EFFECT OF MODEL UNCERTAINTY ON DYNAMIC RESILIENCE

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Abstract

It has been shown previously (Morari, 1983) that the quality of control achievable for a certain system (its dynamic resilience) is limited by the nonminimum phase characteristics of the plant, constraints on the manipulated variables and model uncertainty. Model uncertainty requires the controller to be detuned and performance to be sacrificed. The goal of this paper is to quantify this well known qualitative statement. A general discussion of the model uncertainty problem is followed by the derivation of two sensitivity measures, the condition number and the structured singular value. Both require minimal assumptions about the controller, and are relatively easy to evaluate. Therefore they should be effective tools for screening alternate designs in terms of their resilience characteristics.

Introduction

Increased process integration and tight operating conditions are putting greater demands on plant control system performance than any time in the past. If a plant is designed only on the basis of steady-state considerations, then it is not uncommon that unfavorable process dynamics make it impossible to achieve the expected steady-state performance. For years it has been demanded by representatives of industry and academia that control aspects should be paid attention to during the design phase rather than afterwards. It has been postulated that well designed plants would be easy to control and that advanced control techniques would be uncalled for. What has hampered progress until now is that - short of simulation - no techniques for the assessment of "controllability" of a particular design were available. Simulation is complex and has a number of shortcomings: A complete dynamic model has to be available, specific controller structures, types and parameters have to be assumed, and specific inputs have to be chosen which might or might not correspond to those actually occurring in practice. All these choices might bias the selection of the "most controllable" design in an erroneous manner.

Dynamic Resilience of the Nominal Plant

Morari (1983) suggested to remove the problem of "controllability" assessment from the problem of controller selection. The best closed loop control performance achievable for a system for all possible constant parameter controllers was defined as its dynamic resilience. Thus "dynamic resilience" is an expression of the system inherent limitations to control quality which is not biased by possible controller inadequacies. The evaluation of dynamic resilience was simplified through the use of the Internal Model Control (IMC) structure depicted in Fig. 1B. It can be shown to be equivalent to the classic feedback structure in Fig. 1A through the transformation

$$Q = (I+C\tilde{P})^{-1}C \qquad (1a)$$

$$C = (I-Q\tilde{P})^{-1}Q \qquad (1b)$$

We assume throughout the paper that the nxn plant P with u as its input and y as its output is open loop <u>stable.</u> P is generally different from the model or nominal plant $\tilde{P}$. Q is the IMC controller and C the classic feedback controller. For the IMC structure

$$y = PQ(I+(P-\tilde{P})Q)^{-1}(r-d) + d \qquad (2)$$

and in the case of no model-plant mismatch ($P=\tilde{P}$) we have

$$\begin{aligned} y &= \tilde{P}Q(r-d) + d \\ &= \tilde{H}(r-d) + d \end{aligned} \qquad (3)$$

where we have defined $Q = \tilde{P}^{-1}\tilde{H}$.

Ideally the nominal closed-loop transfer matrix $\tilde{H} = I$ and perfect control would be achieved. However, generally, there are properties inherent in the plant itself which prevent us from choosing $\tilde{H} = I$. From (3) it is clear that for stability Q should be stable. Thus $\tilde{H}$ has to contain all the RHP zeros of $\tilde{P}$ to cancel the RHP poles of $\tilde{P}^{-1}$. Also Q has to be causal which implies that $\tilde{H}$ generally has to include delay terms when they are present in $\tilde{P}$. Finally, Q has to be proper for the control action to be bounded. If $\tilde{P}$ is proper this requires $\tilde{H}$ to roll off sufficiently fast at some frequency, which depends on the tightness of the constraints. These three limitations (RHP zeros, time delays and constraints on the control action) on the quality of control which can be achieved have been discussed in quantitative detail by Morari (1983) and Holt and Morari (1984) and will not be considered further in this paper.

<u>The Effect of Uncertainty</u>

If the actual plant is not equal to the model ($P \neq \tilde{P}$), (2) rather than (3) describes the closed loop relationship and the stability of Q is not sufficient for closed loop stability. For <u>robust stability</u> it is both necessary and sufficient that $\tilde{H}$ is chosen such that

$$(\det(I+(P-\tilde{P})\tilde{P}^{-1}\tilde{H}) \qquad (4)$$

does not encircle the origin as s traverses the Nyquist D contour for any plant P within the family Π of possible plants. Thus, the choice of $\tilde{H}$ is further restricted by the requirement of robust stability. The objective of this article is to present a framework within which the limitations on the nominal response $\tilde{H}$ imposed by model uncertainty can be evaluated.

Let us first discuss the steady state (ω=0) robust stability condition. If there are only performance requirements at $\omega = 0$ then the robust stability requirement at all other frequencies can be satisfied easily by rolling off $\tilde{H}$ sufficiently fast. (For $\tilde{H}$ small enough (4) is always nonzero regardless of ($P-\tilde{P}$)). (4) will always encircle the origin if it starts (ω=0) on the negative real axis. Thus, a robust controller with steady-state offset α for the nominal plant, i.e.

$$\tilde{H}(0) = (1-\alpha)I \qquad (5)$$

exists if and only if

$$\det(\frac{\alpha}{1-\alpha} I + P(0)\tilde{P}(0)^{-1}) > 0 \quad \forall P \varepsilon \Pi \qquad (6)$$

It follows that a robust controller with integral action (α=0) exists for the family Π if and only if

$$\det(P(0)\tilde{P}(0)^{-1}) > 0 \quad \forall P \varepsilon \Pi \qquad (7)$$

Plants for which the determinant of the steady-state gain matrix changes sign, cannot be controlled with controllers containing integral action. (7) also tells us that the only information we need about a plant in order to have perfect steady state control is the sign of the "gain" of the plant expressed as the sign of detP(0).

Nothing has been assumed about the description of the set Π of possible plants. However, in order to get quantitative bounds on $\tilde{H}$ for nonzero frequencies, such descriptions are needed. Before introducing them we will briefly discuss what kinds of uncertainties are typically found in process industries.

Types of Uncertainties in the Process Industry

1. All real processes are nonlinear. The "poor mans method" for dealing with nonlinear systems is to use linear models with "uncertain" coefficients arising from linearization at different points of the operating region. This introduces "uncertainty" over the whole frequency range.

2. Some processes are represented quite accurately by linear models. However, different operating conditions can lead to changes of the linear model parameters. For example throughput/flowrate changes can affect deadtimes and time constants.

3. Finally, even though the underlying process might be essentially linear, the model parameters and even the model order are rarely known precisely. This type of uncertainty is relatively small at low frequencies and tends to increase with frequency.

Uncertainty Descriptions

Associated with each element of a transfer matrix are generally uncertainties of all three types discussed above. The overall result is that the frequency response at each frequency lies in some region in the complex plane. The shapes and sizes of these regions can vary greatly. Most commonly these regions are approximated by discs in the complex plane. These discs have frequency dependent radius $\ell_A(\omega)$ and are centered at the frequency response of the "nominal" plant. Depending on the shape of the "true" region, the discs may include many extra plants, and may thus impose conservativeness on any results derived from such a description.

One attempt to generalize the uncertainty structures is reported by

Chen and Brosilow (1984). They allow regions of completely general shape and not just circles and then perform a "region arithmetic" to test for robust stability and performance. The procedure is completely numerical, also conservative and depends on the choice of a particular controller.

In the following we will study two types of uncertainty descriptions for transfer matrices. The first type is matrix norm bounds, which require no knowledge about bounds on the uncertainty of the individual elements. This is called an "unstructured" uncertainty description by Doyle (1982). The other type of uncertainty description assumes uncertainty norm bounds for the individual elements to be available which makes it possible for the uncertainty to be "localized" in particular elements. The Structured Singular Value (Doyle, 1982) is a valuable tool for analyzing these "structured" uncertainties.

Bounds on $\tilde{H}$ Imposed by Uncertainty

A. **Uncertainty described by matrix norm bounds**

There are three commonly used ways of describing the uncertainty of transfer matrices (Doyle & Stein, 1981); multiplicative output and input uncertainties

$$P = (I+L_O)\tilde{P} \text{ or } L_O = (P-\tilde{P})\tilde{P}^{-1} \tag{8a}$$

$$P = \tilde{P}(I+L_I) \text{ or } L_I = \tilde{P}^{-1}(P-\tilde{P}) \tag{8b}$$

and additive uncertainty

$$P = \tilde{P} + L_A \text{ or } L_A = (P-\tilde{P}) \tag{8c}$$

Three different families Π of plants are obtained by defining norm bounds on these uncertainties

$$\Pi_O = \{P: ||L_O|| < \ell_O(\omega)\} \tag{9a}$$

$$\Pi_I = \{P: ||L_I|| < \ell_I(\omega)\} \tag{9b}$$

$$\Pi_A = \{P: ||L_A|| < \ell_A(\omega)\} \tag{9c}$$

Here $||\cdot||$ denotes the induced 2-norm of a matrix. ℓ_O and ℓ_I are generally small at low frequencies and approach a constant value larger or equal to one for high frequencies when little or nothing is known about the plant. ℓ_A usually has the "opposite" shape. The three families (9) give regions of different "shape" around the nominal plant. This means that the uncertainty descriptions are not equivalent, except for the SISO case. We may start from one uncertainty description and get bounds for another, but any such change of family will introduce conservativeness by adding plants not present in the original family.

Because of the norm type of uncertainties defined above, (4) will encircle the origin for some plant in Π if and only if there is a plant in Π, which makes (4) identically zero for some ω. That is, the closed loop system (2) is robustly stable $\forall P\varepsilon\ \Pi$ iff

$$\det(I+(P-\tilde{P})\tilde{P}^{-1}\tilde{H}) \neq 0 \quad \forall\omega,\ \forall P\varepsilon\ \Pi \tag{10}$$

where Π may be any of the families defined in (9).

At low frequencies within the bandwidth of the closed loop system $\tilde{H} \approx I$ and the robust stability requirement derived from (10) becomes approximately

$$\det(P\tilde{P}^{-1}) \neq 0, \ \forall\omega \text{ for which } \tilde{H} \approx I, \ \forall P \epsilon \Pi \qquad (11)$$

Because $\tilde{P}$ is assumed to be strictly stable $\det \tilde{P}^{-1} \neq 0$, $\forall\omega$ and (11) reduces to

$$\det P \neq 0, \ \forall\omega \text{ for which } \tilde{H} \approx I, \ \forall P \epsilon \Pi \qquad (12)$$

P will be singular ($\det P = 0$) for a particular ω if P has a zero on the imaginary axis or the family of plants has zeros which can move across the imaginary axis. The important conclusion emerging from this discussion is that for robust stability all potential RHP plant zeros have to be well outside the bandwidth of the closed loop system. This is also supported by our earlier remark that perfect control is impossible in the present of RHP zeros.

It should be emphasized that (10) is a necessary condition for robust stability only if convex sets of plants as defined by (9) are considered. For other sets of plants (4) has to be checked directly for encirclements. For example, consider the SISO case with dead-time uncertainty

$$\tilde{H} = 1, \ \tilde{P} \neq 0 \ \forall \ \omega, \quad \Pi_1 = \{P: P = \tilde{P} \ e^{-s\theta}, \ 0 \leq \theta \leq \bar{\theta}\}$$

(11) is met but (4) encircles the origin and therefore the system is unstable. The set Π_1 is not convex and therefore (10) does not apply. $P\tilde{P}^{-1} = e^{-\theta s}$ crosses the negative real axis for $\omega = \pi/\theta$ and (4) then indicates that for stability it is necessary that $|\tilde{H}| < 1$ for $\omega_1 > \pi/\bar{\theta}$.

On the other hand the multiplicative uncertainty for the set of plants Π_1 is $L = e^{-\theta s} - 1$ and one can define a convex set Π_2 of possible plants by norm-bounding L

$$\Pi_2 = \{P: L|(\omega)| < \ell_2\}$$

where

$$\ell_2(\omega) = \begin{cases} (|e^{-i\omega\theta} - 1|) & \omega \leq \pi/\bar{\theta} \\ 2 & \omega \geq \pi/\bar{\theta} \end{cases}$$

This set includes the nominal plant $\tilde{P}$, the "real" plants $P = \tilde{P}e^{-s\theta}$ and clearly many other plants - among them plants which have a zero on the imaginary axis at $s \geq \pi/3\bar{\theta}$ i. This is seen from the Nyquist plot of P at each frequency which will be represented by discs with radius $\ell_2(\omega)\tilde{P}$ centered at the nominal plant $\tilde{P}$. These discs will include the origin (i.e. plants with zeros on the imaginary axis) for $\ell_2(\omega) > 1$, i.e. $\omega > \pi/3\bar{\theta}$. According to (12) it is therefore necessary for stability that $|\tilde{H}| < 1$ for $\omega_2 > \pi/3\bar{\theta}$. Comparing the exact condition ω_1 with ω_2 we note the conservativeness introduced by the norm bounds.

For the case of multiplicative output uncertainty Doyle and Stein

(1981) obtained the following necessary and sufficient condition for (10) (i.e. for robust stability)

$$||\tilde{H}|| < \frac{1}{\ell_o(\omega)} \quad \forall \omega \tag{13}$$

(13) shows that perfect control ($\tilde{H}$=I) of the nominal plant will not be possible because of robustness considerations for frequencies where the uncertainty $\ell_o(\omega) > 1$. Note that $\ell_o(\omega) < 1$ implies that the set Π_o will not include plants with zeros on the imaginary axis.

For multiplicative uncertainty at the plant inputs a necessary and sufficient condition for (10) is

$$||\tilde{P}\ \tilde{H}\ \tilde{P}^{-1}|| < \frac{1}{\ell_I(\omega)} \quad \forall\ \omega \tag{14}$$

A sufficient condition for (14) is

$$||H|| < \frac{1}{\kappa(\tilde{P})}\ \frac{1}{\ell_I} \tag{15}$$

where the condition number κ is defined as

$$\kappa(\tilde{P}) = ||\tilde{P}||\ \ ||\tilde{P}^{-1}|| \tag{16}$$

κ is always greater than unity. (15) indicates that well conditioned plants (small κ) are preferable because they show less model error sensitivity and put less severe constraints on the closed loop transfer matrix $\tilde{H}$. Therefore κ was suggested by Morari (1983) as a property to be considered for the dynamic resilience assessment.

There are some flaws in this idea. First of all, in the absence of detailed uncertainty information ℓ_O is just as good (or as bad) an uncertainty measure as ℓ_I. Therefore, there is no good theoretical reason of why the sufficient condition (15) which suggests the condition number as a sensitivity measure should be paid more attention to than the necessary and sufficient condition (13). Practical experience revealed a number of other deficiencies of (15): 1) As already mentioned error bounds are more naturally defined for individual matrix elements than for the whole matrix in terms of ℓ_I or ℓ_O. It is therefore exceedingly difficult to construct nonconservative estimates of ℓ_I or ℓ_O. 2) κ, ℓ_I and ℓ_O are strongly dependent on the scaling of the system inputs and outputs. 3) A comparison of two plants on the basis of the condition number based on (15) is only meaningful if the conservativeness' introduced by (15) is similar. 4) The associated errors ℓ_I should be similar. In view of the described problems in obtaining ℓ_I it appears difficult to guarantee that this is the case.

B. Robustness, scaling and the condition number

In this section we will try to deal with the main problems of (13) and (15), namely the difficulties in determining ℓ_I and ℓ_O in practical situations and the associated conservativeness, and the scaling dependence of the condition number. We will assume that the uncertainties are additive and described by (8c) and that $\tilde{H} = \tilde{h} \cdot I$, i.e. that the nominal closed loop transfer matrix $\tilde{H}$ is diagonal with identical responses. Starting from the robust stability criterion (10)

$$\det(I+(P-\tilde{P})\tilde{P}^{-1}\tilde{H}) \neq 0, \quad \forall P\epsilon\, \Pi_A, \quad \forall\omega \tag{10}$$

we find the necessary and sufficient conditions for robust stability

$$||(P-\tilde{P})\tilde{P}^{-1}\tilde{H}|| < 1 \quad \forall P\epsilon\, \Pi_A, \quad \forall\omega \tag{17}$$

$$||\tilde{H}|| = |\tilde{h}| < \left[\frac{||P-\tilde{P}||}{||\tilde{P}||}\right]^{-1} \frac{1}{\kappa(\tilde{P})} \quad \forall P\epsilon\, \Pi_A, \quad \forall\omega \tag{18}$$

The first term appears to be a more meaningful definition of a "relative matrix error" than $||L_O||$ or $||L_I||$. From (18) we also learn that perfect control of the nominal plant ($\tilde{H}$=I) may be combined with robust stability if and only if

$$\frac{||P-\tilde{P}||}{||\tilde{P}||} < \frac{1}{\kappa(\tilde{P})} \quad \forall P\epsilon\, \Pi_A, \quad \forall\omega \text{ for which } \tilde{H} \approx I \tag{19}$$

(19) is necessary and sufficient for P to remain nonsingular for $P\epsilon\Pi_A$ and thus it is equivalent to (12).

Uncertainty bounds are most naturally described in terms of the individual elements, but to apply (18) and (19) it is important to be able to construct non-conservative matrix norm-bounded sets Π_A. The least conservative bounds on Π_A may be obtained whenever the absolute errors (uncertainties) Δp_{ij} in all the elements of $\tilde{P}$ are equal. In this case we may choose $\ell_A(\omega) = n\ \Delta p_{ij}$ which will include some more plants than the original set and we derive from (18) a sufficient condition for robust stability

$$|\tilde{h}| < \frac{1}{||\tilde{P}^{-1}||\ \Delta P_{ij}\ n} \quad \forall\ \omega \tag{20}$$

Consequently, in the case of equal absolute errors in the elements, the unscaled minimum singular value $\underline{\sigma}(\tilde{P}) = 1\ /\ ||\tilde{P}^{-1}||$ will be a good measure of the systems sensitivity to uncertainty.

Unfortunately, the absolute error is strongly scaling dependent, and often with elements of different magnitude it is more reasonable to assume the relative errors of the elements, which are unchanged by scaling with diagonal matrices, to be similar. It therefore makes sense to scale the system in order to make all the elements similar in magnitude, because this will correspond to similar absolute errors in the elements of the scaled matrix. Let the scaled system be

$$P_S = S_1 P S_2 \tag{21}$$

Here S_1 and S_2 are real diagonal matrices. For the rescaled system robust stability is insured iff

$$\det[I+(P_S-\tilde{P}_S)\tilde{P}_S^{-1}\ S_1\tilde{H}S_1^{-1}] \neq 0, \quad \forall P\epsilon\, \Pi_A', \quad \forall\omega \tag{22}$$

or because $\tilde{H}$ and S_1 are diagonal, iff

$$\det[I+(P_S-\tilde{P}_S)\tilde{P}_S^{-1}\tilde{H}] \neq 0, \quad \forall P_S \in \Pi_A', \quad \forall\omega \tag{23}$$

Note that we have introduced a different family Π_A' described by an additive norm bound of the type (9c) which we expect to describe the actual set of possible plants less conservatively because of the scaling. From (23) we can find an expression equivalent to (18)

$$||\tilde{H}|| = |\tilde{h}| < \left[\frac{||P_S-\tilde{P}_S||}{||\tilde{P}_S||}\right]^{-1} \frac{1}{\kappa(\tilde{P}_S)} \quad \forall P_S \in \Pi_A', \quad \forall\omega \tag{24}$$

All the heuristic procedures to minimize the condition number of a matrix tend to make the magnitudes of the matrix elements similar. This suggests that the sensitivity of different designs to model/plant mismatch can be compared on the basis of their minimized condition numbers if the following assumptions are satisfied:

1. The desired closed loop transfer function $\tilde{H} = \tilde{h}I$

2. The scaling which minimizes the condition number leads to a "tight" i.e. not too conservative family Π_A'.

3. The "relative errors" $||P_S-\tilde{P}_S|| \,/\, ||\tilde{P}_S||$ are similar for the different designs.

Assumption 2 will probably be satisfied for a transfer matrix that has similar relative errors in the elements, and assumption 3 will be probably be satisfied if the two designs have the same relative error of the elements.

Grosdidier et al. (1984) established a ~~lower~~ upper bound on the minimized conditions number at $\omega = 0$ in terms of the $p = 1$ and ∞ norms of the Relative Gain Array thus avoiding a messy optimization problem.

The idea of comparing minimized κ's was applied successfully by Morari et al. (1984) to a simulation example but did not provide much insight in the experimental work by Levien (1984) unless very large κ's were found and the sensitivity problems were almost obvious. While a κ comparison is very simple (as a first step $\kappa(0)$ evaluated on the basis of the steady-state model might provide some insight), the information obtained from this test is limited and only meaningful when the assumptions stated above are satisfied. However, if at an early design stage nothing specific is known about the model errors and their structures, a comparison of the κ's is probably the only feasible method for screening alternatives. If, on the other hand, error bounds on the individual transfer matrix elements are available, the following method is largely superior.

C. Uncertainty described by norm bounds on individual elements

Matrix norm uncertainty bounds may be very conservative because the uncertainty which might be localized at a single transfer matrix element gets distributed evenly over all elements. The concept of Structured Singular Values (SSV) introduced by Doyle (1982) suggests an approach for overcoming the deficiencies of the matrix norm uncertainty bounds. Let us assume that norm bounds $\ell_{ij}(\omega)$ on the additive errors of the individual transfer matrix elements $\tilde{p}_{ij}$ are available and define the corresponding family of plants Π_E.

$$\Pi_E = \{P: |p_{ij}-\tilde{p}_{ij}| < \ell_{ij}(\omega)\} \tag{25}$$

This kind of uncertainty description is more detailed than the matrix norm bounds discussed previously. Howeever, still some restrictive assumptions about the uncertainty descriptions are needed: 1) The individual elements are norm-bounded, 2) The uncertainty of the elements are independent (while in practice one element might be large only when another is small).

For simplicity in notation we will discuss only 2x2 systems here. The extension to larger systems is conceptually straightforward. For stability considerations the block diagram in Fig. 1B can be cast into the form shown in Fig. 2 where

$$P-\tilde{P} = E\Delta L = \begin{bmatrix} (\Delta_{11}\ell_{11}) & (\Delta_{12}\ell_{12}) \\ (\Delta_{21}\ell_{21}) & (\Delta_{22}\ell_{22}) \end{bmatrix} \tag{26}$$

and the magnitude of the elements Δ_{ij} in the diagonal perturbation matrix Δ is bounded by one ($|\Delta_{ij}| \leq 1$). L contains the information on the maximum additive error of each element (i.e. the radii of the uncertainty discs). We define the set of structured perturbations $X(\delta)$

$$X(\delta) = \{\Delta = \text{diag}\ \{\Delta_i\}: |\Delta_i| < \delta\} \tag{27}$$

The SSV μ of a matrix F is defined as

$$\mu^{-1}(F) = \min\{\delta | \det(I+F\Delta) = 0 \text{ for some } \Delta\varepsilon X(\delta)\} \tag{28}$$

i.e. μ is the inverse of the smallest δ which makes the system singular at each frequency. Because of the convexity of the uncertainty region, some plant $P\varepsilon\Pi_E$ will make (4) encircle the origin if and only if $\det(I+F\Delta)$ is zero for some $P\varepsilon\ \Pi_E$. Thus the system in Fig. 2 with

$$F = L\tilde{P}^{-1}\tilde{H}E \tag{29}$$

is robustly stable $\forall P\varepsilon\ \Pi_E$ iff $\delta \geq 1$, i.e.

$$\mu(F) < 1 \quad \forall\omega \tag{30}$$

In theory (29) and (30) may be used to generate all allowable $\tilde{H}$. Conversely, for a particular chosen closed loop transfer matrix $\tilde{H}$, μ can be interpreted as the sensitivity of the closed loop system to structured perturbations $X(\delta)$. A large μ indicates high sensitivity and implies that the set of structured perturbations for which the closed loop system remains stable is small (δ is small). At the design stage when the relative dynamic resilience is to be determined, $\tilde{H}$ is not known yet but rather the restrictions on $\tilde{H}$ imposed by model error sensitivity are of interest. In order to determine these restrictions from (29) and (30) let us again assume $\tilde{H} = \tilde{h}I$. Then it is easy to show that the system is robustly stable if and only if

$$||\tilde{H}|| = |\tilde{h}| \leq \frac{1}{\mu(L\tilde{P}^{-1}E)} \quad \forall\omega \tag{31}$$

(31) should be compared with (18). Both conditions bound the norm of the closed loop transfer matrix. They indicate how much the controllers have to be detuned because of model uncertainty.

On the basis of (24) which was derived from (18) we suggested to assume the relative model error for different systems to be the same and to compare the minimized condition numbers as a measure of how sensitive they are to these errors. In (31) μ contains information on both the sensitivity and the error. Thus sensitive systems with small model error and insensitive systems with large model error can be compared.

The SSV $\mu(F)$ in (29) and (30) clearly depends on the structure of $\tilde{H}$ and tradeoffs are possible. Even for diagonal $\tilde{H}$ there may be a trade-off between the speeds of the different loops. Unfortunately (31) does not reveal any of these potential trade-offs. If one output is more important than the other it is wise to put weights into E accordingly. For example, if E in our 2x2 example is chosen as

$$E_W = \begin{bmatrix} (\tau_1^2\omega^2+1)^{-1} & 0 & (\tau_1^2\omega^2+1)^{-1} & 0 \\ 0 & (\tau_2^2\omega^2+1)^{-1} & 0 & (\tau_2^2\omega^2+1)^{-1} \end{bmatrix} \tag{32}$$

then with $\tau_1 < \tau_2$ output y_1 can be emphasized over y_2. These weights will decrease μ in the frequency range $\omega > 1/\tau_2$. The performance tradeoff is advantageous if the observed decrease in μ is larger than when a single $\tau = \sqrt{\tau_1\tau_2}$ is used in (33). Note that the resulting $|\tilde{H}|$ has to include the weights in (32) in addition to the constraints imposed by $\mu(L\tilde{P}^{-1}E_W) > 1$ through (31). Allowing interactions in $\tilde{H}$ could also be effective in reducing μ. It does not seem possible to derive an expression like (31) when $\tilde{H}$ has off-diagonal elements.

It is evident that μ offers advantages over κ but at this point we have not discussed how it can be computed. (28) is just a definition which does not suggest a particular computational technique. Doyle (1982) has shown that

$$\max_U \rho(FU) = \mu(F) \leq \min_D \bar{\sigma}(DFD^{-1}) \tag{33}$$

where ρ is the spectral radius, $\bar{\sigma}$ the maximum singular value, U is in general a block diagonal matrix with each block unitary and D is a diagonal matrix with positive elements. In our case where Δ is diagonal, U is diagonal with complex elements on the unit circle. The maximization of ρ is nonconvex. The minimization of $\bar{\sigma}$ is convex but equality is achieved only for $\dim(F) \leq 3$; though experience has shown the bound to be quite tight.

An alternate method is to use circular arithmetic (Henrici) which is a generalization of interval analysis (Moore, 1966) to the complex plane. Circular arithmetic defines the usual arithmetic operations (addition, multiplication and division) for circular regions in the complex plane. With each of the Δ_i a disc, $\det(I+F\Delta)$ defines a region in the complex plane. A circle containing this region can be found by evaluating det(I+FΔ) according to the rules of circular arithmetic. Decreasing δ decreases the size of the resulting circle. The largest δ for which the resulting circle does not include the origin is a lower bound on μ^{-1}. This technique is very simple; we are currently testing its convervativeness.

Robust Performance

Assume that we are comparing the dynamic resilience of two systems by evaluating their respective SSV's μ_1 and μ_2 as defined by Eq. (31). Our preceding discussion suggests that if $\mu_1 < \mu_2$, $\forall\omega$ system 1 shows less model error sensitivity and should lead to superior performance. Though this is likely to happen, especially when μ_1 and μ_2 are significantly different, the conclusion is not entirely correct. If we select $\tilde{H} = \tilde{h}I$ satisfying

$$|\tilde{h}| = \mu^{-1} \tag{34}$$

then the nominal response of system 1 will indeed be superior. Recall however that (31) is necessary and sufficient for robust stability in case the family of plant is truly given by norm bounds on the additive uncertainties of the elements and $\tilde{H} = \tilde{h}I$. This means that there will be at least one plant P in the family Π_E of permitted plants for which the closed loop system will be on the limit to instability. To achieve reasonable performance for all plants in Π_E, $|\tilde{h}|$ might have to be selected significantly less than μ^{-1}. Thus, a measure of the best possible robust performance achievable for the family of plants Π_E would be a more useful measure of dynamic resilience than μ. Though this is true in principle, this idea is not very practical at the design stage where often many different systems have to be evaluated. The first difficulty is that for a performance assessment to be meaningful frequency dependent weights have to be assigned to each input/output pair because other frequencies in addition to "crossover" are important for performance. The second problem is that, unlike the computation of μ, determining the optimal robust performance requires an optimally robust controller to be designed, at present even for 2x2 systems an unresolved problem.

Conclusions

Model uncertainty requires feedback controllers to be detuned and performance to be sacrificed. The amount of detuning necessary and the resulting performance deterioration depend on the sensitivity of the plant. Different measures for this sensitivity were proposed. The condition number of the plant can be evaluated without specifying the expected model uncertainty. If it is minimized over all scaling matrices and if it is reasonable to assume that the additive model uncertainty of the scaled system can be norm bounded without too much conservativeness it may be a good measure. Furthermore, to allow a comparison of different systems on the basis of the condition number, their "relative errors" should be equal. The structured singular value μ is a much better sensitivity measure but requires for its computation the expected uncertainty to be specified. Furthermore, the tools for computing μ are quite limited at present.

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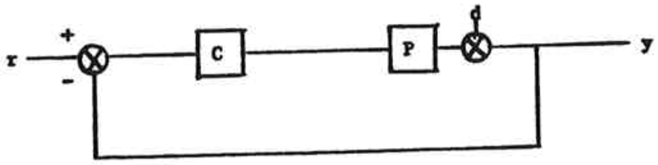

Figure 1A

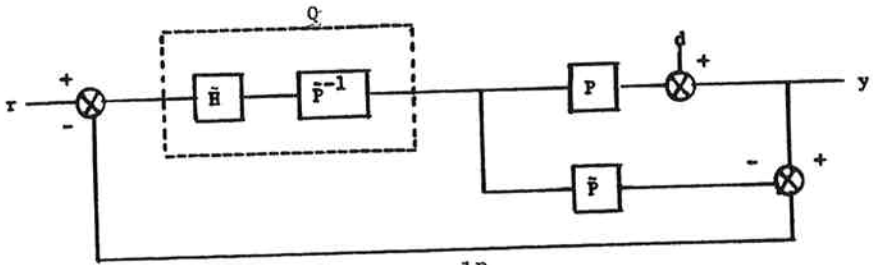

Figure 1B

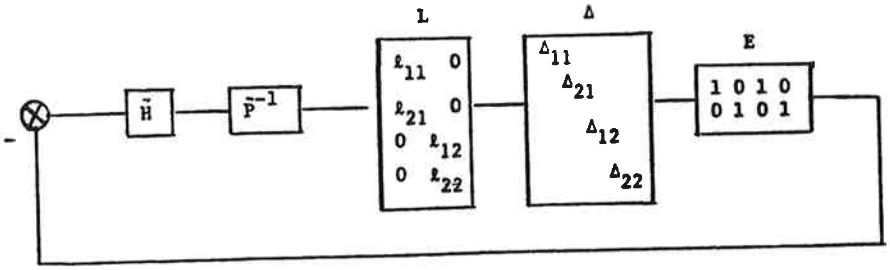

Figure 2