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MODEL UNCERTAINTY, PROCESS DESIGN, AND PROCESS CONTROL

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Abstract

It has been shown previously (Morari, 1983) that the quality of control achievable for a certain system (its dynamic resilience) is limited by the nonminimum phase characteristics of the plant, constraints on the manipulated variables and model uncertainty. Model uncertainty requires the controller to be detuned and performance to be sacrificed. The goal of this paper is to quantify this well known qualitative statement. A general discussion of the model uncertainty problem is followed by the derivation of simple bounds on the nominal performance imposed by the robust stability condition for some uncertainty descriptions. These bounds are relatively easy to evaluate and should be effective tools for screening alternate designs in terms of their resilience characteristics. It is shown that the use of the minimized condition number as a sensitivity measure implicity assumes that the relative errors of the transfer matrix elements are independent and have similar magnitude bounds.

I. INTRODUCTION

Most chemical plants are designed on the basis of steady state considerations, and the control system is designed separately in a later stage of the project. This separation is acceptable provided there exists methods which can be used at the design stage to assess the "controllability" of the plant, i.e. to indicate if it will be possible later on to design a control system which yields reasonable closed loop performance. Until recently such methods did not exist, and as a result the expected performance is often not achieved in the operating plant. In some cases a simple change at the initial design stage could have resulted in a "controllable" plant.

Previously, the "controllability" assessment has been based on simulations. This approach is complex and requires a complete dynamic model of the plant. Usually a number of case studies are performed with different choice of inputs, disturbances, operating conditions, controller structures and controller parameters. All those choices might bias the "controllability" assessment in an erroneous manner.

Morari (1983a) suggested to make the problem of "controllability" assessment independent of the problem of controller selection. This is done by finding the best closed loop control performance achievable for a plant for all possible constant parameter linear controllers. This target or bound on the achievable closed loop performance is defined as the plants dynamic resilience. Thus "dynamic resilience" is an expression of the system inherent limitation on the dynamic response of the closed loop system which is not biased by specific choices for controllers.

The limitations imposed by non-minimum phase elements (RHP zeros, time delays) and constraints have been discussed in quantitative detail by Morari (1983a) and Holt and Morari (1985). To achieve perfect control the plant has to be invertible. Non-minimum phase elements make it impossible to invert the plant and retain (internal) stability of the closed-loop system. The effect of constraints on performance is related to how close the plant is to being singular. If the minimum singular value of the plant, $\underline{\sigma}(\tilde{P})$, is small, the system is nearly singular. This means that the plant has a very small gain for a particular input direction. To achieve perfect control the controller will have to provide very high input signals in this direction, thus possibly violating the constraints on the size of the inputs.

The objective of this paper is to study the effect of model uncertainty on the dynamic resilience. Model uncertainty requires the controller to be detuned and performance to be sacrificed. The goal is to quantify this well known qualitative statement and to derive expressions relating achievable closed loop performance and uncertainty. It is important that these performance bounds are simple in order to give the engineer insight into what may be causing the problem and how to alter the design to get a more "controllable" plant.

The paper is organized as follows: Section II gives an introduction to how uncertainty affects stability and performance, Section III introduces a general framework for handling uncertainties and states necessary and sufficient conditions for robust stability in terms of the structured singular value μ (Doyle, 1982a,b). Section IV discusses the use of the condition number as a sensitivity measure. An overview of notation is given in Appendix 1, and some properties of μ are given in Appendix 2.

II. Uncertainty, Stability and Performance

1. The Effect of Uncertainty

Before discussing how uncertainty limits the achievable performance (dynamic resilience), a digression on why feedback is used for control in the first place is of interest. Obviously, for stable plants if there were no uncertainties of any kind, feedforward control would be all that was needed. Feedback is used to be able to control the plant in spite of unmeasured disturbances and model uncertainty. One particular example is the use of integral action in order to get perfect steady state control. Without knowing the steady state gain exactly, perfect control may be achieved through feedback. Also, it is well known (Horowitz, 1963), that the sensitivity of the output of the system with respect to model uncertainty may be reduced (at least over some frequency range) by using feedback.

However, even though feedback may be used to reduce the effect of uncertainty, it is intuitively obvious that there is a limit on how much uncertainty we can tolerate before we have to detune the system and sacrifice performance. Thus uncertainty may impose limitations on the achievable performance (dynamic resilience). We want to find quantitatively how uncertainty affects closed-loop performance.

Let us first define some terms:

Model uncertainty: We assume the plant is linear and time invariant, but that its exact mathematical description is unknown. However, it is known to be in a specified "neighborhood" of the "nominal" system, whose mathematical "model" is available. This neighborhood defines the "set of possible plants".

Performance: "Performance" is the measure with which we rate the closed loop system (Fig. 1). High performance is desirable and low performance undesirable. "Achievable performance" is the "upper bound" on performance which is possible for any controller under some set of conditions. A "lower bound" on performance is some minimum performance requirements the system has to meet.

Nominal stability/performance: The closed loop stability/performance of the nominal system.

Robust stability: The closed loop system is stable for all plants in the "uncertainty set".

Robust performance: The closed loop system satisfies some performance specifications for all plants in the "uncertainty set".

In the context of how uncertainty affects performance, there are at least three problems of interest:

Problem 1: Effect of robust stability requirement on nominal performance: How does the requirement of stability for all plants in the uncertainty set limit the nominal performance?

Problem 2: Effect of robust performance specification on nominal performance: How does the requirement of a given lower bound on performance for all plants in the uncertainty set, bound the nominal performance?

Problem 3: Achievable robust performance: Design the best possible controller: What is the best performance which can be achieved by all plants in the uncertainty set.

In Problem 1 and 2 a lower bound on robust performance is specified (for Problem 1 this lower bound is simply the requirement of robust stability), and we are considering the effect of this on the nominal performance. The goal is to derive some simple bounds on the nominal system which when satisfied will give the desired robust performance. Such bounds are thus intended to assist the engineer in designing a controller for the nominal system while achieving the specified performance for all plants in the uncertainty set.

In Problem 3 we do not care in particular about the performance of the nominal system. The problem in this case is to find a upper bound on robust performance using any linear controller. This problem is addressed by Doyle

(1984). The solution involves actually finding the optimal controller by maximizing the performance, i.e. by minimizing a weighted "μ-norm" of the transfer matrix from disturbances to errors. This is a complicated mathematical and numerical problem which will not be addressed in this paper.

This paper will be concerned with Problem 1. The problem is important in the case the plant is operating most of the time close to its nominal point, but with occasional perturbations. In this case we may not care about the performance when perturbations occur as long as the system remains stable. Furthermore, for Problem 1 we will be able to derive reasonably simple bounds on the achievable nominal performance. Simplicity is desired in order for the engineer to gain insight into why a particular design is sensitive to uncertainty.

2. Causes of Model Uncertainty

To find quantitative bounds on the achievable performance imposed by uncertainty a description of the uncertainty is needed. This is in general not a trivial problem, but the usefulness of the bounds which are derived is obviously closely related to how well the modelled uncertainty captures to actual uncertainty. Before going into detail on how to describe the uncertainty mathematically, a qualitative discussion on the causes of the uncertainty is of interest.

All real processes are nonlinear. In this paper linear transfer functions are used to represent the plant, and "uncertainty" is introduced by linearizing the nonlinear plant at various operating points. This may lead to a linear model with "uncertain" coefficients.

In other cases the process may be represented quite accurately by linear models. However, different operating conditions can lead to changes of the parameters in the linear model. For example increased throughput/flowrates will usually result in smaller deadtimes and time constants.

Consequently, in many cases parts of the "uncertainty" are well known. However, there will always exist "true" uncertainties even though the underlying process is essentially linear: The model parameters are never known exactly and at high frequencies even the model order is unknown. This last form may often be approximated in some crude manner using "unstructured" uncertainty (Doyle and Stein, 1981).

Associated with each element of a transfer matrix are generally uncertainties of all the three kinds discussed above. The overall result is that for each element the frequency response (i.e. magnitude phase) at each frequency lies in some region in the complex plane. The shapes and sizes of these regions vary greatly, and the usefulness of the derived performance limitations obviously depends on how well the modelled uncertainty matches the actual region. Furthermore, the uncertainties in the elements may not be independent, i.e. there will be a correlation at each frequency between where each element is in its region. Such correlations will arise frequently, e.g. when a linear constant parameter model is used to represent different operating (equilibrium) points. As shown by examples later, the correlation of the uncertainties must be represented properly in order to avoid excessively conservative performance bounds.

III. Mathematical Framework for Handling Uncertainty and Robust Stability

This section will address Problem 1 stated in the previous section: How does the robust stability requirement limit the nominal performance? The main objective is to familiarize the reader with the work of Doyle (1982a, 1982b, 1984). His framework for describing uncertainty is very useful because it provides necessary and sufficient conditions for robust stability (and robust performance).

From the qualitative discussion of uncertainty in the previous section, we conclude that uncertainty models for each process will vary greatly, and that a general framework for describing uncertainty is needed.

The most general description is to define the set Π of all possible plants. The set may contain a finite number of plants, or it may be an infinite set. For a given controller, stability and performance may be checked for each plant $P \varepsilon \Pi$. However, this kind of uncertainty description is obviously too general to be useful in most cases.

Doyle's framework handles many uncertainty descriptions found in practice except cases with finite sets of plants. It is assumed that the set of possible plants may be written in terms of perturbations (uncertainty) on the nominal system. Each perturbation Δ_i is assumed to be norm-bounded

$$\bar{\sigma}(\Delta_i) < 1 \qquad \forall\ \omega \tag{1}$$

i.e. Δ_i is any stable rational transfer matrix which is bounded as in (1). For perturbations of size 1x1 the perturbation has to be confined to the unit disc

$$|\Delta_i| < 1 \qquad \forall\ \omega \tag{2}$$

This means that without any reformulations real parameter variations must be handled by complex parameter variations. The use of the singular value $\bar{\sigma}$ to bound Δ_i is required to obtain necessary and sufficient conditions in the theorem which follows.

The perturbations (uncertainties) which may occur at different places in the feedback system (e.g. Fig. 2) can be collected and put into one large block diagonal perturbation matrix

$$\Delta = \text{diag}\{\Delta_1, \ldots \Delta_n\} \tag{3}$$

for which we have

$$\bar{\sigma}(\Delta) < 1 \qquad \forall\ \omega \tag{4}$$

The blocks Δ_i in (3) can have any size and may also be repeated for example in order to handle correlations between the uncertainties in different elements. The interconnection matrix M contains the transfer functions from the output of the perturbations Δ to their inputs as shown in Figure 3. Constructing M is conceptually straightforward, but may be tedious for specific problems.

We want to derive conditions on M in order to guarantee robust stability. Assume the nominal system with no perturbations ($\Delta=0$) is stable, i.e. assume in particular M is stable. Then the closed loop system will be robustly stable if and only if the system in Fig. 3 is stable for all perturbations, i.e. if and only if $\det(I+\Delta M)$ does not encircle the origin as s traverses the Nyquist D contour for all possible Δ. Because the set of plants is norm bounded this is equivalent to

$$\det(I+\Delta M) \neq 0 \qquad \forall\omega, \quad \forall\Delta, \quad \bar{\sigma}(\Delta) < 1 \tag{5}$$

Equation (5) by itself is not very useful since it is only a yes/no condition which must be tested for all possible perturbations. What is desired, is a condition on the matrix M, preferably on some norm of M. This is supplied by the following theorem (Doyle, 1982b).

Theorem 1. Necessary and Sufficient Condition for Robust Stability. Assume the nominal system ($\Delta=0$) is stable. Then the closed loop system is stable for all Δ, $\bar{\sigma}(\Delta) < 1$ if and only if

$$\mu(M) \leq 1 \qquad \forall\omega \tag{6}$$

The function μ, called the Structured Singular Value (SSV), is defined in order to get the tightest possible bound on M such that (5) is satisfied. The SSV μ was introduced by Doyle (1982a), and a more precise definition of μ and some of its properties are given in Appendix 1. It is important to note that $\mu(M)$ will depend both on the matrix M and on the structure of the perturbations Δ. $\mu(M)$

is a generalization of the spectral radius $\rho(M)$ and maximum singular value $\bar{\sigma}(M)$ in that $\mu(M) = \rho(M)$ when the perturbation Δ is totally structured ($\Delta=\delta I, |\delta| \leq 1$) and $\mu(M) = \bar{\sigma}(M)$ when the perturbation is unstructured (Δ is a full matrix).

The matrix M is a function of the nominal system only and the condition $\mu(M) \leq 1$ limits the possible nominal transfer functions. However, if M is not a transfer function of particular interest to the engineer, then the bound $\mu(M) \leq 1$ does not provide much insight. Also, to find M a model of the perturbations is needed. In Section IV we will look at cases where the bound $\mu(M) \leq 1$ reduces to simple conditions in terms of the nominal plant $\tilde{P}$.

At this point it is not at all obvious that the uncertainty description (1)-(4) above, indeed gives a useful framework for handling uncertainty, also it is not clear how to find the matrix M. Hopefully this will become clearer through the following two examples.

Example 1. Input and Output Multiplicative Uncertainty

Consider the system in Fig. 2 which has input and output multiplicative uncertainty with respect to the model of the plant $\tilde{P}$. The perturbation block Δ_I represents the multiplicative input uncertainty, possibly due to uncertain actuator (valve) dynamics. $\delta_I(\omega)$ is a scalar weighting function which gives the size and frequency dependency of the input uncertainty. The block Δ_O represents the multiplicative output uncertainty, e.g. due to measurement uncertainty or uncertainty in the dead time involved in one or more of the measurements. From what was assumed above, Δ_I and Δ_O will probably be diagonal perturbation matrices, since there is little reason to assume that the actuators or measurements influence each other. However, some of the unmodelled dynamics in the plant $\tilde{P}$ itself, which has crossterms, may be approximated by putting them into Δ_I or Δ_O, thus making either one of them a "full" matrix.

Note that for MIMO systems using multiplicative uncertainty at the input (Δ_I) or the output (Δ_O) of the plant makes a difference, i.e. these two uncertainty descriptions give sets with different "shape" around the nominal plant. This means that we may start from one uncertainty description and get bounds for the other, but that any such change will introduce conservativeness by adding plants not present in the original set. For SISO systems multiplicative input and output uncertainty cannot be distinguished.

To examine the performance constraints imposed by this uncertainty description, let $\Delta = \text{diag}\{\Delta_I, \Delta_O\}$ and rearrange the system in Fig. 2 into the form in Fig. 3. We find the interconnection matrix M

$$M = \begin{bmatrix} -\delta_I(\omega)C\tilde{P}(I+C\tilde{P})^{-1} & -\delta_I(\omega)C(I+\tilde{P}C)^{-1} \\ \delta_O(\omega)\tilde{P}(I+C\tilde{P})^{-1} & -\delta_O(\omega)\tilde{P}C(I+\tilde{P}C)^{-1} \end{bmatrix} \tag{7}$$

and robust stability is guaranteed for $\bar{\sigma}(\Delta) < 1$ if and only if $\mu(M) \leq 1$ at all frequencies. Although this condition indirectly limits the allowed nominal closed loop transfer functions, it does not give a transparent bound which gives insight into how uncertainty limits performance. Such transparent bound may be found for example, when we assume that there is only one kind of uncertainty occurring, i.e. if $\Delta_I = 0$ or $\Delta_O = 0$.

<u>Case 1. Multiplicative Output Uncertainty</u>. In this case $\Delta_I = 0$ and using (7) yields the following condition

$$\text{Robust stability } \forall \Delta_O, \bar{\sigma}(\Delta_O) < 1$$

$$\text{iff} \quad \mu(\tilde{H}) \leq \frac{1}{\delta_O(\omega)} \quad \forall\omega \tag{8}$$

where

$$\tilde{H} = \tilde{P}C(I+\tilde{P}C)^{-1} \tag{9}$$

Here $\tilde{H}$ is the nominal transfer function from setpoints (r) to outputs (y) (see Fig. 1). Condition (8) gives a "clean" bound in terms of $\tilde{H}$, which is a transfer matrix directly related to performance, and this is one of the reasons output uncertainty is a "popular" choice for describing uncertainty. $\mu(\tilde{H})$ depends both on $\tilde{H}$ and the structure of Δ_0. If the perturbation Δ_0 is a diagonal matrix, $\mu(\tilde{H})$ must be used. In this case use of $\bar{\sigma}(\tilde{H})$ may discard acceptable designs as shown in Fig. 4. However, if Δ_0 is a "full" matrix $\mu(\tilde{H}) = \bar{\sigma}(H)$ and we get a non-conservative condition in terms of $\bar{\sigma}(\tilde{H})$:

$$\text{Robust Stability}, \ \forall \ \Delta_O, \ \Delta_O \text{ full matrix}, \ \bar{\sigma}(\Delta_O) < 1$$

$$\text{iff} \quad \bar{\sigma}(\tilde{H}) \leq \frac{1}{\delta_0(\omega)} \quad \forall\omega \tag{10}$$

This condition was first presented by Doyle and Stein (1981). Condition (10) indicates that the system has to be detuned and performance sacrificed whenever $\delta_O(\omega) > 1$. (8) and (10) are necessary conditions only if the actual uncertainty description fits the modelled description. In other cases they may be arbitrarily conservative.

Case 2. Multiplicative Input Uncertainty. In this case $\Delta_O = 0$, and assuming Δ_I is a full matrix, (7) yields the condition

$$\text{Robust Stability} \ \forall \ \Delta_I, \ \Delta_I \text{ full matrix}, \ \bar{\sigma}(\Delta_I) < 1$$

$$\text{iff} \quad \bar{\sigma}(C\tilde{P}(I+C\tilde{P})^{-1}) \leq \frac{1}{\delta_I(\omega)} \quad \forall\omega \tag{11}$$

Here $C\tilde{P}(I+C\tilde{P})^{-1}$ is the transfer matrix from a disturbance entering at the input of the process to the output of the controller. Usually this transfer matrix is not of primary interest for judging the overall performance. To get a more "useful" bound the following identity may be used

$$C\tilde{P}(I+C\tilde{P})^{-1} = \tilde{P}^{-1}\tilde{H}\tilde{P} \tag{12}$$

and the following condition which is only sufficient for robust stability can be derived from (11)

$$\text{Robust Stability} \ \forall\Delta_I, \ \Delta_I \text{ full matrix}, \ \bar{\sigma}(\Delta_I) < 1$$

$$\text{if} \quad \bar{\sigma}(\tilde{H}) \leq \frac{1}{\gamma(\tilde{P})} \frac{1}{\delta_I(\omega)} \quad \forall\omega \tag{13}$$

Here $\gamma(\tilde{P}) = \bar{\sigma}(\tilde{P})/\underline{\sigma}(\tilde{P})$ is the condition number of the plant. This bound may be arbitrarily conservative even if we make the restrictive assumption that the uncertainty is actually described in terms of a norm bounded input uncertainty. (13) has been used to introduce the condition number as a sensitivity measure with respect to uncertainty (Morari, 1983), but this is at best misleading. Note that the condition number drops out entirely and we get a much tighter bound if we assume $\tilde{H} = \tilde{h}I$ or if the system is SISO. We will discuss later in which situations the condition number may be used as a sensitivity measure.

Example 2: Independent Uncertainty in the Transfer Matrix Elements

In many cases the uncertainties are most easily described in terms of uncertainties of the individual transfer matrix elements. This kind of uncertainty description may arise from an experimental identification of the system. If the uncertainties vary greatly from one element to another, an uncertainty description of the multiplicative type (as in the last example) will tend to be very conservative as much additional uncertainty has to be included to arrive at the appropriate set.

The simplest form of element uncertainty is to assume that each element p_{ij} in the plant P is independent, but confined to a disc with radius $a_{ij}(\omega)$ around $\tilde{p}_{ij}$ in the Nyquist plane (Fig. 5), i.e.

$$|p_{ij}-\tilde{p}_{ij}| < a_{ij}(\omega) \qquad \forall\omega \tag{14}$$

This corresponds to treating each element as an independent SISO plant. For SISO systems additive and multiplicative (relative) uncertainty are equivalent. The two main limitations of the uncertainty description (14) are

1. The discs shape is potentially conservative, e.g. for pure time delay error.
2. Correlations between the elements cannot be handled (potentially <u>very</u> conservative).

By "conservative" we mean that the modelled set of uncertainties will be larger than the actual set.

Defining the complex perturbation Δ_{ij}:

$$\Delta_{ij} = \frac{p_{ij}-\tilde{p}_{ij}}{a_{ij}} \;,\; |\Delta_{ij}| < 1 \tag{15}$$

we may represent the uncertainty as a weighted additive uncertainty on the nominal plant

$$P - \tilde{P} = \begin{bmatrix} \Delta_{11}a_{11} & \Delta_{12}a_{12} & \cdots \\ \Delta_{12}a_{12} & & \\ \vdots & & \Delta_{nn}a_{nn} \end{bmatrix} \tag{16}$$

or equivalently (Fig. 6)

$$P - \tilde{P} = E\Delta L \tag{17}$$

where $\Delta \varepsilon\ C^{n^2 x n^2}$ is diagonal

$$\Delta = \mathrm{diag}\{\Delta_{11}\Delta_{12} \cdots \Delta_{nn}\} \;,\; |\Delta_{ij}| < 1 \tag{18a}$$

and $E \varepsilon\ R^{n x n^2}$ and $L \varepsilon\ R^{n^2 x n}$

$$E = \begin{bmatrix} 11\cdots1 & & \\ & 11\cdots1 & \\ & & 11\cdots1 \end{bmatrix} ,\; L = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ & & & a_{nn} \end{bmatrix} \tag{18b}$$

Using this uncertainty description the system in Fig. 6 may be rearranged into the form in Fig. 3 with the interconnection matrix

$$M = LC(I+\tilde{P}C)^{-1}E = L\tilde{P}^{-1}\tilde{H}E \tag{19}$$

From Theorem 1 follows the necessary and sufficient condition for robust stability

$$\mu(L\tilde{P}^{-1}\tilde{H}E) \leq 1 \qquad \forall\omega \tag{20}$$

In principle this equation may be used to generate all nominal closed loop transfer matrices $\tilde{H}$ for which the closed loop system is robustly stable. Alternatively, it may be used to check if particular designs meet the robust stability requirement. However, at the design stage when the dynamic resilience is to be determined, $\tilde{H}$ is not known, but rather the restrictions on $\tilde{H}$ imposed by the uncertainty are of interest. In order to obtain an explicit bound on $\tilde{H}$ from (20) assume $\tilde{H} = \tilde{h}I$, i.e. choose $C = c(s)\tilde{P}^{-1}$. This controller choice yields a decoupled closed loop transfer function with identical responses. From (20) follows:

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$$\text{iff} \quad \bar{\sigma}(\tilde{H}) = |\tilde{h}| \leq \frac{1}{\mu(L\tilde{P}^{-1}E)} \quad \forall\omega \tag{21}$$

Again, this bound shows that the system has to be detuned and performance sacrificed when the uncertainty is large, i.e. in this case when $\mu(L\tilde{P}^{-1}E) > 1$. $\mu(L\tilde{P}^{-1}E)$ is a measure of the dynamic resilience which takes into account both the size of the uncertainty and the sensitivity of the plant to uncertainty.

IV. Additive Norm Bounded Uncertainty

The robust stability condition $\mu(M) < 1$ stated in Theorem 1 is in principle all that is needed to study robust stability for systems with norm bounded uncertainty. However, as stated before, we are looking for simpler conditions which may be used to gain additional insight into why and when uncertainty causes robustness problems for a particular system. It is hoped that this insight will indicate to the design engineer how the system has to be modified to improve its resilience characteristics.

In the following we will consider a less general uncertainty description given in terms of weighted additive uncertainty on the nominal plant. We will derive conditions which are consequently less general than Theorem 1, but which yield additional insight, and may be used to interpret the condition number as a sensitivity measure. The results in this section also fit in nicely with previous work (Morari, 1983, Grosdidier, et al., 1985) and show that those results may be extended and placed into the framework used in this paper.

Consider a generalization of the uncertainty description presented in Example 2. Assume that the uncertainty may be written as an additive weighted norm bounded uncertainty on the plant (Fig. 6), i.e.

$$P - \tilde{P} = E\Delta L \ , \ \bar{\sigma}(\Delta) < 1 \tag{22}$$

where E and L are weighting transfer matrices. E and L do not have to be square or nonsingular but do generally have to be stable. We may define a corresponding norm bounded set of plants, Π_A,

$$\Pi_A = \{P: P-\tilde{P} = E\Delta L, \ \ \forall\Delta \text{ such that } \bar{\sigma}(\Delta) < 1\} \tag{23}$$

As shown in Example 2, any uncertainties given in terms of norm bounds on the individual elements (i.e. $|p_{ij}-\tilde{p}_{ij}| < a_{ij}$) may be written in this form. Also uncertainties which are not by themselves additive perturbations on the plant may in some cases be written in the form (22). For example, the multiplicative input uncertainty of Example 1 may be written in this form if we assume the input uncertainty to be the only kind of uncertainty and choose the weights $E = \tilde{P}$ and $L = I$.

The major restrictions on writing the uncertainty in the form (22)

1. Uncertainties entering at several places in the system cannot be handled. In particular this implies:
   - 1a. Different kinds of uncertainty descriptions cannot be combined. The multiplicative input and output uncertainty together (Example 1) cannot be put in the form of (22).
   - 1b. Most uncertainties stemming from real parameter variations cannot be written in the form of (22).
2. Some uncertainty descriptions with only one kind of uncertainty cannot be handled, e.g. the uncertainty description in Fig. 7.

1. The Determinant Condition

The determinant condition in terms of $P-\tilde{P}$ follows directly from Theorem 1.

Theorem 2. Determinant Condition for Robust Stability. Assume the uncertainty may be written as a norm bounded set of plants Π_A, i.e. $P-\tilde{P} = E\Delta L$. Assume the nominal system is closed loop stable. Then robust stability is achieved for all $\bar{\sigma}(\Delta) < 1$ if and only if

$$\det(I+(P-\tilde{P})\tilde{P}^{-1}\tilde{H}) \neq 0 \quad \forall\omega, \quad \forall P\epsilon\ \Pi_A \tag{24}$$

$$\text{or} \quad \text{iff} \quad \mu(L\tilde{P}^{-1}\tilde{H}E) \leq 1 \quad \forall\omega \tag{25}$$

Proof: See Appendix 3.

The sufficiency of condition (24) for this kind of uncertainty is obvious from the Nyquist criterion when applied to the IMC structure (Morari et. al., 1985). The necessity is a consequence of the assumed norm bounds on the additive uncertainty. Condition (24) itself is not very useful but it will be used later to derive conditions involving the condition number of $\tilde{P}$. For $\tilde{H} = \tilde{h}I$ we may simplify (25) and derive a necessary and sufficient bound on $|\tilde{h}|$ for robust stability which shows how performance must be sacrificed in the case of uncertainty

$$|\tilde{h}| \leq \frac{1}{\mu(L\tilde{P}^{-1}E)} \qquad \forall\omega \tag{26}$$

As the uncertainty expressed in terms of L and E increases, μ generally increases, and $|\tilde{h}|$ has to be reduced and performance deteriorates. A result which gives further insight into what kind of uncertainties cause problems can be derived directly from Thm. 2 for the mathematical idealization of "perfect control":

Corollary 2.1. Perfect Control ($\tilde{H}$=I): Robust stability $\forall P\varepsilon\ \Pi_A$

$$\text{iff} \quad \det(P\tilde{P}^{-1}) \neq 0 \quad \forall\omega, \quad \forall P\varepsilon\ \Pi_A \tag{27}$$

$$\text{iff} \quad \mu(L\tilde{P}^{-1}E) \leq 1 \quad \forall\omega \tag{28}$$

Condition (27) indicates that in the case of a norm bounded uncertainty on the plant perfect control is possible if and only if P is nonsingular (detP $\neq$ 0) for $\forall P\varepsilon\ \Pi_A$, $\forall\omega$. P will be singular (detP = 0) for a particular ω if P has a zero on the imaginary axis. It is well known (Morari, 1983) that perfect control ($\tilde{H}$=I) of the nominal plant $\tilde{P}$ is possible only if $\tilde{P}$ is minimum phase. Corollary 2.1 confirms that for perfect control of the actual plant P to be possible, zeros must not cross into the RHP in the frequency range where $\tilde{H} \approx$ I.

For stable plants the following necessary and sufficient condition for robust stability with integral control may be derived from Corollary 2.1.

Corollary 2.2. Integral Control. ($\tilde{H}(0)$=I): Assume P is stable. Robust stability may be achieved

$$\text{iff} \quad \det(P(0)\tilde{P}(0)^{-1}) > 0 \quad \forall P\varepsilon\ \Pi_A \tag{29a}$$

$$\text{iff} \quad \mu(L\tilde{P}^{-1}E) \leq 1, \quad \omega = 0 \tag{29b}$$

Proof. See Appendix 3.

Note that Corollary 2.2 uses only steady state information about the plant. This is important since this is often the only information which is available to the engineer. Robust stability under integral control implies that the steady state performance will be perfect for any plant P in the uncertainty set $P \epsilon \Pi$, i.e. robust performance at steady state is guaranteed. This is in contrast to the other conditions derived above which were only in terms of the nominal performance, meaning that performance might be arbitrarily poor for $P \neq \tilde{P}$.

2. The Condition Number as a Sensitivity Measure

It has been argued previously in a qualitative manner (Morari, 1983, Grosdidier et al., 1985) that the minimized condition number $\gamma^*(\tilde{P})$ is a measure of sensitivity to model uncertainty. Furthermore, there is a direct relationship between large elements in the Relative Gain Array (RGA) and $\gamma^*(\tilde{P})$ (Grosdidier et al., 1985), and large elements in the RGA are often claimed to indicate sensitivity to model uncertainty. In this section it will be shown that the minimized condition number $\gamma^*(\tilde{P})$ is a useful measure only if the relative errors of the transfer matrix elements are independent and have similar magnitude bounds. This is a restrictive assumption in many cases.

Theorem 3. Condition Number Criterion. Assume $\tilde{H} = \text{diag}\{\tilde{h}_i\}$. Robust stability is achieved

$$\text{if} \quad |\tilde{h}_i| \leq \frac{1}{r_{max}\gamma_a^*(\tilde{P})} \qquad \forall\omega, \quad \forall i \tag{30a}$$

which is satisfied

$$\text{if} \quad |\tilde{h}_i| \leq \frac{1}{r_{max}\sqrt{n}\ \gamma^*(\tilde{P})} \qquad \forall\omega, \quad \forall i \tag{30b}$$

Proof: See Appendix 3.

r_{max} is the largest relative error (uncertainty) bound on the magnitude of any of the elements of the transfer matrix $\tilde{P}$:

$$r_{max}(\omega) = \max_{ij} \; r_{ij}(\omega) \tag{31a}$$

where

$$r_{ij} = \max_{P \varepsilon \Pi_A} \left| \frac{p_{ij}-\tilde{p}_{ij}}{\tilde{p}_{ij}} \right| \tag{31b}$$

$\gamma^*(\tilde{P})$ is the minimized condition number and $\gamma^*_a(\tilde{P})$ is the minimized "absolute" condition number as defined in Appendix 1. The minimized condition numbers are similar in magnitude since

$$\gamma^*_a(\tilde{P})/\sqrt{n} \leq \gamma^*(\tilde{P}) \leq \gamma^*_a(\tilde{P}) \tag{32}$$

In general $\gamma^*_a(\tilde{P})$ and $\gamma^*(\tilde{P})$ may be found by numerical optimization, but this is undesirable. For 2x2 systems the following analytical expression is available for $\gamma^*_a(\tilde{P})$ (Appendix 3):

$$\gamma^*_a(\tilde{P}) = \frac{1+|A|^{1/2}}{(|A|-2|A|^{1/2}\cos\phi/2+1)^{1/2}} \tag{33}$$

and for $\gamma^*(\tilde{P})$ when $\tilde{P}$ is real (ω=0) (Grosdidier et al., 1985)

$$\gamma^*(\tilde{P}) = \begin{cases} \dfrac{1+A^{1/2}}{|1-A^{1/2}|}, & A > 0 \\ 1, & A \leq 0 \end{cases} \tag{34}$$

where A is Rijnsdorps interaction measure (Rijnsdorp, 1965)

$$A = \frac{\tilde{p}_{12}\tilde{p}_{21}}{\tilde{p}_{11}\tilde{p}_{22}} = |A|e^{j\phi} \tag{35}$$

A relationship between $\gamma^*(\tilde{P})$ and the induced 1- and ∞-norms of the RGA has been conjectured by Grosdidier et al. (1985):

$$\gamma^*(\tilde{P}) \leq 2\max[\,||RGA||_1,\ ||RGA||_\infty] \tag{36}$$

Generally, this bound is violated for systems of dimension 4x4 or higher. However, for 2x2 systems this bound holds even for $\gamma^*_a(\tilde{P})$ and we have the stronger result:

Theorem 4 (2x2): $\gamma^*_a(\tilde{P}) \leq ||RGA||_a$ (37)

Proof: See Appendix 3.

$||\cdot||_a$ is the norm defined as the sum of all the magnitudes of the elements, i.e.

$$||G||_a = \sum_{i,j} |g_{ij}| \quad (38)$$

Note that for 2x2 systems $||RGA||_a = 2||RGA||_1 = 2||RGA||_\infty$. Numerical examples for 3x3 and 4x4 systems support the following extension to systems with higher dimensions:

Conjecture 1 (nxn): $\gamma^*_a(\tilde{P}) \leq ||RGA||_a + 2$ (39)

For real matrices and high condition numbers $||RGA||_a$ approaches $\gamma^*_a(\tilde{P})$. The bound (39) appears to be most conservative for small condition numbers.

Theorems 3 and 4 and Conjecture 1 provide at least a partial explanation of why ill conditioned multivariable systems with large RGA should be avoided already at the design stage: When $\gamma^*_a(\gamma^*)$ or equivalently $||RGA||_a$ is large, then the performance mesured in terms of $|\tilde{h}|$ is very restricted (c.f. (30)) even if the model uncertainty r_{max} is small. The uncertainty description (31) can be very conservative and inappropriate at times. However a small fraction of any model error is always "random" and for that type of error (31) is suited quite well. Conditions (30) are attractive because they are independent of the scaling of the system inputs and outputs. An expression similar to (30b) which introduces $\underline{\sigma}(\tilde{P})$ as a sensitivity measure for additive errors on the elements was presented by Morari and Skogestad (1985). However, contrary to the relative error, the additive error is strongly dependent on the scaling of the system. Defining uncertainty in terms of relative error of the elements is simple and convenient since it is the kind of uncertainty description immediately understandable to the engineer.

The disadvantage of Thm. 3 is that it is only sufficient. If condition (30a) were very conservative use of the condition number as a sensitivity measure could be quite misleading. Theorem 5 shows that, at least for 2x2 systems, this is not the case when the relative error bounds are equal:

Theorem 5 (2x2). Assume $\tilde{H} = \tilde{h}I$ and assume the relative errors of the elements of $\tilde{P}$ are independent and have equal magnitude bounds r. Then condition (30a) in Theorem 3 is necessary and sufficient for robust stability, i.e.

$$\text{Robust Stability iff} \quad |\tilde{h}| \leq \frac{1}{r\gamma^*_a(\tilde{P})} \quad \forall\omega \tag{40}$$

Proof: See Appendix 3.

Conjecture 2 (nxn): Theorem 5 holds also for systems of higher dimensions.

Conjecture 2 is based on numerical results for 3x3 and 4x4 systems. An equivalent statement of Conjecture 2 is

$$\gamma^*_a(\tilde{P}) = \mu_{rc}(\tilde{P}) \tag{41}$$

where $\mu_{rc}(\tilde{P}) = \mu(L\tilde{P}^{-1}E)$ for complex perturbations with equal relative errors r = 1. Note from (31) that the perturbations on the elements of $\tilde{P}$ are assumed to be complex. This is reasonable at nonzero frequencies but does not make much physical sense at steady state (ω=0) where $\tilde{P}$ is real. Conjecture 2 will obviously be conservative at $\omega = 0$ since complex perturbations cannot occur. Fortunately, for 2x2 systems it turns out that $\gamma^*(\tilde{P})$ gives the desired measure of sensitivity with respect to real perturbations, i.e.

$$(2x2:) \quad \gamma^*(\tilde{P}) = \mu_{rr}(\tilde{P}) \tag{42}$$

where $\mu_{rr}(\tilde{P}) = \mu(LP^{-1}E)$ for real perturbations with equal relative errors r =1. Corollary 2.2 and (42) may be combined into the following theorem:

Theorem 6. (2x2) Integral Control ($\tilde{H}(0)=I$)

Assume the relative error of the elements in $\tilde{P}(0)$ are independent and real and have equal magnitude bounds r. Then for open loop stable systems

$$\text{Robust stability may be achieved iff} \quad \gamma^*(\tilde{P}(0)) < 1/r \tag{43}$$

Proof: See Appendix 3.

Theorem 6 is unique to 2x2 systems as numerical examples show no such relationship for systems of higher dimensions. If the magnitude bounds on the relative errors are not equal and r_{max} is substituted for r, Theorem 6 and Conjecture 2 provide sufficient conditions for robust stability.

Theorem 6 and Conjecture 2 give very clear interpretations of the minimized condition numbers as sensitivity measures: $\gamma^*(\tilde{P}(0))$ and $\gamma^*{}_a(\tilde{P}(j\omega))$ are accurate measures of sensitivity only if the plant uncertainties are given in terms of independent norm-bounded elements with equal relative error bounds. For other uncertainty structures the minimized condition number may be a very misleading sensitivity measure, and bounds on the uncertainties such as (30) may be arbitrarily conservative. This is illustrated in the following example.

Example 3. Integral Control of High Purity Distillation Column

This example represents a binary distillation column with inputs L (reflux flowrate) and V (vapor flowrate from reboiler). The feed composition Z_F = 0.424, reflux ratio L/D = 17, number of trays N = 110, relative volatility α = 1.15, and the nominal top and bottom compositions (outputs) are y_D = 0.993 and x_B = 0.005. Using the approximate relationships of Shinskey (1984) we derive the linearized gains for the nominal plant

$$\tilde{P}(0) = \begin{bmatrix} 0.8031 & -0.8054 \\ 1.2376 & -1.2345 \end{bmatrix} = \begin{bmatrix} (\frac{\partial x_B}{\partial L})_V & (\frac{\partial x_B}{\partial V})_L \\ (\frac{\partial y_D}{\partial L})_V & (\frac{\partial y_D}{\partial V})_L \end{bmatrix}$$

and

$$||RGA||_a = 745.2, \quad \gamma^*(\tilde{P}(0)) = 745.2, \quad \gamma(\tilde{P}(0)) = 815.1$$

From the high condition number, one might conclude that the closed loop performance is very sensitive to model uncertainty. This would be true if the uncertainty had the form of independent element errors, but not necessarily otherwise. Let us look at conditions for using integral action ($\tilde{H}(I)=0$) for two different assumptions about the uncertainty. To illustrate this point:

Case 1: The elements are assumed independent and norm bounded with equal relative error r. As expected from Theorem 6 the SSV μ (real perturbations) and the γ^*-test both give the same result: Robust stability with integral action is possible if and only if $r < 1.34 \times 10^{-3}$.

Case 2: A more realistic uncertainty description for this high purity distillation column is

$$P-\tilde{P} = \begin{bmatrix} d & -d \\ -d & d \end{bmatrix}, \quad |d| \leq \delta$$

The reason for the highly structured uncertainty is that any upset in the column will keep $x_B(1-y_D)$ nearly constant (Skogestad et al., 1985).

Using the SSV μ (Corollary 2.2 with real perturbations) we find that a control system with integral action can be designed for which the closed loop system is robustly stable if and only if $\delta < 6.7$, i.e. the elements may even change sign. (This may not be true in practice because the uncertainty description above is an approximation). Thus despite the high condition number the system is not sensitive at all to this physically motivated model error.

V. Conclusions

To guarantee robust stability model uncertainty requires feedback controllers to be detuned and performance to be sacrificed. To which extent detuning is necessary depends on the size of the uncertainty and the sensitivity of the plant. The Structured Singular Value $\mu(M)$ is by definition the best measure of the effect of uncertainty on performance:

$$\text{Robust stability iff} \qquad \mu(M) \leq 1 \qquad \forall\omega \tag{6}$$

However, the issue here is not control system design but process design. From this point of view systems whose closed loop stability and performance are very sensitive to model error are undesirable because they are either impossible to control or require that enormous effort be put into the design of the control system. Condition (6) assumes that a control system has already been designed and is therefore unsuitable for screening purposes at the design stage. If some mild assumptions are made on the type of model uncertainty and the control structure, achievable performance can be related directly to characteristics of the system itself. We will assume throughout the following summary that the nominal closed loop system is decoupled ($\tilde{H}=\tilde{h}I$) with identical responses. This is a reasonable assumption at low frequencies, and leads to the least conservative bounds.

I) Uncertainty: $P - \tilde{P} = E\Delta L$

$\Delta = \text{diag}\{\Delta_i\}$, $\bar{\sigma}(\Delta_i) < 1$

E, L stable

Robust stability iff

$$|\tilde{h}| < \frac{1}{\mu(L\tilde{P}^{-1}E)} \qquad \forall\omega \tag{21}$$

II) Uncertainty: $\left|\frac{p_{ij}-\tilde{p}_{ij}}{\tilde{p}_{ij}}\right| < r_{ij}$

$$r_{max} = \max_{ij} \; r_{ij}$$

1. Robust stability if

$$|\tilde{h}| < \frac{1}{r_{max}\gamma^*_a(\tilde{P})} \qquad \forall\omega \tag{30a}$$

2. 2x2 systems, $r_{ij} = r \;\; \forall i,j$.

   Robust stability iff

$$|\tilde{h}| < \frac{1}{r\gamma^*{}_a(\tilde{P})} \qquad \forall\omega \tag{40}$$

3. 2x2 systems, $r_{ij} = r \ \ \forall i,j$, integral control ($\tilde{H}(0)=I$).

Robust stability may be achieved iff

$$|\tilde{h}(0)| = 1 < \frac{1}{r\gamma^*(\tilde{P}(0))} \tag{43}$$

The minimized condition number $\gamma^*(\gamma^*{}_a)$ or equivalently the RGA is a reliable indicator of closed loop sensitivity to model uncertainty only if the relative errors of the transfer matrix elements are independent and have similar magnitude bounds. Otherwise the uncertainty should be modeled as suggested in case I). Then the appropriate SSV μ is a good indicator of the performance deterioration caused by model uncertainty.

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Appendix 1, Notation (see Fig. 1)

C(s) - rational transfer matrix of fixed-parameter controller

P(s) - nxn square rational transfer matrix of actual plant = $\{p_{ij}\}$

$\tilde{P}(s)$ - nxn square rational transfer matrix of nominal plant = $\{\tilde{p}_{ij}\}$

Π - set of all possible plants, i.e. $P \in \Pi$

|G| - matrix G with all elements replaced by their absolute value

RGA(G) - $G(j\omega) \times (G^{-1}(j\omega))^T$ - Relative Gain Array of G. x represents the Schur or Haddemard product (element by element multiplication).

$\rho(G)$ - spectral radius of G, i.e. magnitude of largest eigenvalue

$\bar{\sigma}(G)$ - maximum singular value or spectral norm of the transfer matrix G, which is equal to the induced 2-norm

$$\bar{\sigma}(G(j\omega)) = \max_u \frac{||Gu||_2}{||u||_2} (j\omega)$$

Here $||\cdot||_2$ denotes the usual Eucledian spatial norm

$\underline{\sigma}(G)$ - minimum singular value of G

$$\underline{\sigma}(G(j\omega)) = \min_u \frac{||Gu||_2}{||u||_2} (j\omega)$$

We have the property $\underline{\sigma}(G) = 1/\bar{\sigma}(G^{-1})$

$\gamma(G)$ = $\bar{\sigma}(G)/\underline{\sigma}(G)$ - condition number of G

$\gamma_a(G)$ = $\bar{\sigma}(|G|)/\underline{\sigma}(G)$ - absolute condition number

$\gamma^*(G)$ - minimized condition number, $\gamma^*(G) = \min_{S_1,S_2} \gamma(S_1 G S_2)$, where S_1 and S_2 are real diagonal matrices

$\mu(G)$ - structured singular value (see Appendix 2).

The Laplace variable s or $j\omega$ is omitted in most cases.

Appendix 2. The SSV μ and its properties

Definition (Doyle, 1982a). The function $\mu(M)$, called the structured singular value (SSV) is defined at each frequency such that $\mu^{-1}(M)$ is equal to the smallest δ needed to make $(I+\Delta M)$ singular, i.e.

$$\mu^{-1}(M) = \min_{\delta}\{\delta \mid \det(I+\Delta M) = 0 \text{ for some } \Delta,\ \bar{\sigma}(\Delta) \leq \delta(\omega)\} \qquad (A2\text{-}1)$$

Δ is a block diagonal perturbation matrix. $\mu(M)$ depends on the matrix M and the structure of the perturbations Δ. The definition of μ may be extended by restricting Δ to a smaller set, e.g. Δ real. The above definition is not in itself useful for computing μ since the optimization problem implied by it does not appear to be easily solvable. Fortunately, Doyle (1982) has proven several properties of μ which makes it more useful in applications.

Properties of μ (Doyle, 1982a)

1. The following bounds exists for μ:

$$\rho(M) \leq \mu(M) \leq \bar{\sigma}(M) \qquad (A2\text{-}2)$$

   $\mu(M) = \rho(M)$ in the case $\Delta = \delta I$. $\mu(M) = \bar{\sigma}(M)$ in the case Δ is "unstructured", i.e. Δ is a full matrix.

2. Let $\mathcal{U}$ be the set of all unitary matrices with the same structure as Δ, then

$$\max_{U\in\mathcal{U}} \rho(UM) = \mu(M) \qquad (A2\text{-}3)$$

   This optimization problems is in general not convex.

3. Let $\mathcal{D}$ be the set of real positive diagonal matrices $D = \{\mathrm{diag}(d_i I_i)\}$ where the size of each block (size of I_i) is equal to the size of the blocks Δ_i. Then for 3 or fewer blocks

$$\min_{D\in\mathcal{D}} \bar{\sigma}(DMD^{-1}) = \mu(M) \qquad (A2\text{-}4)$$

   For 4 or more blocks numerical evidence suggest that this gives a tight upper bound on $\mu(M)$.

4. $\mu(\alpha M) = |\alpha|\mu(M)$, α is a scalar. (A2-5)

5. For real matrices M with real, non-repeated perturbations, the search in (A2-3) may be performed with real matrices U only, and only the cornerpoints ("± 1") need to be considered. For Eq. (29a) in Corollary 2.2 this implies that only cornerpoints for the possible P(0)'s need to be checked.

Appendix 3

Proof of Theorem 2: Construct the interconnection matrix M for this uncertainty description (rearrange Fig. 6 to get Fig. 3):

$$M = LC(I+\tilde{P}C)^{-1} E = L\tilde{P}^{-1}\tilde{H}E \qquad \text{(A3-1)}$$

The transfer matrix $\tilde{P}^{-1}\tilde{H}$ is stable because the nominal system with $\Delta = 0$ is assumed to be stable. Substituting (A3-1) into (5) we find

$$\det(I+\Delta M) = \det(I+\Delta L\tilde{P}^{-1}\tilde{H}E) = \det(I+E\Delta L\tilde{P}^{-1}\tilde{H}) = \det(I+(P-\tilde{P})\,\tilde{P}^{-1}\tilde{H}) \neq 0 \qquad \text{QED}$$

Proof of Corollary 2.2:

Necessity: Cor. 2.1 provides the necessary condition

$$\det(P(0)\tilde{P}(0)^{-1}) \neq 0 \qquad \forall P \in \Pi_A \qquad \text{(A3-2)}$$

Because Π_A is convex and $\tilde{P} \in \Pi_A$ (A3-2) is equivalent to

$$\det(P(0)\tilde{P}(0)^{-1}) > 0 \qquad \forall P \in \Pi_A \qquad \text{(29a)}$$

Sufficiency: If (29a), then conditions (24) and (25) of Thm. 2 and (26) are satisfied for $\omega = 0$ with $\tilde{H}(0) = I$. For frequencies $\omega \neq 0$ and stable plants, $\tilde{H}$ can always be selected $\tilde{H} = \tilde{h}I$ such that (26) holds.

Proof of Theorem 3

Assume a weighted additive uncertainty of the form $P - \tilde{P} = E\Delta L$. Then we have robust stability (Theorem 2) if and only if

$$\det(I+(P-\tilde{P})\tilde{P}^{-1}\tilde{H}) \neq 0 \qquad \forall\omega, \quad \forall P\varepsilon\, \Pi_A \tag{24}$$

Define the scaled plant by

$$\tilde{P}_S = S_1\tilde{P}S_2 \ , \ P_S = S_1PS_2 \tag{A3-3}$$

where S_1 and S_2 are real diagonal matrices. Substitute (A3-3) into (24) and use the property $\det(I+AB) = \det(I+BA)$ to get

$$\det(I+(P_S-\tilde{P}_S)\tilde{P}_S^{-1}S_1\tilde{H}S_1^{-1}) \neq 0, \quad \forall P\varepsilon\, \Pi_A \tag{A3-4}$$

To eliminate S_1 and S_1^{-1} assume $\tilde{H}$ is diagonal and apply the Small Gain Theorem to (A3-4) to get

Robust stability for $\tilde{H} = \text{diag}\{\tilde{h}_i\}$ if

$$|\tilde{h}_i| < \left[\max_{P_S\varepsilon\Pi_A} \frac{\bar{\sigma}(P_S-\tilde{P}_S)}{\bar{\sigma}(\tilde{P}_S)}\right]^{-1} \frac{1}{\gamma(\tilde{P}_S)} \qquad \forall\omega \ , \quad \forall i \tag{A3-5}$$

The following Lemma bounds the "relative error" of a matrix in terms of the relative errors of the elements

Lemma 1

Consider any set of plants Π. Define the maximum and minimum relative error in the elements

$$r_{max}(\omega) = \{ \max_{ij} r_{ij}(\omega)\} \tag{A3-6a}$$

$$r_{min}(\omega) = \{ \min_{ij} r_{ij}(\omega)\} \tag{A3-6b}$$

$$r_{ij}(\omega) = \max_{P\varepsilon\,\Pi} \left| \frac{P_{ij}-\tilde{P}_{ij}}{\tilde{P}_{ij}} \right| \tag{A3-6c}$$

Then

$$\max_{P\varepsilon\Pi} \frac{\bar{\sigma}(P-\tilde{P})}{\bar{\sigma}(\tilde{P})} \leq r_{max} \frac{\bar{\sigma}(|\tilde{P}|)}{\bar{\sigma}(\tilde{P})} \leq r_{max}\sqrt{n} \tag{A3-7}$$

$|\tilde{P}|$ is the matrix formed by taking the absolute value of the elements in $\tilde{P}$. If the set of plants corresponds to a norm bounded set Π_E with independent elements we also have a lower bound

$$\max_{P\in\Pi_E} \frac{\bar{\sigma}(P-\tilde{P})}{\bar{\sigma}(\tilde{P})} \geq r_{min} \frac{\bar{\sigma}(|\tilde{P}|)}{\bar{\sigma}(\tilde{P})} \geq r_{min} \tag{A3-8}$$

Consequently if the relative error bounds of the elements are equal we have equality

$$\max_{P\in\Pi_E} \frac{\bar{\sigma}(P-\tilde{P})}{\bar{\sigma}(\tilde{P})} = r \frac{\bar{\sigma}(|\tilde{P}|)}{\bar{\sigma}(\tilde{P})} \tag{A3-9}$$

Proof of Lemma 1:

The upper bound is found by defining a larger norm bounded family Π_L

$$\Pi_L = \{P: \left|\frac{p_{ij}-\tilde{p}_{ij}}{\tilde{p}_{ij}}\right| \leq r_{max}(\omega)\} \tag{A3-10}$$

We get

$$P-\tilde{P} = r_{max} \begin{bmatrix} |\tilde{p}_{11}|\Delta_{11} & |\tilde{p}_{12}|\Delta_{21} & \cdots & |\tilde{p}_{1n}|\Delta_{1n} \\ \vdots & & & \vdots \\ |\tilde{p}_{n1}|\Delta_{n1} & & & |\tilde{p}_{nn}|\Delta_{nn} \end{bmatrix} \tag{A3-11}$$

and $\bar{\sigma}(P-P)$ has it maximum when all $\Delta_{ij} = 1$. Consequently, since $\Pi \subseteq \Pi_L$:

$$\max_{P\in\Pi} \bar{\sigma}(P-\tilde{P}) \leq \max_{P\in\Pi_L} \bar{\sigma}(P-\tilde{P}) = r_{max}\, \bar{\sigma}(|\tilde{P}|) \tag{A3-12}$$

To get the right hand side of (A3-7) use the general property (Stone, 1962)

$$\frac{1}{\sqrt{n}} ||A||_F \leq \bar{\sigma}(A) \leq ||A||_F \tag{A3-13}$$

where $||\cdot||_F$ denotes the Frobenius norm of the matrix A,

$$||A||_F = (\sum_{i,j} |a_{ij}|^2)^{1/2} \tag{A3-14}$$

Using the obvious property $||A||_F = ||\,|A|\,||_F$ we get

$$\frac{\bar{\sigma}(|\tilde{P}|)}{\bar{\sigma}(\tilde{P})} \leq \frac{|| \, |\tilde{P}| \, ||_F}{||\tilde{P}||_F\sqrt{n}} = \sqrt{n} \tag{A3-15}$$

The lower bound (A3-8) is found by defining a smaller norm bounded family of plants Π_S which is included in Π_E

$$\Pi_S = \{P: \left|\frac{p_{ij}-\tilde{p}_{ij}}{\tilde{p}_{ij}}\right| \leq r_{min}(\omega)\} \tag{A3-16}$$

and since $\Pi_E \supseteq \Pi_S$ we have

$$\max_{P \epsilon \Pi_E} \bar{\sigma}(P-\tilde{P}) \geq \max_{P \epsilon \Pi_S} \bar{\sigma}(P-\tilde{P}) = r_{min}\bar{\sigma}(|\tilde{P}|) \tag{A3-17}$$

This concludes the proof of Lemma 1.

Using the fact that relative errors are unchanged by the input and output scaling defined in (A3-3), we use (A3-7) to conclude that the robust stability condition (A3-5) will be satisfied if

$$|\tilde{h}_i| < \frac{1}{r_{max}} \frac{\bar{\sigma}(\tilde{P}_S)}{\bar{\sigma}(|\tilde{P}_S|)} \frac{1}{\gamma(\tilde{P}_S)} = \frac{1}{r_{max}} \frac{1}{\gamma_a(\tilde{P}_S)} \tag{A3-18}$$

This condition will guarantee robust stability for any choice of scaling. Since r_{max} is independent of scaling we can make (A3-18) least conservative by minimizing $\gamma_a(\tilde{P}_S)$ over all possible scaling matrices. This proves the first condition (30a) in Theorem 3. The second statement is proved by applying (A3-15).

Proof of Theorem 4:

For 2x2 systems the RGA becomes

$$RGA = \begin{bmatrix} \lambda_{11} & 1-\lambda_{11} \\ 1-\lambda_{11} & \lambda_{11} \end{bmatrix}, \quad \lambda_{11} = \frac{\tilde{p}_{11}\tilde{p}_{22}}{\tilde{p}_{11}\tilde{p}_{22}-\tilde{p}_{12}\tilde{p}_{21}} = \frac{1}{1-A} \tag{A3-19}$$

Consequently

$$||RGA||_a = 2(|\lambda_{11}|+|1-\lambda_{11}|) = 2\,\frac{1+|A|}{|1-A|} \tag{A3-20}$$

Using Eq. (33)

$$\gamma^*_a = \frac{1+|A|^{1/2}}{(|A|-2|A|^{1/2}\cos\phi/2+1)^{1/2}} \leq \frac{1+2|A|^{1/2}+|A|}{|1-A|} \leq ||RGA||_a , \qquad \text{QED(A3-21)}$$

Proof of Theorem 5

Let $\tilde{P}$ be a nonsingular 2x2 transfer matrix and consider the case of independent elements with equal relative errors r.

$$P = \begin{bmatrix} \tilde{p}_{11}(1+r\Delta_{11}) & \tilde{p}_{12}(1+r\Delta_{12}) \\ \tilde{p}_{21}(1+r\Delta_{21}) & \tilde{p}_{22}(1+r\Delta_{22}) \end{bmatrix} , \quad |\Delta_{ij}| < 1 \qquad \text{(A3-22)}$$

First prove Eq. (41) which (using Corollary 2.1) is equivalent to the following statement:

"the smallest r which makes detP = 0 is $r = 1/\gamma^*_a(\tilde{P})$" (A3-23)

Again define

$$A = \frac{\tilde{p}_{12}\tilde{p}_{21}}{\tilde{p}_{11}\tilde{p}_{22}} = |A|\, e^{j\phi} \qquad \text{(A3-24)}$$

and use

$$\det P = 0 \text{ iff } \quad (1+r\Delta_{11})(1+r\Delta_{22}) = A(1+r\Delta_{12})(1+r\Delta_{21}) \qquad \text{(A3-25)}$$

The smallest r which satisfies (A3-25) is found for $\Delta_{11} = \Delta_{22} = \Delta_1$, $\Delta_{12} = \Delta_{21} = \Delta_2$ and we get

$$(1+r\Delta_1)^2 = A(1+r\Delta_2)^2$$

Introduce $\Delta_1 = e^{j\phi_1}$, $\Delta_2 = e^{j\phi_2}$ to find:

$$1-|A|^{1/2}e^{j\phi/2} = r|A|^{1/2}\, e^{j(\frac{\phi}{2}+\phi_2)} - r\, e^{j\phi_1}$$

Using geometrical arguments we see that the smallest r satisfying this expression is found when the two terms on the right side are aligned, and we get

$$r = \frac{(|A|-2|A|^{1/2}\cos\phi/2 + 1)^{1/2}}{1+|A|^{1/2}} \qquad \text{(A3-26)}$$

The derivation of the expression for $\gamma^*_a(\tilde{P})$ is very tedious but straightforward and follows the derivation for $\gamma^*(\tilde{P})$ (Grosdidier et. al, 1985). The derivation shows that r given in (A3-26) is equal to $1/\gamma^*_a(\tilde{P})$ which proves (41). Combining Eq. (41) and Eq. (21) gives Theorem 5.

Proof of Theorem 6

The proof is similar to that of Theorem 5, but the perturbations are assumed to be real ($-1<\Delta_{ij}<1$) and all the elements in $\tilde{P}$ are assumed to be real. We want to find the smallest r which satisfies (A3-25).

Case 1: $A < 0$

In this case (A3-25) cannot be satisfied for any $r < 1$, but it may clearly be satisfied if $r = 1$ (e.g. choose $\Delta_{12} = -1$ and $\Delta_{11} = -1$). Consequently, the smallest r which makes $\det P = 0$ in this case is $r = 1$, and since $\gamma^*(\tilde{P}) = 1$ for $A < 0$ we have $r = 1/\gamma^*(\tilde{P})$.

Case 2a: $A > 1$

Only cornerpoints of (A3-25) need to be checked (see Appendix 2). Then it is obvious that the smallest r which satisfies (A3-25) for $A > 1$ is the solution of (choose $\Delta_{11} = \Delta_{22} = 1$, $\Delta_{12} = \Delta_{21} = -1$):

$$(1+r)^2 = A(1-r)^2$$

which has as its smallest root

$$r = \frac{\sqrt{A}-1}{\sqrt{A}+1} = \frac{\left|1-\sqrt{A}\right|}{1+\sqrt{A}} \qquad \text{(A3-27)}$$

Case 2b: $0 < A < 1$

The smallest r which satisfies (A3-25) in this case is a solution of

$$(1-r^2) = A(1+r)^2$$

which has as its smallest root

$$r = \frac{1-\sqrt{A}}{1+\sqrt{A}} = \frac{\left|1-\sqrt{A}\right|}{1+\sqrt{A}} \qquad \text{(A3-28)}$$

This is in fact equal to $1/\gamma^*(\tilde{P})$ for $A > 0$ and proves statement (42). QED

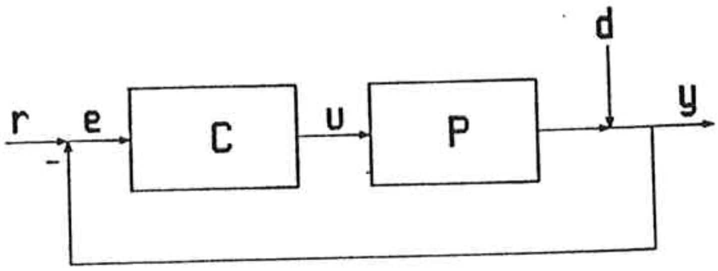


Figure 1. Feedback system with controller C and plant P.

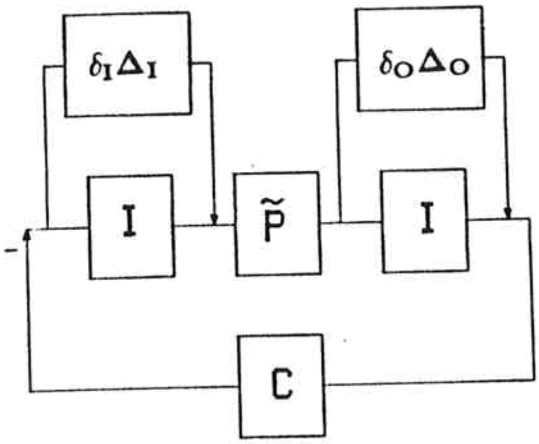


Figure 2. System with input and output uncertainties. This may be rearranged into the form in Fig. 3.

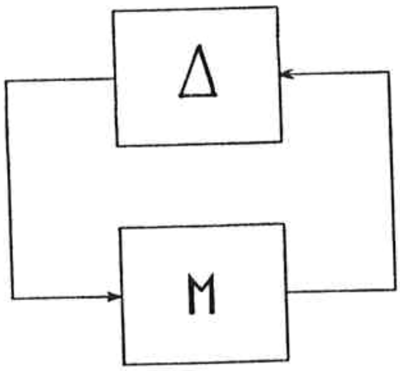


Figure 3. Interconnection structure for uncertainties.

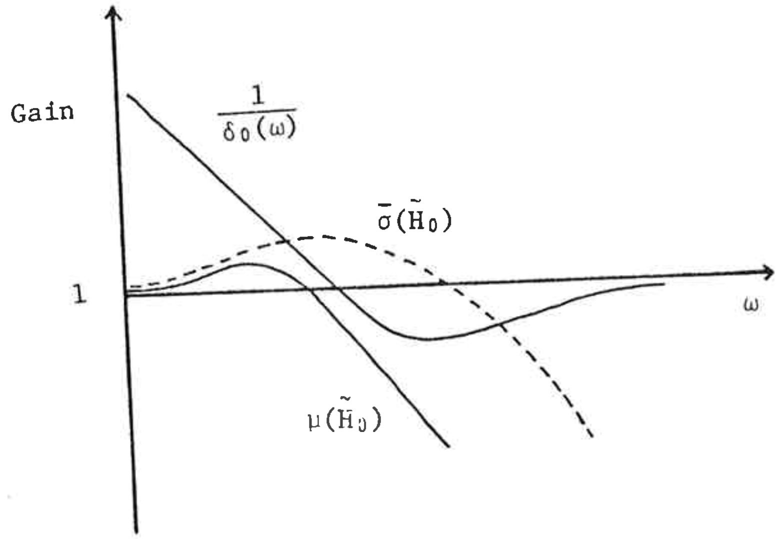


Figure 4. $\mu(\tilde{H})$ is limited by the inverse of the magnitude of the output multiplicative uncertainty $(1/\delta_o(\omega))$. Use of $\bar{\sigma}(\tilde{H})$ requires a more conservative controller design or less uncertainty.

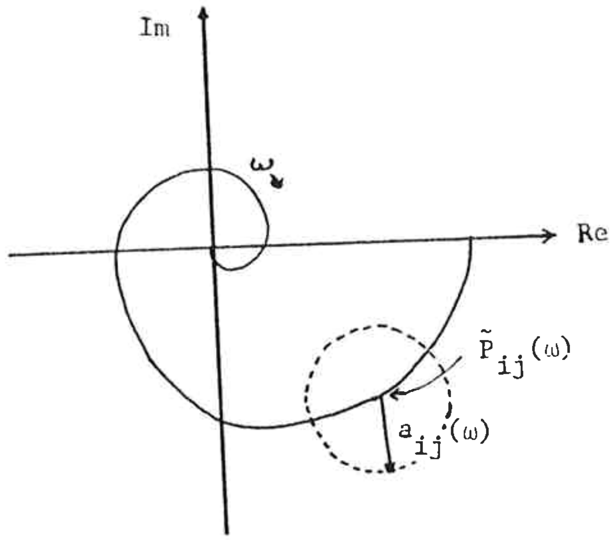


Figure 5. Additive uncertainty on the elements. The disc represents the set of possible p_{ij} at a given frequency.

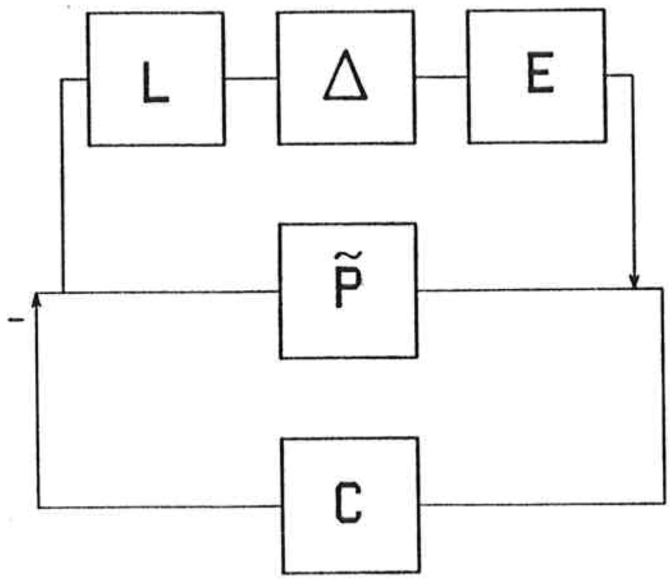
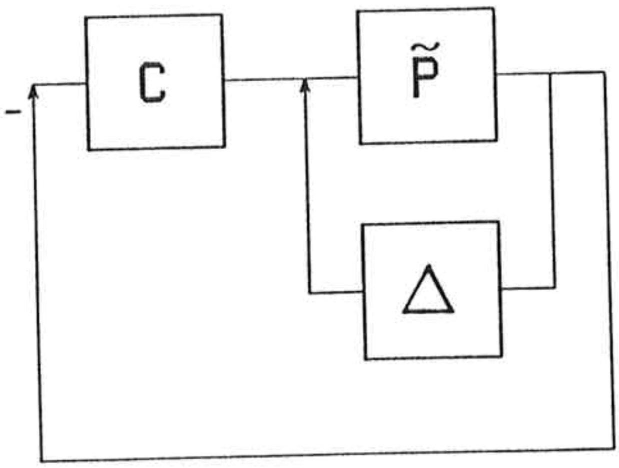


Figure 6. System with weighted additive uncertainty.



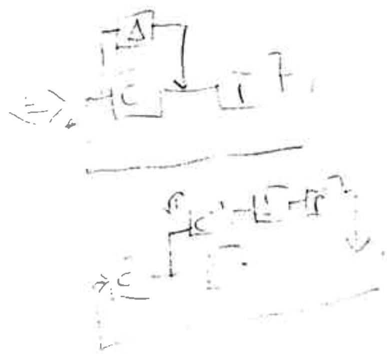


Figure 7. Example of uncertainty which cannot be represented by additive uncertainty as in Fig. 6. (with L and E independent of C).