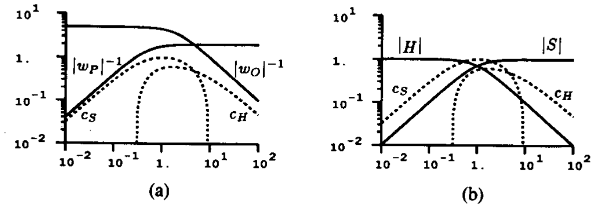

Fig. 3. Graphical representation of conditions (33) and (34). R.P. is guaranteed since $|S| < c_s$ for $\omega < 2$ and $|H| < c_H$ for $\omega > 1.4$.

The step from (A1) to (A2) follows from $M = N_{11} + N_{12}(I - TN_{22})^{-1}TN_{21}$ and Schurs formula

$$\det(A - BD^{-1}C) = \det\begin{bmatrix} A & B \\ C & D \end{bmatrix} \Big/ \det D \qquad \text{(A3)}$$

and the assumption $\det D = \det(I - TN_{22}) \neq 0$.

*Lemma 1:* An equivalent statement of the lemma is as follows: Let $\tilde{\Delta} = \text{diag}\{\Delta_1, \Delta_2\}$ where $\Delta_1$ and $\Delta_2$ have the same *size* as $N_{11}$ and $N_{22}$, respectively. ($\Delta_1$ and $\Delta_2$ may have additional structure.) Then

$$\mu_{\tilde{\Delta}}\begin{bmatrix} 0 & N_{12} \\ cN_{21} & 0 \end{bmatrix} = \sqrt{c}\,\mu_{\tilde{\Delta}}\begin{bmatrix} 0 & N_{12} \\ N_{21} & 0 \end{bmatrix}. \qquad \text{(A4)}$$

*Proof of (A4):*

$$\mu_{\tilde{\Delta}}\begin{bmatrix} 0 & N_{12} \\ cN_{21} & 0 \end{bmatrix} \leq 1/k_1 \qquad \text{(A5)}$$

$$\Leftrightarrow \det\left(I + k_1\begin{bmatrix} \Delta_1 & \\ & \Delta_2 \end{bmatrix}\begin{bmatrix} 0 & N_{12} \\ cN_{21} & 0 \end{bmatrix}\right) \neq 0$$

$$\Leftrightarrow \det\begin{bmatrix} I & k_1\Delta_1 N_{12} \\ k_1 c\Delta_2 N_{21} & I \end{bmatrix} \neq 0 \qquad \text{(A6)}$$

$$\Leftrightarrow \det(I - k_1^2 c\Delta_1 N_{12}\Delta_2 N_{21}) \neq 0 \qquad \text{(A7)}$$

$$\Leftrightarrow \det\begin{bmatrix} I & \sqrt{k_1^2 c}\,\Delta_1 N_{12} \\ \sqrt{k_1^2 c}\,\Delta_2 N_{21} & I \end{bmatrix} \neq 0 \qquad \text{(A8)}$$

$$\Leftrightarrow \mu_{\tilde{\Delta}}\begin{bmatrix} 0 & N_{12} \\ N_{21} & 0 \end{bmatrix} \leq 1/\sqrt{k_1^2 c}. \qquad \text{(A9)}$$

The conditions involving det ( ) $\neq 0$ must hold for $\forall\Delta_1$ s.t. $\bar{\sigma}(\Delta_1) < 1$ and $\forall\Delta_2$ s.t. $\bar{\sigma}(\Delta_2 < 1)$. The step from (A6) to (A7) and back to (A8) follows from (A3). Since (A5) and (A9) must hold for *any* value of $k_1$, (A4) follows.

*Theorem 2:* From Theorem 1 and Lemma 1 for the case $N_{11} = N_{22} = 0$

$$\mu_\Delta(N_{12}TN_{21}) \leq k \text{ if } \bar{\sigma}(T)\mu_{\tilde{\Delta}}^{\,2}\begin{bmatrix} 0 & N_{12} \\ N_{21} & 0 \end{bmatrix} \leq k. \qquad \text{(A10)}$$

Since (A10) must hold for *any* choice of $k$ it is equivalent to

$$\mu_\Delta(N_{12}TN_{21}) \leq \bar{\sigma}(T)\mu_{\tilde{\Delta}}^{\,2}\begin{bmatrix} 0 & N_{12} \\ N_{21} & 0 \end{bmatrix}.$$

Theorem 2 follows by choosing $N_{12} = A$, $N_{21} = B$.

*Special Cases of Theorem 2:* Let $\Delta_1$ and $\Delta_2$ have the same *structure* as $\Delta$ and $T$ in Theorem 2. Define $\tilde{\Delta} = \text{diag}\{\Delta_1, \Delta_2\}$. Then

$$\mu_{\tilde{\Delta}}^{\,2}\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \leq 1/k \qquad \text{(A11)}$$

$$\Leftrightarrow \mu_{\tilde{\Delta}}\begin{bmatrix} 0 & kA \\ B & 0 \end{bmatrix} \leq 1 \qquad \text{(A12)}$$

$$\Leftrightarrow \det\left(I + \begin{bmatrix} \Delta_1 & \\ & \Delta_2 \end{bmatrix}\begin{bmatrix} 0 & kA \\ B & 0 \end{bmatrix}\right) \neq 0 \;\; \forall\Delta_1, \Delta_2$$

$$\Leftrightarrow \det(I - k\Delta_2 B\Delta_1 A) = \det(I - k\Delta_1 A\Delta_2 B) \neq 0 \;\; \forall\Delta_1, \Delta_2$$

$$\Leftrightarrow \mu_{\Delta_2}(B\Delta_1 A) \leq 1/k \;\; \forall\Delta_1 \qquad \text{(A13)}$$

$$\Leftrightarrow \mu_{\Delta_1}(A\Delta_2 B) \leq 1/k \;\; \forall\Delta_2 \qquad \text{(A14)}$$

$$\Leftrightarrow \rho(\Delta_1 A\Delta_2 B) = \rho(\Delta_2 B\Delta_1 A) \leq 1/k \;\; \forall\Delta_1, \Delta_2. \qquad \text{(A15)}$$

By $\forall\Delta_i$ is understood all $\Delta_i$ s.t. $\bar{\sigma}(\Delta_i) < 1$. The step from (A11) to (A12) follows from (A4).

*Case 1):* Follows from (A15): Use the SVD of $A = U_A\Sigma_A V_A^H$ and $B = U_B\Sigma_B V_B^H$. Since $\Delta_1$ and $\Delta_2$ are full, $\Delta_1$ may be chosen such that $\Delta_1 U_A = V_B$ and $\Delta_2$ such that $V_A^H\Delta_2 = U_B^H$. Then $\rho(\Delta_1 A\Delta_2 B) = \rho(V_B\Sigma_1\Sigma_2 V_B^H) = \rho(\Sigma_1\Sigma_2) = \bar{\sigma}(A)\bar{\sigma}(B)$. [The generalization to the case when $A$ and $B$ are nonsquare is straightforward and involves lining up the directions corresponding to $\bar{\sigma}(A)$ and $\bar{\sigma}(B)$.]

*Case 2):* Follows from (A14).

*Case 3):* Follows from (A13).

*Cases 4), 5):* Follow from (A15).

ACKNOWLEDGMENT

The authors are thankful to J. C. Doyle for numerous useful discussions and remarks. The idea of treating $T$ as a perturbation (which subsequently led to the derivation of Theorem 1) was first presented by Grosdidier and Morari [4] in their derivation of the $\mu$-interaction measure.

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## Recursive Algorithm for the Computation of the $H^\infty$-Norm of Polynomials

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***Abstract*—A recursive algorithm for computing the $H^\infty$-norm of polynomials is developed. The algorithm is shown to converge monotonically and the convergence rate is also established. Some examples are presented to illustrate the algorithm.**

### I. INTRODUCTION

In recent years, $H^\infty$-norm and its optimization have been used more and more frequently in many areas of control theory and applications. For example, $H^\infty$-norm optimal controller synthesis approach [1], [2], model/

Manuscript received October 6, 1987; revised January 28, 1988.
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IEEE Log Number 8823316.

controller reduction, and even some problems in system identification are closely related to $H^\infty$-norms. The model/controller reduction is often best posed as a frequency weighted $H^\infty$ optimal approximation problem [3]. For a given transfer function $G(z)$, many approaches give a reduced-order transfer function $G_r(z)$, normally, which is not optimal in the $H^\infty$ sense. Certainly, it is desirable to know the value of the $H^\infty$-approximation error $\|G(z) - G_r(z)\|_\infty$. In system identification, if a monic polynomial $C(z)$ is the moving average noise process transfer function in an ARMAX model, it is well known that for the convergence of the extended least-squares algorithm, a key condition is that $C^{-1}(z) - 1/2$ is strictly positive real (e.g., [4], [5]). It is easy to see that this condition is equivalent to the requirement $\|C(z) - 1\|_\infty < 1$. However, in practice, to calculate the value of the $H^\infty$-norm is not a pleasant task. It is usually done by a rather trivial method, i.e., plotting the absolute value of the function concerned on the unit circle.

In this note, we propose a theoretical recursive algorithm for the computation of the $H^\infty$-norm of polynomials or FIR transfer functions (Section II). In Section III we give the derivation of the algorithm and show that the guaranteed convergence rate of the algorithm is $O(\log n/n)$. Simulation results of some examples are provided in Section IV. Section V concludes the note with some remarks.

Before pursuing further, we need some concepts and definitions as follows.

Let $f(z)$ be a complex-valued function on the unit circle bounded almost everywhere; the set of all such functions is denoted by $L^\infty$, with norm

$$\|f(z)\|_\infty = \operatorname*{ess\,sup}_{\theta\in[0,2\pi]} |f(e^{i\theta})|. \quad (1)$$

The Hardy space $H^\infty$ consists of all complex-valued functions which are analytic and of bounded modulus on $|z| < 1$, with norm

$$\|f(z)\|_\infty = \sup_{|z|<1} |f(z)|. \quad (2)$$

It is known that each $f$ in $H^\infty$ yields a unique $L^\infty$ boundary function with the two norms equal. The set of such boundary functions is the subspace of $L^\infty$-functions with Fourier coefficients zero for negative indexes, and we can regard $H^\infty$ as a closed-subspace of space $L^\infty$.

We also need the concept of space $L^p$ $(p > 0)$. It consists of all measurable complex functions $f(z)$ defined on the unit circle $|z| = 1$ such that $|f(e^{i\theta})|p$ is integrable with respect to Lebesgue measure, with norm

$$\|f(z)\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right\}^{1/p}. \quad (3)$$

## II. Algorithm Description and Main Results

Let $C(z)$ be a polynomial with real coefficients and with degree $r$

$$C(z) = C_0 + C_1 z + \cdots + C_r z^r, \qquad C_0 C_r \neq 0. \quad (4)$$

Define a function $f(z)$ as

$$f(z) = C(z)C(z^{-1}) \triangleq \gamma_0 + \sum_{j=1}^{r} \gamma_j (z^j + z^{-j}) \quad (5)$$

where

$$\gamma_j = \sum_{k=0}^{r} C_k C_{k+j}, \qquad (C_k = 0, \text{ for } k > r). \quad (6)$$

To describe our algorithm, we need the following auxiliary variables:

$$\{\chi_i(n),\ 1 \leqslant i \leqslant 2r,\ n \geqslant 1\} \text{ and } \{T(n),\ n \geqslant 1\}$$

which are recursively defined by

$$\chi_{k+r}(n-1) = \left[ \sum_{j=1}^{r} (nj-k)\gamma_j \chi_{k-j}(n-1) - \sum_{j=0}^{r-1} (nj+k)\gamma_j \chi_{k+j}(n-1) \right] \Big/ (nr+k)\gamma_r \qquad 1 \leqslant k \leqslant r \quad (7)$$

$$\chi_k(n) = (n/k) \sum_{j=1}^{r} j\gamma_j [\chi_{k-j}(n-1) - \chi_{k+j}(n-1)] \cdot \left[ \gamma_0 + 2\sum_{j=1}^{r} \gamma_j \chi_j(n-1) \right]^{-1} \qquad 1 \leqslant k \leqslant r \quad (8)$$

$$T(n) = \frac{n-1}{n} T(n-1) + \frac{1}{2n} \log \left[ \gamma_0 + 2\sum_{j=1}^{r} \gamma_j \chi_j(n-1) \right] \quad (9)$$

where by definition

$$\chi_0(n) = 1 \text{ and } \chi_{-i}(n) = \chi_i(n), \qquad 1 \leqslant i \leqslant 2r,\ n \geqslant 1$$

and where the initial conditions are

$$\chi_j(1) = \gamma_j/\gamma_0,\ 1 \leqslant j \leqslant r; \qquad T(1) = \frac{1}{2}\log \gamma_0.$$

The $n$th approximation for the norm $\|C(z)\|_\infty$ is defined by

$$J(n) = \exp\{T(n)\}, \qquad n \geqslant 1. \quad (11)$$

The asymptotic properties of the above algorithm are summarized in the following theorem.

*Theorem 1:* For any polynomial $C(z)$ defined as in (4), the quantity $J(n)$ given by (7)–(11) increases monotonically and converges to $\|C(z)\|_\infty$ as $n \to \infty$, with convergence rate

$$\|C(z)\|_\infty - J(n) \leqslant (\|C(z)\|_\infty) \frac{\log n}{2n} + 0(1/n). \quad (12)$$

## III. Convergence Analysis

For the proof of Theorem 1, we first establish the following lemmas.

*Lemma 1:* For $T(n)$ given by (9)

$$T(n) = \log(\|C(z)\|_{2n})$$

holds for any $n \geqslant 1$.

*Proof:* Define

$$M_k(n) = \frac{1}{2\pi} \int_0^{2\pi} f^n(e^{i\theta}) e^{ki\theta} \, d\theta \quad (13)$$

for $n \geqslant 1$, $-2r \leqslant k \leqslant 2r$, where $f(e^{i\theta})$ is given by (5).

It is easy to see that for any $n \geqslant 1$

$$M_{-k}(n) = M_k(n), \qquad k = 0, 1, \cdots, 2r \quad (14)$$

and

$$M_0(n) = \frac{1}{2\pi} \int_0^{2\pi} f^n(e^{i\theta}) \, d\theta = \|f^n(z)\|_n^n = \|C(z)\|_{2n}^{2n}. \quad (15)$$

So for the proof of the lemma, we need only to show that

$$T(n) = \frac{1}{2n} \log M_0(n). \quad (16)$$

We proceed as follows.

By (5), (13), and (14)

$$\begin{aligned} M_0(n) &= \frac{1}{2\pi}\int_0^{2\pi} f^{n-1}(e^{i\theta})f(e^{i\theta})\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f^{n-1}(e^{i\theta})\left[\gamma_0+\sum_{k=1}^{r}\gamma_k(e^{ki\theta}+e^{-ki\theta})\right]d\theta \\ &= \gamma_0 M_0(n-1)+2\sum_{j=1}^{r}\gamma_j M_j(n-1) \\ &= M_0(n-1)\left\{\gamma_0+2\sum_{j=1}^{r}\gamma_j[M_j(n-1)/M_0(n-1)]\right\} \end{aligned} \tag{17}$$

consequently, we have

$$\left[\frac{1}{2n}\log M_0(n)\right]=\frac{n-1}{n}\left[\frac{1}{2(n-1)}\log M_0(n-1)\right]+\frac{1}{2n}\log\left\{\gamma_0+2\sum_{j=1}^{r}\gamma_j[M_j(n-1)/M_0(n-1)]\right\}.$$

Comparing this to (9), we see that for (16) it suffices to show that

$$\chi_j(n)=M_j(n)/M_0(n),\qquad 1\leqslant j\leqslant r. \tag{18}$$

Now, by integral parts from (13) and the fact that $f(z) = f(z^{-1})$, we have

$$\begin{aligned} M_k(n) &= \frac{1}{2\pi}\int_0^{2\pi} f^n(e^{-i\theta})e^{ki\theta}\,d\theta \\ &= \frac{1}{2\pi ki}\int_0^{2\pi} f^n(e^{-i\theta})\,de^{ki\theta} \\ &= \frac{1}{2\pi ki}\left\{f^n(e^{-i\theta})e^{ki\theta}\Big|_0^{2\pi} - \int_0^{2\pi}(e^{ki\theta})\,d[f^n(e^{-i\theta})]\right\} \\ &= \frac{n}{2\pi k}\int_0^{2\pi} f^{n-1}(e^{-i\theta})\cdot f'(e^{-i\theta})e^{(k-1)i\theta}\,d\theta \end{aligned} \tag{19}$$

where

$$\begin{aligned} f'(e^{-i\theta}) &\frac{df(z)}{dz}\bigg|_{z=e^{-i\theta}} \\ &= \sum_{j=1}^{r} j\gamma_j[e^{(1-j)i\theta}-e^{(j+1)i\theta}]. \end{aligned} \tag{20}$$

For (19) and (20) we immediately have $(1 \leqslant k \leqslant r)$

$$M_k(n)=\frac{n}{k}\sum_{j=1}^{r} j\gamma_j[M_{k-j}(n-1)-M_{k+j}(n-1)]. \tag{21}$$

Multiplying $1/M_0(n)$ on both sides of this equality and using (17), we know that the recursion (8) is true with $\chi_k(n)$ replaced by $M_k(n)/M_0(n)$.

To conclude (18), we still need to show that the recursion (7) also holds with $\chi_k(n)$ replaced by $M_k(n)/M_0(n)$. To this end, consider the following decomposition for $f'(e^{-i\theta})$:

$$f'(e^{-i\theta})=g_1(e^{-i\theta})-rf(e^{-i\theta})e^{i\theta} \tag{22}$$

where

$$g_1(e^{-i\theta})=r\gamma_0e^{i\theta}+\sum_{j=1}^{r}\gamma_j[(r+j)e^{(1-j)i\theta}+(r-j)e^{(j+1)i\theta}]. \tag{23}$$

Substituting (22) into (19), we get

$$\begin{aligned} M_k(n) &= \frac{n}{2\pi k}\int_0^{2\pi} f^{n-1}(e^{-i\theta})[g_1(e^{-i\theta}) - rf(e^{-i\theta})e^{i\theta}]e^{(k-1)i\theta}\,d\theta \\ &= \frac{-nr}{2\pi k}\int_0^{2\pi} f^n(e^{-i\theta})e^{ki\theta}\,d\theta + \frac{n}{2\pi k}\int_0^{2\pi} f^{n-1}(e^{-i\theta})g_1(e^{-i\theta})e^{(k-1)i\theta}\,d\theta \\ &= \frac{-nr}{k}M_k(n)+\frac{n}{2\pi k}\int_0^{2\pi} f^{n-1}(e^{-i\theta})\cdot g_1(e^{-i\theta})e^{(k-1)i\theta}\,d\theta. \end{aligned}$$

By this identity, we obtain for $1 \leqslant k \leqslant r$

$$\begin{aligned} M_k(n) &= \frac{n}{nr+k}\cdot\frac{1}{2\pi}\int_0^{2\pi} f^{n-1}(e^{-i\theta})g_1(e^{-i\theta})e^{(k-1)i\theta}\,d\theta \\ &= \frac{n}{nr+k}\left\{r\gamma_0M_k(n-1)+\sum_{j=1}^{r}\gamma_j[(r+j)M_{k-j}(n-1) + (r-j)M_{k+j}(n-1)]\right\} \end{aligned} \tag{24}$$

which in conjunction with (21) gives the recursive formula for $M_{k+r}(n-1)$

$$M_{k+r}(n-1)=\frac{1}{(nr+k)\gamma_r}\left[\sum_{j=1}^{r}(nj-k)\gamma_jM_{k-j}(n-1) - \sum_{j=0}^{r-1}(nj+k)\gamma_jM_{k+j}(n-1)\right].$$

From here it is easy to see that (7) is true with $\chi_k(n-1)$ replaced by $M_k(n-1)/M_0(n-1)$. This proves the assertion (18), and hence the conclusion of the lemma.

*Lemma 2:* Let a complex function $f(z)\epsilon L^\infty$, if $d/d\theta\,[|f(e^{i\theta})|^2]\epsilon L^\infty$; then

$$0\leqslant\|f(e^{i\theta})\|_\infty-\|f(e^{i\theta})\|_n\leqslant(\|f(e^{i\theta})\|_\infty)\frac{\log n}{n}+O\left(\frac{1}{n}\right).$$

*Proof:* By (1) and (3) it is evident that for any $n \geqslant 1$

$$\|f(e^{i\theta})\|_n\leqslant\|f(e^{i\theta})\|_\infty.$$

Now, denote

$$g(\theta)=|f(e^{i\theta})|^2,\qquad \theta\in[0,2\pi].$$

Since $g(\theta)$ is a continuous function of $\theta$, there exists a $\theta_0\epsilon[0, 2\pi]$ such that

$$g(\theta_0)=\max_{\theta\in[0,2\pi]}g(\theta)=\|f(e^{i\theta})\|_\infty^2.$$

Without loss of generality, assume $\theta_0\epsilon(0, 2\pi)$.

By the Taylor expansion we know that

$$g(\theta)=g(\theta_0)+g'(\xi)(\theta-\theta_0)$$

where $\xi$ is some point between $\theta$ and $\theta_0$.

From here we have, for sufficiently large $n$,

$$\begin{aligned}
&\|f(e^{i\theta})\|_n \\
&= \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^n \, d\theta\right\}^{1/n} \\
&= \left\{\frac{1}{2\pi}\int_0^{2\pi} [g(\theta)]^{n/2} \, d\theta\right\}^{1/n} \\
&= \left\{\frac{1}{2\pi}\int_0^{2\pi} g^{n/2}(\theta_0)\left[1+\frac{g'(\xi)}{g(\theta_0)}(\theta-\theta_0)\right]^{n/2} d\theta\right\}^{1/n} \\
&= [g(\theta_0)]^{1/2}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left[1+\frac{g'(\xi)}{g(\theta_0)}(\theta-\theta_0)\right]^{n/2} d\theta\right\}^{1/n} \\
&\geqslant \|f(z)\|_\infty \left\{\frac{1}{2\pi}\int_{\theta_0-1/n}^{\theta_0+1/n}\left[1+\frac{g'(\xi)}{g(\theta_0)}(\theta-\theta_0)\right]^{n/2} d\theta\right\}^{1/n} \\
&\geqslant \|f(z)\|_\infty \left\{\frac{1}{2\pi}\int_{\theta_0-1/n}^{\theta_0+1/n}\left[1-\frac{\|g'(\theta)\|_\infty}{g(\theta_0)}\cdot\frac{1}{n}\right]^{n/2} d\theta\right\}^{1/n} \\
&= \|f(z)\|_\infty \left(\frac{1}{\pi n}\right)^{1/n}\left[1-\frac{\|g'(\theta)\|_\infty}{g(\theta_0)}\cdot\frac{1}{n}\right]^{1/2} \\
&= \|f(z)\|_\infty \cdot \exp\left\{\frac{1}{n}\log\frac{1}{n\pi}\right\}\cdot\left[1+O\left(\frac{1}{n}\right)\right] \\
&= \|f(z)\|_\infty \cdot \left[1-\frac{\log \pi n}{n}+O\left(\frac{\log^2 n}{n^2}\right)\right]\left[1+O\left(\frac{1}{n}\right)\right] \\
&= \|f(z)\|_\infty - (\|f(z)\|_\infty)\frac{\log n}{n}+O\left(\frac{1}{n}\right).
\end{aligned}$$

This completes the proof of the lemma.

*Proof of Theorem 1:* By (11) and Lemma 1, we know that

$$J(n) = \|C(z)\|_{2n}. \tag{25}$$

By the Hölder inequality, it is easy to see that the $L^p$-norm $\|\cdot\|_p$ is monotonically increasing in $p$, and hence $J(n)$ is monotonically increasing in $n$. The other results follow from (25) and Lemma 2.

## IV. Example Studies

To illustrate the algorithm works, two examples are studied. They are as follows:

i) $C(z) = 1 - z - z^2$

ii) $C(z) = 1 + 2z + 3z^2$.

It is easy to show in example ii) that $\|C(z)\|_\infty = 6$. However, it is not straightforward to see in example i) that $\|C(z)\|_\infty = \sqrt{5}$. After 1500 iterations, the $H_\infty$-norm is approximated with relative error under 0.00154 in both cases, which are depicted in Figs. 1 and 2, respectively.

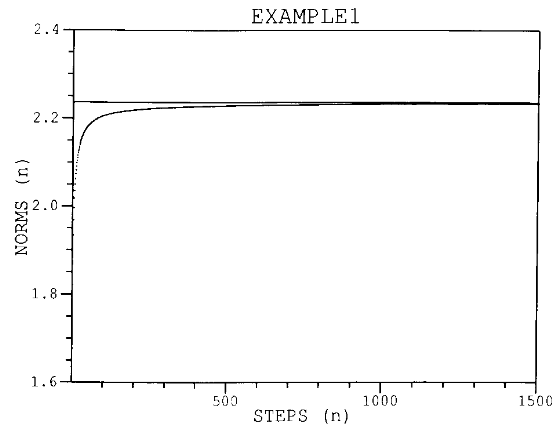


Fig. 1.

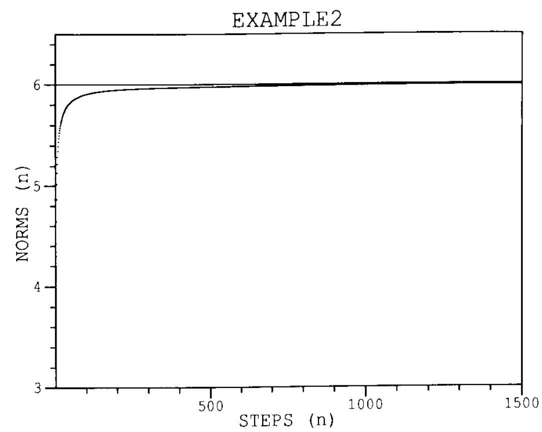


Fig. 2.

## V. Conclusions and Remarks

a) The proposed algorithm has itself theoretical interests as well as its application importance. Various algorithms for minimization (maximization) of functions exist [6]–[8], but to the authors' knowledge, theoretical algorithms for computing the $H^\infty$-norm have not yet been studied elsewhere.

b) It is interesting to note that the principal part of the relative error of the algorithm is independent of the polynomial $C(z)$ (i.e., $(\log n)/2n$). Furthermore, the error is monotonically decreasing to zero. So, for a given relative error, we can roughly decide the iteration step $n$ to achieve the desired accuracy.

c) In this note, we have only considered the scalar polynomial case. Of course, for a given stable scalar rational function, one can first approximate it by an $r$th-order polynomial (with exponential decaying error $0(\lambda^r)$, $0 < \lambda < 1$) and then use the above method to approximate the $H^\infty$-norm of the rational function. It is desirable to extend the results of this note to the general matrix transfer function case.

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