Effect of Disturbance Directions on Closed-Loop Performance

Sigurd Skogestad[†] and Manfred Morari*

Chemical Engineering, 206-41, California Institute of Technology, Pasadena, California 91125

The effectiveness of disturbance suppression in a multivariable control system can depend strongly on the direction of the disturbance. The "disturbance condition number" is introduced to quantify the effect of disturbance direction on closed-loop performance. As an example a two-point composition control system for a distillation column is analyzed for various disturbance and set-point changes.

I. Introduction

Disturbance rejection is often the main objective of process control. For multivariable systems, usually each disturbance affects all the outputs. As an example, consider a distillation column. A feed composition disturbance corresponding to an increased amount of light component in the feed leads to an *increase* of *both* product compositions y_D and x_B. (Here y_D and x_B correspond to the mole fraction of light component in the top and bottom products.) In this paper we define as "disturbance direction" the direction of the system output vector resulting from a specific disturbance. As we will show, some disturbance directions may be easily counteracted by the control system, while others may not. This has also been pointed out by Shimizu and Matsubara (1985) and Stanley et al. (1985). These papers are discussed in some detail later. The aim of this paper is to develop simple measures which may be used to indicate how the disturbances are "aligned" with the plant and thus how well they can be rejected.

Consider the linear control system in Figure 1. The process model is

$$\begin{aligned} \boldsymbol{y}(s) &= \mathbf{G}(s)\boldsymbol{m}(s) + \mathbf{G}_d(s)\boldsymbol{z}(s) \\ &= \mathbf{G}(s)\boldsymbol{m}(s) + \boldsymbol{d}(s) \end{aligned} \tag{1}$$

where $\boldsymbol{y}$ is the output vector, $\boldsymbol{m}$ is the manipulated input vector, and $\boldsymbol{d}$ represents the *effect* of the disturbances $\boldsymbol{z}$ on the outputs. The square transfer matrix $\mathbf{G}(s)$ is the process model, and $\mathbf{G}_d(s)$ is the disturbance model expressing the relationship between the physical disturbances z_i and their effect on the output. For a distillation column, the components $\boldsymbol{z} = (z_1, ..., z_i, ..., z_n)^T$ may correspond to disturbances in feed rate, feed composition, boilup rate, etc. The column vector $\boldsymbol{g}_{d_i}$ of $\mathbf{G}_d$ represent the disturbance model for each disturbance z_i. The effect of a particular disturbance (z_i) on the process output is $\boldsymbol{d}_i$,

$$\boldsymbol{d}_i = \boldsymbol{g}_{d_i} z_i \tag{2}$$

The direction of the vector $\boldsymbol{d}_i$ will be referred to as the direction of disturbance i. The overall effect of all disturbances (z_i) on the output is $\boldsymbol{d}$,

$$\boldsymbol{d} = \sum_i \boldsymbol{d}_i = \sum_i \boldsymbol{g}_{d_i} z_i = \mathbf{G}_d \boldsymbol{z} \tag{3}$$

In most cases, we will consider the effect of one particular disturbance, z_i. To simplify notation, we will usually drop the subscript i, and $\boldsymbol{d} = \boldsymbol{g}_d z$ will then denote the effect of this *single* disturbances $z_i = z$ on the outputs. We will also be referring to $\boldsymbol{d}$ as a "disturbance", although in general it represents the *effect* of the physical disturbance.

We will consider two different effects of disturbance directions. One is in terms of the magnitude of the manipulated variables $\boldsymbol{m}$ needed to cancel the influence of the disturbance on the process output completely at steady state. It is independent of the controller $\mathbf{C}$. This may be used to identify problems with constraints at steady state. However, the issue of constraints at steady state is not really a *control* problem, but rather a plant *design* problem. Any well-designed plant should be able to reject disturbances at steady state. The second and most important effect of disturbance directions is on closed-loop performance. Here we mean by performance the behavior of the controlled outputs $\boldsymbol{y}$ in the presence of disturbances.

II. Singular Value Decomposition

Throughout this paper we will make use of the singular value decomposition (SVD) of a matrix (Klema and Laub, 1980). Any complex $n \times n$ matrix $\mathbf{A}$ can be written in the form

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H \tag{4}$$

where $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices ($\mathbf{U}^H = \mathbf{U}^{-1}$) and $\boldsymbol{\Sigma}$ is a diagonal matrix with real nonnegative entries

$$\boldsymbol{\Sigma} = \text{diag}\{\sigma_j\} \tag{5}$$

The superscript H denotes complex conjugate transpose. The set of $\{\sigma_j\}$ are the singular values of $\mathbf{A}$, and we have

$$\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \geq 0$$

The number of nonzero singular values is equal to the rank of matrix $\mathbf{A}$. If matrix $\mathbf{A}$ is nonsingular, all singular values are greater than zero, and this will be assumed in the following. The maximum singular value $\sigma_1 = \sigma_{max}$ and the minimum singular value $\sigma_n = \sigma_{min}$ are of particular interest because of the properties

$$\max_{v \neq 0} \frac{\|\mathbf{A}\boldsymbol{v}\|_2}{\|\boldsymbol{v}\|_2} = \sigma_{max}(\mathbf{A}) \tag{6a}$$

and

$$\min_{v \neq 0} \frac{\|\mathbf{A}\boldsymbol{v}\|_2}{\|\boldsymbol{v}\|_2} = \sigma_{min}(\mathbf{A}) \tag{6b}$$

Here $\|\cdot\|_2$ denotes the Euclidean vector norm.

$$\|\boldsymbol{x}\|_2 = (\sum_i \boldsymbol{x}_i^2)^{1/2}$$

Consequently, σ_{max} corresponds to the largest amplification by matrix $\mathbf{A}$ and σ_{min} to its smallest amplification. Matrix $\mathbf{U}$ consists of the left singular vectors $\{\boldsymbol{u}_j\}$, $\|\boldsymbol{u}_j\|_2 = 1$, and matrix $\mathbf{V}$ of the right singular vectors $\{\boldsymbol{v}_j\}$, $\|\boldsymbol{v}_j\|_2 = 1$. For each right singular vector $\boldsymbol{v}_j$, we have

$$\mathbf{A}\boldsymbol{v}_j = \sigma_j(\mathbf{A})\boldsymbol{u}_j \tag{7}$$

* To whom all correspondence should be addressed.
[†] Present address: Chemical Engineering, Norwegian Institute of Technology, N-7034 Trondheim, Norway.

0888-5885/87/2626-2029$01.50/0

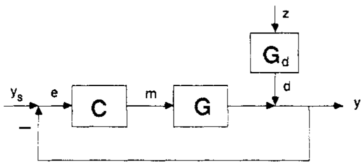

Figure 1. Block diagram of linear control system. The physical disturbances $\boldsymbol{z}$ have the effect $\boldsymbol{d} = \mathbf{G}_d\boldsymbol{z}$ on the outputs.

and in particular for the singular vectors associated with the maximum and minimum singular value

$$\mathbf{A}\boldsymbol{v}_{max} = \sigma_{max}(\mathbf{A})\boldsymbol{u}_{max} \tag{8a}$$

$$\mathbf{A}\boldsymbol{v}_{min} = \sigma_{min}(\mathbf{A})\boldsymbol{u}_{min} \tag{8b}$$

$\boldsymbol{v}_{max}(\mathbf{A})$ therefore corresponds to the direction of the input which undergoes the largest amplification, and $\boldsymbol{v}_{min}(\mathbf{A})$ to the direction with the smallest amplification. Furthermore,

$$\mathbf{A}^{-1} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^H \tag{9}$$

which is the SVD of $\mathbf{A}^{-1}$, but with the order of the singular values reversed. Let $l = n - j + 1$. It then follows from (9) that

$$\sigma_j(\mathbf{A}^{-1}) = 1/\sigma_l(\mathbf{A}) \tag{10a}$$

$$\boldsymbol{u}_j(\mathbf{A}^{-1}) = \boldsymbol{v}_l(\mathbf{A}) \tag{10b}$$

$$\boldsymbol{v}_j(\mathbf{A}^{-1}) = \boldsymbol{u}_l(\mathbf{A}) \tag{10c}$$

and in particular

$$\sigma_{max}(\mathbf{A}^{-1}) = 1/\sigma_{min}(\mathbf{A}) \tag{11a}$$

$$\boldsymbol{u}_{max}(\mathbf{A}^{-1}) = \boldsymbol{v}_{min}(\mathbf{A}) \tag{11b}$$

$$\boldsymbol{u}_{min}(\mathbf{A}^{-1}) = \boldsymbol{v}_{max}(\mathbf{A}) \tag{11c}$$

III. Effect of Disturbance Direction on Manipulated Variables

Constraints. Assume the disturbance model and the process model have been scaled such that at steady state $-1 \le z_i \le 1$ corresponds to the expected range of each disturbance and $-1 \le m_j \le 1$ corresponds to the acceptable range for each manipulated variable. For process control, $m_j = -1$ may correspond to a closed valve and $m_j = 1$ to a fully open valve. The *steady-state* process model is

$$\boldsymbol{y} = \mathbf{G}\boldsymbol{m} + \mathbf{G}_d\boldsymbol{z} \tag{12}$$

For complete disturbance rejection ($\boldsymbol{y} = 0$), we require

$$\boldsymbol{m} = -\mathbf{G}^{-1}\mathbf{G}_d\boldsymbol{z} \tag{13}$$

Let $\|\boldsymbol{x}\|_\infty$ denote the largest component of the vector $\boldsymbol{x}$. To avoid problems with constraints, we have to require

$$\|\boldsymbol{m}\|_\infty \le 1 \quad \text{for} \quad \text{all} \quad \|\boldsymbol{z}\|_\infty \le 1$$

Mathematically this is equivalent to requiring

$$\|\mathbf{G}^{-1}\mathbf{G}_d\|_{i\infty} \le 1 \tag{14}$$

$\|\mathbf{A}\|_{i\infty}$ is the induced ∞-norm of the matrix $\mathbf{A}$ which is equal to its largest row sum:

$$\|\mathbf{A}\|_{i\infty} = \max_i \left(\sum_j |a_{ij}|\right) \tag{15}$$

Whether (14) is violated and constraints cause problems depends both on the process model $\mathbf{G}$ and the disturbance model $\mathbf{G}_d$. Even if $\|\mathbf{G}^{-1}\|_{i\infty}$ is "large", $\|\mathbf{G}^{-1}\mathbf{G}_d\|_{i\infty}$ can be "small" if $\mathbf{G}_d$ is "aligned" with $\mathbf{G}^{-1}$ in a certain manner. We will discuss this in more detail later.

It should be stressed that the issue of constraints (or more generally the magnitude of the manipulated inputs) at steady state is mainly a plant *design* problem rather than a *control* problem, and one should use such arguments for discriminating between various control systems only with great care. One example where such arguments are highly misleading is for choosing control configurations for distillation columns: The manipulated variables for a distillation column are, irrespective of the control configuration, the distillate product (D), the bottom product (B), the reflux (L), the overhead vapor (V_T), and the boilup (V). To reject disturbances at steady state, the same changes in these flows are needed *irrespective* of the chosen control configuration. For example, a 10% increase in the feed rate is rejected by a 10% increase in all the flows. It does not matter whether we use the LV configuration (L and V used for composition control), DV configuration, or any other configuration; the steady state changes in the flows for perfect disturbance rejection are the same and cannot be used to choose the best configuration. Nevertheless, this is exactly what is done by Shimizu and Matsubara (1985). For example, they claim that at steady state the LV configuration can handle larger disturbances in the feed rate than the DV configuration. Obviously, this is not correct—there is no difference whatsoever. (The reason for their misinterpretation is that they only look at two flows at a time (for example, L and V for the LV configuration) and do not take into account that the other flows also have to change). This does not mean that all configurations handle disturbances equally well, but this issue should be addressed in the framework of control *performance* as discussed below.

Disturbance Condition Number. Even when constraints are not causing any problems, it is of interest to investigate the magnitude of the manipulated variable necessary to compensate for the effect of a disturbance. In this context it is more reasonable to use the Euclidean (2-norm) norm as a measure of magnitude because it "sums up" the deviations of all manipulated variables rather than accounting for the maximum deviation only (like the ∞-norm). Consider a particular disturbance $\boldsymbol{d} = \boldsymbol{g}_d z$. For complete disturbance rejection of this disturbance

$$\mathbf{m} = -\mathbf{G}^{-1}\boldsymbol{d} \tag{16}$$

The quantity

$$\|\boldsymbol{m}\|_2/\|\boldsymbol{d}\|_2 = \|\mathbf{G}^{-1}\boldsymbol{d}\|_2/\|\boldsymbol{d}\|_2 \tag{17}$$

depends only on the direction of the disturbance $\boldsymbol{d}$ but not on its magnitude. It measures the magnitude of $\boldsymbol{m}$ needed to reject a disturbance $\boldsymbol{d}$ of unit magnitude which enters in a particular direction expressed by $\boldsymbol{d}/\|\boldsymbol{d}\|_2$.

The "best" disturbance direction, requiring the *least* action by the manipulated variables, is that of the left singular vector $\boldsymbol{u}_{max}(\mathbf{G})$ associated with the largest singular value of $\mathbf{G}$.

$$\boldsymbol{d} = \boldsymbol{v}_{min}(\mathbf{G}^{-1}) = \boldsymbol{u}_{max}(\mathbf{G})$$

In this case we find that by use of (11b)

$$\|\mathbf{G}^{-1}\boldsymbol{d}\|_2/\|\boldsymbol{d}\|_2 = \|\mathbf{G}^{-1}\boldsymbol{v}_{min}(\mathbf{G}^{-1})\|_2 = \sigma_{min}(\mathbf{G}^{-1}) = 1/\sigma_{max}(\mathbf{G}) \tag{18}$$

By normalizing (17) with this "best" disturbance, we obtain the following measure which we call the *disturbance condition number of the plant* $\mathbf{G}$

$$\gamma_d(\mathbf{G}) = \frac{\|\mathbf{G}^{-1}\boldsymbol{d}\|_2}{\|\boldsymbol{d}\|_2}\sigma_{max}(\mathbf{G}) \tag{19a}$$

or equivalently

$$\gamma_d(\mathbf{G}) = \frac{\|\mathbf{G}^{-1}\boldsymbol{g}_d\|_2}{\|\boldsymbol{g}_d\|_2}\sigma_{max}(\mathbf{G}) \tag{19b}$$

It measures the magnitude of the manipulated variables needed to reject a disturbance in the direction $\boldsymbol{d}$ relative to rejecting a disturbance with the same magnitude, but in the "best" direction.

The "worst" disturbance direction is

$$\boldsymbol{d} = \boldsymbol{v}_{max}(\mathbf{G}^{-1}) = \boldsymbol{u}_{min}(\mathbf{G})$$

and in this case we get

$$\gamma_d(\mathbf{G})_{max} = \sigma_{max}(\mathbf{G}^{-1})\sigma_{max}(\mathbf{G}) = \gamma(\mathbf{G}) \tag{20}$$

where $\gamma(\mathbf{G})$ is the condition number of the plant. It follows that

$$1 \le \gamma_d(\mathbf{G}) \le \gamma(\mathbf{G})$$

and $\gamma_d(\mathbf{G})$ may be viewed as a generalization of the condition number $\gamma(\mathbf{G})$ of the plant, which also takes into account the direction of the disturbances. The disturbance condition number $\gamma_d(\mathbf{G})$ is clearly scaling dependent since $\gamma(\mathbf{G})$ is scaling dependent. We know that ill-conditioned plants ($\gamma(\mathbf{G})$ large) indicate control problems (Morari and Doyle, 1986; Skogestad and Morari, 1987a). A large value of $\gamma(\mathbf{G})$ indicates a large degree of directionality in the plant $\mathbf{G}$, which may have to be compensated for by the controller in order to get good response. We used "may" in the last sentence because this also depends on the disturbance direction: If $\gamma_d(\mathbf{G})$ is small for all disturbances, then it really does not matter if $\gamma(\mathbf{G})$ is large.

In the next section we will look at closed-loop performance and show explicitly the physical significance of $\gamma_d(\mathbf{G})$ in this context. However, let us first look at another measure which has been suggested for measuring disturbance directionality.

Relative Disturbance Gain. We will show that the Relative Disturbance Gain (RDG) introduced by Stanley et al. (1985) is similar to the disturbance measure $\gamma_d(\mathbf{G})$ defined above, but with a different normalization. One advantage of the RDG is that it is scaling independent, while $\gamma_d(\mathbf{G})$ is scaling dependent. On the other hand, the physical significance of the RDG is less clear than that of $\gamma_d(\mathbf{G})$.

For a particular disturbance z, the RDG, β_l, is defined for each manipulated variable, m_l, as (Stanley et al., 1985)

$$\beta_l = \frac{(\partial m_l/\partial z)_{y_j}}{(\partial m_l/\partial z)_{y_l,m_{j\neq l}}} \tag{21}$$

$(\partial m_l/\partial z)_{y_j}$ is the change in manipulated variable m_l needed for perfect disturbance rejection. $(\partial m_l/\partial z)_{y_l,m_{j\neq l}}$ is the change in manipulated variable m_l needed for perfect disturbance rejection for the corresponding output y_l, while keeping all other manipulated variables constant. To find the relationship between β_l and $\gamma_d(\mathbf{G})$, the following identities (Grosdidier, 1985) are used

$$\left[\frac{\partial m_l}{\partial z}\right]_{y_j} = -(\mathbf{G}^{-1}\boldsymbol{g}_d)_l \tag{22}$$

$$\left[\frac{\partial m_l}{\partial z}\right]_{y_l,m_{j\neq l}} = -((\mathbf{G}_{diag})^{-1}\boldsymbol{g}_d)_l \tag{23}$$

which follow trivially from the linear relationship (Figure 1) $d\boldsymbol{y} = \boldsymbol{g}_d\, dz + \mathbf{G}\, d\boldsymbol{m}$. Here $\mathbf{G}_{diag}$ denotes the matrix consisting of the diagonal elements in $\mathbf{G}$. Using $\boldsymbol{d} = \boldsymbol{g}_d z$, the definition of β_l (eq 21) may be rewritten as

$$\beta_l = \frac{(\mathbf{G}^{-1}\boldsymbol{d})_l}{((\mathbf{G}_{diag})^{-1}\boldsymbol{d})_l} \tag{24}$$

Equation 24 is similar to the definition of $\gamma_d(\mathbf{G})$ in (19a), but with the diagonal plant as the normalization factor instead of the "best" disturbance. Note that $\beta_l = 1$ for any disturbance if $\mathbf{G}$ is diagonal.

We can also define a RDG matrix for the case when we have several disturbances $\boldsymbol{z}$ as

$$\mathrm{RDG} = \mathbf{G}^{-1}\mathbf{G}_d/(\mathbf{G}_{diag})^{-1}\mathbf{G}_d \tag{25}$$

where the division in this case denotes element by element division. Note the resemblance with the Relative Gain Array (RGA) which may be defined as the matrix

$$\mathrm{RGA} = \mathbf{G} \times (\mathbf{G}^{-1})^T \tag{26}$$

where × denotes element by element (Schur) multiplication. The RGA is also scaling invariant.

Stanley et al. (1985) claim that the RDG can be used to investigate the effect of decoupling. However, (21) and (24) clearly show that the RDG is independent of the controller which may or may not include decoupling. The meaning of (24) when $\mathbf{G}$ is replaced by $\mathbf{GH}$, where $\mathbf{H}$ denotes a decoupler, is not clear from this definition. The variables m_l will then be some internal variable in the controller with no direct physical significance.

Below we will derive an alternative physical interpretation for β_l in terms of closed-loop performance which retains it significance when $\mathbf{G}$ is replaced by $\mathbf{GH}$ in (24).

IV. Effect of Disturbance Direction on Closed-Loop Performance

In section III we derived measures of how the magnitude of the manipulated variables depends on the disturbance direction. In this section we will rederive these measures in terms of closed-loop performance. This will give us a more powerful interpretation of these measures and will allow us to define dynamic measures and to include "decouplers".

Disturbance Condition Number. One objective of the control system (Figure 1) is to minimize the effect of the disturbances on the outputs $\boldsymbol{y}$. Consider a particular disturbance $\boldsymbol{d}(s) = \boldsymbol{g}_d(s)z(s)$. The closed-loop relationship between this disturbance and the outputs is

$$\boldsymbol{y}(s) = (\mathbf{I} + \mathbf{GC}(s))^{-1}\boldsymbol{d}(s) = \boldsymbol{S}(s)\boldsymbol{d}(s) \tag{27}$$

where the sensitivity operator is

$$\mathbf{S}(s) = (\mathbf{I} + \mathbf{G}(s)\mathbf{C}(s))^{-1} \tag{28}$$

Let $\|\boldsymbol{y}(j\omega)\|_2$ denote the Euclidean norm of $\boldsymbol{y}$ evaluated at each frequency. The quantity

$$\alpha(\omega) = \|\mathbf{S}\boldsymbol{d}(j\omega)\|_2/\|\boldsymbol{d}(j\omega)\|_2 \tag{29}$$

depends only on the disturbance direction but not on its magnitude. $\alpha(\omega)$ measures the magnitude of the output vector $\boldsymbol{y}(j\omega)$ resulting from a sinusoidal disturbance $\boldsymbol{d}(j\omega)$ of unit magnitude and frequency ω.

The "best" disturbance direction causing the smallest output deviation is that of the right singular vector $\boldsymbol{v}_{min}(\mathbf{S})$ associated with the smallest singular value $\sigma_{min}(\mathbf{S})$ of $\mathbf{S}$. By normalizing $\alpha(\omega)$ with this best disturbance, we obtain the disturbance condition number of $\mathbf{S}^{-1}$

$$\gamma_d(\mathbf{S}^{-1}) = \frac{\|\mathbf{S}\boldsymbol{d}\|_2}{\sigma_{min}(\mathbf{S})\|\boldsymbol{d}\|_2}(j\omega) \tag{30}$$

($\mathbf{S}^{-1}$ is used in the argument of γ_d for consistency with the previously defined $\gamma_d(\mathbf{G})$ in (19)). Again

$$1 \le \gamma_d(\mathbf{S}^{-1}) \le \gamma(\mathbf{S}^{-1}) = \gamma(\mathbf{S}) \tag{31}$$

At low frequencies where the controller gain is high, we have

$$\mathbf{S}(j\omega) \approx (\mathbf{GC}(j\omega))^{-1} \tag{32}$$

In particular, this expression is exact at steady state ($\omega = 0$) if we have integral action. On the basis of this approximation, we derive the disturbance condition number of **GC**

$$\gamma_d(\mathbf{GC}) = \frac{\|(\mathbf{GC})^{-1}\boldsymbol{d}\|_2}{\|\boldsymbol{d}\|_2}\sigma_{max}(\mathbf{GC})(j\omega) \quad (33)$$

As stated above, this measure has physical significance only when $\sigma_{min}(\mathbf{GC}) \gg 1$. To avoid problems with evaluating the measure at $\omega = 0$, write

$$\mathbf{C}(s) = k(s)\mathbf{H}(s) \quad (34)$$

where $k(s)$ is a scalar transfer function which includes any integral action. $\mathbf{H}(s)$ may be viewed as a "decoupler". We have

$$\gamma_d(\mathbf{GC}) = \frac{\|(\mathbf{GH})^{-1}\boldsymbol{d}\|_2}{\|\boldsymbol{d}\|_2}\sigma_{max}(\mathbf{GH})(j\omega) \quad (35)$$

To evaluate how the disturbance direction is aligned with the plant **G** itself, choose $\mathbf{H} = \mathbf{I}$ (i.e., the controller is $k(s)\mathbf{I}$) and rederive the disturbance condition number of **G**

$$\gamma_d(\mathbf{G}) = \frac{\|\mathbf{G}^{-1}\boldsymbol{d}\|_2}{\|\boldsymbol{d}\|_2}\sigma_{max}(\mathbf{G}) \quad (36)$$

$\gamma_d(\mathbf{G})$ can be interpreted in terms of closed-loop performance as follows: If a scalar controller $\mathbf{C} = k(s)\mathbf{I}$ is chosen (which keeps the directionality of the plant unchanged), then $\gamma_d(\mathbf{G})$ measures the magnitude of the *output* $\boldsymbol{y}$ for a particular disturbance $\boldsymbol{d}$, compared to the magnitude of the output if the disturbance were in the "best" direction (corresponding to the large plant gain). If $\gamma_d(\mathbf{G}) = \gamma(\mathbf{G})$, the disturbance has all its components in the "bad" direction, corresponding to low plant gain and low bandwidth. If $\gamma_d(\mathbf{G}) = 1$, the disturbance has all its components in the "good" direction, corresponding to high plant gain and high bandwidth.

Though a large value of $\gamma_d(\mathbf{G})$ does not necessarily imply bad performance, it usually does: In principle we could choose an inverse-based compensator **C** which makes $\gamma_d(\mathbf{GC}) = 1$ for all disturbances. However, this controller often leads to serious robustness problems. For example, it is shown by Skogestad and Morari (1987c) that one should never use an inverse-based compensator for plants with large elments in the Relative Gain Array (RGA) because of the presence of uncertainty on the manipulated inputs. For a detailed analysis, the reader is referred to Morari and Doyle (1986) and Skogestad and Morari (1987a,c).

Decomposition of *d* along Singular Vectors. The objective here is to gain insight into the type of dynamic response which is to be expected when disturbances along a particular direction affect a system with a high degree of directionality ($\gamma(\mathbf{S})$ is "large"). The right singular vectors $\boldsymbol{v}_j(\mathbf{S})$ of **S** form an orthonormal basis. The disturbance vector $\boldsymbol{d}$ can be represented in terms of this basis

$$\boldsymbol{d} = \sum_{j=1}^{n}(\boldsymbol{v}_j(\mathbf{S})^T\cdot\boldsymbol{d})\boldsymbol{v}_j(\mathbf{S}) \quad (37)$$

where · denotes the usual scalar vector product. Then the output $\boldsymbol{y}$ is described by

$$\boldsymbol{y}(j\omega) = \mathbf{S}\boldsymbol{d}(j\omega) \quad (38a)$$

$$\boldsymbol{y}(j\omega) = \sum_{j=1}^{n}\mathbf{S}\boldsymbol{v}_j(\mathbf{S})(\boldsymbol{v}_j(\mathbf{S})^T\cdot\boldsymbol{d})(j\omega) \quad (38b)$$

$$\boldsymbol{y}(j\omega) = \sum_{j=1}^{n}\sigma_j(\mathbf{S})\boldsymbol{u}_j(\mathbf{S})(\boldsymbol{v}_j(\mathbf{S})^T\cdot\boldsymbol{d})(j\omega) \quad (38c)$$

$$\boldsymbol{y}(j\omega) = \sum_{j=1}^{n}\sigma_j(\mathbf{S})\boldsymbol{d}^j(j\omega) \quad (38d)$$

where we have defined the new "disturbance components" as

$$\boldsymbol{d}^j = (\boldsymbol{v}_j(\mathbf{S})^T\cdot\boldsymbol{d})\boldsymbol{u}_j(\mathbf{S}) \quad (39)$$

Equation 38d shows that the response to a particular disturbance can be viewed as the sum of responses to the disturbances $\boldsymbol{d}^j$ passing through the scalar transfer function $\sigma_j(\mathbf{S})$. The magnitude of $\boldsymbol{d}^j$ depends on the alignment of the disturbance $\boldsymbol{d}$ with the singular vector $\boldsymbol{v}_j(\mathbf{S})$. The characteristics (speed) of the response to $\boldsymbol{d}^j$ depend on $\sigma_j(\mathbf{S})$.

For the controller

$$\mathbf{C}(s) = k(s)\mathbf{H}(s) \quad (40)$$

with integral action in $k(s)$, the approximation

$$\mathbf{S}(j\omega) \approx \frac{1}{k}(\mathbf{GH})^{-1}(j\omega) \quad (41)$$

is valid for small ω. If $l = n - j + 1$ is defined and (10) is used, (38d) becomes

$$\boldsymbol{y}(j\omega) = \sum_{l=1}^{n}\frac{1}{k\sigma_l(\mathbf{GH})}\tilde{\boldsymbol{d}}^l \quad (42)$$

where

$$\tilde{\boldsymbol{d}}^l = \boldsymbol{d}^{n-l+1} = (\boldsymbol{u}_l(\mathbf{GH})^T\cdot\boldsymbol{d})\boldsymbol{v}_l(\mathbf{GH}) \quad (43)$$

The magnitude of $\tilde{\boldsymbol{d}}^l$ is given by the component of $\boldsymbol{d}$ in the direction of the singular vector $\boldsymbol{u}_l(\mathbf{GH})$ and $\tilde{\boldsymbol{d}}^l$ affects the output along the direction of the singular vector $\boldsymbol{v}_l(\mathbf{GH})$. If the loop transfer matrix **GH** has a high gain in this direction (i.e., $\sigma_l(\mathbf{GH})$ is large), then the control will be quick and good. If the gain is low, the response will be slow and poor. If **GH** is ill-conditioned ($\gamma(\mathbf{GH})$ large), the widely different response characteristics for different disturbance components will result in unusual overall system responses. These issues will become clearer from the example at the end of this paper.

Performance Interpretation of the RDG. The process response to a particular disturbance $\boldsymbol{d} = \boldsymbol{g}_d z$ is given by

$$\boldsymbol{y}(s) = (\mathbf{I} + \mathbf{GC}(s))^{-1}\boldsymbol{d}(s) \quad (44)$$

Let controller **C** be given by

$$\mathbf{C}(s) = \mathbf{H}(s)\mathbf{K}(s) \quad (45)$$

where $\mathbf{K}(s)$ is diagonal and includes integral action in all channels. Since at low frequencies $\sigma_{min}(\mathbf{GC})(j\omega) \gg 1$, (44) can be approximated by

$$\boldsymbol{y}(j\omega) \approx (\mathbf{GHK})^{-1}\boldsymbol{d}(j\omega) \quad (46)$$

and in particular for output $\boldsymbol{y}_l$

$$y_l(j\omega) = \frac{1}{k_l(j\omega)}[(\mathbf{GH})^{-1}\boldsymbol{d}(j\omega)]_l \quad (47)$$

Normalize $\boldsymbol{y}_l(j\omega)$ with respect to the response that would occur if the off-diagonal elements in the system **GH** were neglected:

$$\beta_l(\mathbf{GH})(j\omega) = \frac{[(\mathbf{GH})^{-1}\boldsymbol{d}(j\omega)]_l}{[((\mathbf{GH})_{diag})^{-1}\boldsymbol{d}]_l}(j\omega) \quad (48)$$

Comparing this with the definition of the RDG in eq 24, we see that eq 48 gives a performance interpretation to the RDG and extends it to frequencies other than zero. More importantly, this definition provides a justification for using RDG to evaluate the effect of decouplers **H**.

Table I. Steady-State Data for Distillation Column

Given: Binary Separation, Liquid Feed, Constant Molar Flows	
relative volatility	$\alpha = 1.5$
no. of theoretical trays	$N = 40$
feed tray location (1 = reboiler)	$N_F = 21$
feed composition	$x_F = 0.5$
product compositions	$y_D = 0.99$, $x_B = 0.01$
external flow rates	$F = 1$, $B = 0.5$, $D = 0.5$
Computed	
reflux ratio	$L/D = 5.41$

Gains Using L and V as Inputs (Linearized Tray-by-Tray Model)

$$\begin{bmatrix} \Delta y_d \\ \Delta x_B \end{bmatrix} = \begin{bmatrix} 0.878 & 0.864 \\ 1.082 & 1.096 \end{bmatrix} \begin{bmatrix} \Delta L \\ -\Delta V \end{bmatrix}$$

The normalization using $(\mathbf{GH})_{diag}$ makes the RDG scaling independent, which might be viewed as an advantage over γ_d. (In particular, the RDG is the same for any diagonal controller **K**.) However, contrary to γ_d, a physical interpretation becomes difficult or impossible. If a disturbance does not affect y_l at all, one finds $\beta_l = \infty$. For example, for a full 2 × 2 system with

$$\boldsymbol{d}^T = [0 \quad 1]$$

we find (Stanley et al., 1985)

$$\text{RDG} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \infty \\ \lambda(\mathbf{GH}) \end{bmatrix}$$

where λ is the 1,1 element of the RGA of **GH**. (Also note that if **GH** were diagonal, then β_1 would be undefined for this specific $\boldsymbol{d}$.) Consequently, β_l may range in magnitude from $-\infty$ to ∞, and contrary to $\gamma_d(\mathbf{G})$ the magnitude of β_l by itself may not be very informative.

V. Example: *LV* Distillation Column

Consider the distillation column in Table I with L and V as manipulated variables and the product compositions y_D and x_B as controlled outputs. The steady-state gain matrix is (Skogestad and Morari, 1987b)

$$\mathbf{G} = \begin{bmatrix} 0.878 & 0.864 \\ 1.082 & 1.096 \end{bmatrix} \tag{49}$$

We assume there will be no problems with constraints. We want to study how well the system rejects various disturbances using a diagonal controller $\mathbf{C}(s) = k(s)\mathbf{I}$. Since we are only concerned about the outputs (y_D and x_B), the scaling does not matter provided the outputs are scaled such that an output of magnitude one is equally "bad" for both y_D and x_B. We have

$$\sigma_{max}(\mathbf{G}) = 1.972, \qquad \sigma_{min}(\mathbf{G}) = 0.0139,$$
$$\gamma(\mathbf{G}) = 141.7, \qquad \lambda(\mathbf{G}) = 35.1$$

Consider disturbances z of unit magnitude in feed composition, x_F, feed flow rate, F, feed liquid fraction, q_F, and boilrate, $-V_d$. The linearized steady-state disturbance models are

$$\boldsymbol{d} = \boldsymbol{g}_d = \begin{bmatrix} 0.881 \\ 1.119 \end{bmatrix}, \begin{bmatrix} 0.394 \\ 0.586 \end{bmatrix}, \begin{bmatrix} 0.868 \\ 1.092 \end{bmatrix}, \text{ and } \begin{bmatrix} 0.864 \\ 1.096 \end{bmatrix} \tag{50a}$$

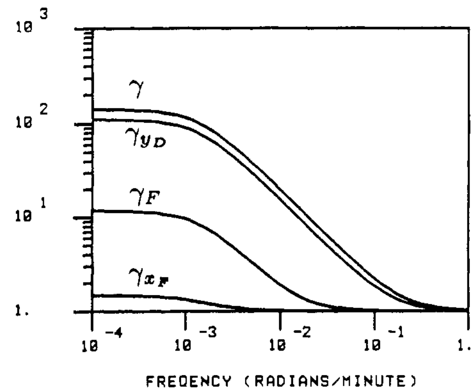

Figure 2. Disturbance condition number of $\mathbf{S}^{-1}$ for disturbances in feed rate F, feed composition x_F, and set-point change in y_D. $\mathbf{C}(s) = 0.1/s\mathbf{I}$.

Also consider set-point changes in y_D and x_B of magnitude one. These are mathematically equivalent to disturbances with

$$\boldsymbol{d} = \boldsymbol{g}_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{50b}$$

The steady-state values of the RDG, $\beta_l(\mathbf{G})$, and the disturbance condition number, $\gamma_d(\mathbf{G})$, are given for these disturbances in Table II. The disturbance condition number of $\mathbf{S}^{-1}$, using the controller described below, is shown as a function of frequency in Figure 2. From these data, we see that disturbances in x_F, q_F, and V are very well "aligned" with the plant, and there is little need for using a decoupler to change the directions of **G**. The feed flow disturbance is clearly the "worst" disturbance, but even it has its largest effect in the "good" direction.

A decoupler is clearly desirable if we want to follow set-point changes which have a large component in the bad direction corresponding to low plant gains. However, a decoupler is *not* recommended for this distillation column because of severe robustness problems caused by uncertainty (Skogestad and Morari, 1987c). Therefore, it may be difficult to obtain acceptable set-point tracking for this *LV* configuration. Other configurations which are less sensitive to input uncertainty may be better (Skogestad and Morari, 1987b). If set-point changes are of little or no interest, the *LV* configuration using a diagonal controller may be a good choice. The response to a feed rate disturbance is then expected to be somewhat sluggish because of the high value of $\gamma_d(\mathbf{G})$.

Time Responses. We will now confirm the predictions based on the data in Table II by studying some time responses. Assume the plant $\mathbf{G}(s)$ has no dynamics; i.e., $\mathbf{G}(s)$ is as given in (49) at all frequencies. This is obviously unrealistic, but the dominating dynamics are often similar in all the elements of $\mathbf{G}(s)$, and we can make the crude assumption that these dynamics are exactly compensated for by the dynamics in the controller. This also assumes that the magnitude of the disturbances is small, such that a linear approximation with constant time constants for

Table II. Disturbance Measures for Distillation Example

	disturbance				set-point change	
	x_F	F	q_F	$-V_d$	y_{DS}	x_{BS}
$\boldsymbol{d}$	$\begin{bmatrix} 0.881 \\ 1.119 \end{bmatrix}$	$\begin{bmatrix} 0.394 \\ 0.586 \end{bmatrix}$	$\begin{bmatrix} 0.868 \\ 1.092 \end{bmatrix}$	$\begin{bmatrix} 0.864 \\ 1.096 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
RDG (eq 24)	$\begin{bmatrix} -0.05 \\ 1.04 \end{bmatrix}$	$\begin{bmatrix} -6.05 \\ 6.01 \end{bmatrix}$	$\begin{bmatrix} 0.29 \\ 0.72 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 35.1 \\ -\infty \end{bmatrix}$	$\begin{bmatrix} \infty \\ 35.1 \end{bmatrix}$
$\gamma_d(\mathbf{G})$ (eq 19)	1.48	11.75	1.09	1.41	110.7	88.5

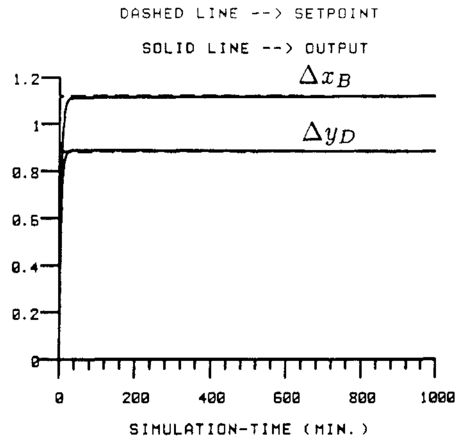

Figure 3. Step change in set point = $(0.881, 1.119)^T$, $\gamma_d(G) = 1.48$ (closed-loop response to "disturbance" in x_F).

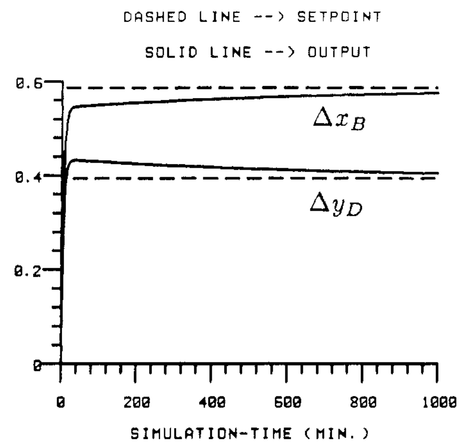

Figure 4. Step change in set point = $(0.394, 0.586)^T$, $\gamma_d(G) = 11.75$ (closed-loop response to "disturbance" in F).

the column is valid. We use a diagonal controller of the form

$$\mathbf{C}(s) = k(s)\mathbf{I}$$

where $k(s)$ is a simple integrator with gain 0.1 (min^{-1})

$$k(s) = \frac{0.1}{s}$$

(In practice $k(s)$ may be a PI controller, $k(s) = (1 + Ts)/s$, with integral time T equal to the time constant of the distillation column.)

Time domain simulations are shown for "disturbances" in x_F and F and for a set-point change in y_D in Figures 3–5. We have simulated all responses as step set-point changes of size $\boldsymbol{d}$ (eq 50) to make comparisons easier. All simulations are linear, and readers who are concerned about nonphysical values for y_D and x_B may assume, for example, that the deviations, Δy_D and Δx_B, from the initial steady state are in ppm. Dynamics have not been included in the disturbances for x_F and F, which is clearly unrealistic, but this has been done to make the example simpler. The time responses confirm what could be predicted based on the disturbance measures in Table II with respect to which disturbances are the worst. However, the measures in Table II give no direct way of predicting the shape of the responses. The responses are odd-looking, and one might almost expect that the system is nonlinear. This is obviously not the case, and the response may in fact be easily explained by decomposing the disturbances along the singular vector directions of the closed-loop system, as shown before. For each disturbance, the closed-loop frequency response at low frequencies can be approximated by

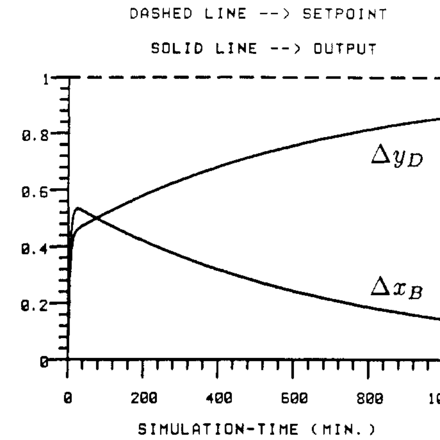

Figure 5. Step change in set point = $(1, 0)^T$, $\gamma_d(G) = 110.7$ (closed-loop response to set-point change for y_D).

Table III. $\tilde{d}^1$ and $\tilde{d}^2$ for Distillation Example

	disturbance		set-point change
	x_F	F	y_{DS}
$\boldsymbol{d}$	$\begin{bmatrix}0.881\\1.119\end{bmatrix}$	$\begin{bmatrix}0.394\\0.586\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$
$\tilde{d}^1$ (eq 52a)	$\begin{bmatrix}1.00\\1.00\end{bmatrix}$	$\begin{bmatrix}0.50\\0.50\end{bmatrix}$	$\begin{bmatrix}0.44\\0.44\end{bmatrix}$
$\tilde{d}^2$ (eq 52b)	$\begin{bmatrix}-0.008\\0.008\end{bmatrix}$	$\begin{bmatrix}-0.04\\0.04\end{bmatrix}$	$\begin{bmatrix}0.55\\-0.55\end{bmatrix}$

$$y(j\omega) \approx \frac{1}{k}\mathbf{G}^{-1}\boldsymbol{d}(j\omega)$$

By decomposing $\boldsymbol{d}$ along the "directions" of G as in (42) and (43), we may write this response as the sum of two SISO responses

$$y(j\omega) \cong \left[\frac{1}{k\sigma_{\max}(\mathbf{G})}\tilde{d}^1 + \frac{1}{k\sigma_{\min}(\mathbf{G})}\tilde{d}^2 \right] \quad (51)$$

where

$$\tilde{d}^1 = (\boldsymbol{u}_{\max}(\mathbf{G})^T\cdot\boldsymbol{d})\boldsymbol{v}_{\max}(\mathbf{G}) \quad (52a)$$

$$\tilde{d}^2 = (\boldsymbol{u}_{\min}(\mathbf{G})^T\cdot\boldsymbol{d})\boldsymbol{v}_{\min}(\mathbf{G}) \quad (52b)$$

Thus, each disturbance response will consist of two responses: one fast in the direction $\tilde{\boldsymbol{d}}^1$ and one slow in the direction of $\tilde{\boldsymbol{d}}^2$. The singular value decomposition $\mathbf{G} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H$ gives

$$\boldsymbol{\Sigma} = \begin{bmatrix}\sigma_{\max} & 0\\ 0 & \sigma_{\min}\end{bmatrix} = \begin{bmatrix}1.972 & 0\\ 0 & 0.01391\end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix}\boldsymbol{u}_{\max} & \boldsymbol{u}_{\min}\end{bmatrix} = \begin{bmatrix}0.625 & 0.781\\ 0.781 & -0.625\end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix}\boldsymbol{v}_{\max} & \boldsymbol{v}_{\min}\end{bmatrix} = \begin{bmatrix}0.707 & 0.708\\ 0.708 & -0.707\end{bmatrix}$$

$\tilde{\boldsymbol{d}}^1$ and $\tilde{\boldsymbol{d}}^2$ are given in Table III for the cases simulated in Figures 3–5.

The decomposition in (51) and (52), which applies at low frequencies, explains the actual responses very well: Initially there is a very fast response in the direction of $\boldsymbol{v}_{\max}^T = [0.707, 0.708]$. This response has overall open-loop transfer function $k\sigma_{\max}(\mathbf{G}) = 0.197/s$ corresponding to a first-order response with time constant $1/0.197 = 5.1$ min. Added to this is a slow first-order response with time constant $(k\sigma_{\min}(\mathbf{G}))^{-1} = (0.1 \times 0.01391)^{-1} = 720$ min in the direction of $\boldsymbol{v}_{\min}^T = [0.708, -0.707]^T$.

Note that in this example the slow disturbance component $\tilde{\boldsymbol{d}}^2$ is the "error" at $t \approx 40$ min, because the fast response has almost settled at this time. As an example, consider the disturbance in feed rate F (Figure 4). At $t \approx 40$ min, the deviation from the desired set point, $(0.394, 0.586)^T$, is approximately equal to $\tilde{d}^2 = (-0.04, 0.04)^T$. Similarly, for the set-point change in y_D (Figure 5), the deviation from desired set point, $(1, 0)^T$, at $t \approx 40$ min is approximately equal to $\tilde{d}^2 = (0.55, -0.55)^T$.

VI. Summary and Conclusions

The disturbance condition number of matrix **A** with respect to a disturbance with direction $\boldsymbol{d}$ is defined as

$$\gamma_d(\mathbf{A}) = \frac{\|\mathbf{A}^{-1}\boldsymbol{d}\|_2}{\|\boldsymbol{d}\|_2}\sigma_{max}(\mathbf{A})$$

The maximum singular value $\sigma_{max}(\mathbf{A}) = [\rho(\mathbf{A}^H\mathbf{A})]^{1/2}$ (ρ is the magnitude of the largest eigenvalue) and is easily computed by using the eigenvalue (or preferably singular value) routines in IMSL, EISPACK, or LINPACK. For nonsquare **A** (e.g., plants with more inputs than outputs), one should replace $\|\mathbf{A}^{-1}\boldsymbol{d}\|_2$ by $\{\min \|m\|_2$, subject to $\mathbf{A}\boldsymbol{m} = -\boldsymbol{d}\}$; that is, one should replace $\mathbf{A}^{-1}$ by $\mathbf{A}^{\#}$ (the pseudoinverse of **A**).

In this paper, we have used the disturbance condition number of the plant **G** (eq 19), of $\mathbf{S}^{-1}$ (eq 30), and of **GC** (eq 35). Of these, the first has the advantage of being independent of the controller, but it must be interpreted with some care for exactly the same reason. The disturbance condition number is a measure of control performance and therefore *must* be scaling *dependent*: Performance is defined as a weighted average of the outputs (for example, the 2-norm corresponds to the mean square average), and any measure involving performance should depend on how we choose to weigh the outputs. (On the other hand, the issue of stability is independent of scaling, and any measure used as a tool for evaluating a system's stability should be independent of scaling.)

Uses of γ_d(G): 1. Discriminating between Process Alternatives and Selecting Controlled and/or Manipulated Inputs. Plants with large values of $\gamma_d(\mathbf{G})$ are not necessarily bad, but if other factors are equal (e.g., RGA values, RHP zeros), we should prefer a design with a low value of $\gamma_d(\mathbf{G})$. This measure may therefore be used as *one* criterion for selecting controlled/manipulated variables and discriminating among alternatives.

2. Selecting Variable Pairings. The measure $\gamma_d(\mathbf{G})$ is invariant to permutations of the inputs and outputs and is therefore not useful in this respect.

3. Selecting *Controller* Structure (E.g., Diagonal or Multivariable Inverse-Based Controller). An inverse-based controller is generally of the form $\mathbf{C} = k(s)\mathbf{G}(s)^{-1}$. Examples of such controllers include steady-state and dynamic decouplers and IMC controllers. $\gamma_d(\mathbf{G})$ is useful in this respect, for example when used in conjunction with the Relative Gain Array (RGA). This is discussed by Skogestad and Morari (1986c) who present the following guidelines for the preferred controller structure:

RGA elements	$\max_d \gamma_d(\mathbf{G})$: large	$\max_d \gamma_d(\mathbf{G})$: small
large	(diagonal)	diagonal
small	inverse-based	inverse-based (diagonal)

Acknowledgment

Partial support from the National Science Foundation and Norsk Hydro is gratefully acknowledged.

Nomenclature

C = controller
$\boldsymbol{d}$ = effect of disturbances $\boldsymbol{z}$ on outputs, $=\mathbf{G}_d\boldsymbol{z}$
G = plant transfer matrix
$\mathbf{G}_d$ = disturbance transfer matrix, $= \{\boldsymbol{g}_{d_1}, ..., \boldsymbol{g}_{d_n}\}$
H = decoupler
$\boldsymbol{m}$ = manipulated variable
S = sensitivity matrix, eq 28
U = matrix of left singular vectors
V = matrix of right singular vectors
$\boldsymbol{y}$ = plant output
$\boldsymbol{z}$ = disturbance vectors, $= (z_1, ..., z_n)^T$

Greek Symbols

β_l = Relative Disturbance Gain (RDG) for manipulated variable m_l, eq 24
$\gamma(\mathbf{A})$ = condition number of **A**, $= \sigma_{max}(\mathbf{A})/\sigma_{min}(\mathbf{A})$
$\gamma_d(\mathbf{A})$ = disturbance condition number of **A**, $= (\|\mathbf{A}^{-1}\boldsymbol{d}\|_2/\|\boldsymbol{d}\|_2)\sigma_{max}(\mathbf{A})$
$\lambda(\mathbf{A})$ = 1,1 element of Relative Gain Array of **A**
σ_{max} = maximum singular value
σ_{min} = minimum singular value
Σ = matrix of singular values, = diag $\{\sigma_i\}$
ω = frequency

Literature Cited

Grosdidier, P., personal communication, Caltech, 1985.
Morari, M. *Chem. Eng. Sci.* **1983**, *38*(11) 1881–1891.
Morari, M.; Doyle, J. D. "A Unifying Framework for Control System Design Under Uncertainty and its Implications for Chemical Process Control" In *Chemical Process Control*; Morari, M., McAvoy, T. J., Eds.; CACHE-Elsevier: Amsterdam, 1986, CPC 3.
Klema, V. C.; Laub, A. J. *IEEE Trans. Autom. Control* **1980**, *AC-25*(2), 164–176.
Shimizu, K.; Matsubara, M. *Chem. Eng. Commun.* **1985**, *37*, 67–91.
Skogestad, S.; Morari, M. "Effect of Model Uncertainty on Dynamic Resilience", *Chem. Eng. Sci.* **1987a**, in press.
Skogestad, S.; Morari, M. "A Systematic Approach to Distillation Column Control", Distillation and Absorption 87, Brighton, Sept 7–9, 1987, also in *Ind. Chem. Eng., Symp. Ser.* **1987b**, *104*, A71–A86.
Skogestad, S.; Morari, M. "Implications of Large RGA Elements on Control Performance", *Ind. Eng. Chem. Res.* **1987c**, in press.
Stanley, G.; Marino-Galarraga, M.; McAvoy, T. J. *Ind. Eng. Chem. Process Des. Dev.* **1985**, *24*(4), 1181–1188.

Received for review June 13, 1986
Revised manuscript received May 2, 1987
Accepted May 24, 1987