The step from (A7) to (A8) follows $M = N_{11} + N_{12}(\mathrm{I} - TN_{22})^{-1}TN_{21}$ and Schurs formula

$$\det(A - BD^{-1}C) = \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \Big/ \det D \tag{A9}$$

and the assumption $\det D = \det(\mathrm{I} - TN_{22}) \neq 0$.