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Adaptive Control Design of Uncertain Piecewise-Linear Systems

Nanzhu Lin
School of Electrical Engineering
Jilin University
Changchun, China, 130000
Email: nzl.jlu@gmail.com

Xin Wang
and Yanan Zhang
Changchun University of Technology
Changchun, China, 130000
Email: {ynzhang,xwang}@ccut.edu.cn

Chunze Wang and Shu Diao
School of Electrical Engineering
Jilin University
Changchun, China, 130000
Email: {wangcz6509,diaoshu}@jlu.edu.cn

***Abstract*—In this paper, we study adaptive control design problem of piecewise-linear systems with matching parametric uncertainties. The basic idea is to construct a piecewise control law and a piecewise parameter adaptation law in such a way that a piecewise quadric Lyapunov function can be used to establish the global stability. All of the synthesis conditions are formulated as Linear Matrix Inequities and can therefore be efficiently solved. Moreover, the possibility of sliding motion at the boundary of polytopic regions is considered, and the results are demonstrated by application to control a linear motor.**

I. Introduction

Piecewise linear(PWL) systems are switched linear systems with state space-partition-based switching. PWL systems are very important in representing many practical systems including power electronics [1], [2], robots [3], [4] and biology regulatory networks [5]. In additional, PWL systems can approximate nonlinear systems to any degree of accuracy and therefore provide a useful framework for the analysis and synthesis of a large class of nonlinear systems.

Remarkable progress on the PWL systems have been achieved thanks to the dedication of many researchers over the last decades. Important achievements include stability and stabilization [6]–[11], state and output tracking control [12]–[14] and robust control of PWL systems [15]–[20] among others.

Although the proposed robust control methods [15]–[20] can be applied to deal with the uncertain PWL systems, model uncertainties coming from parametric uncertainties can not be reduced. In order to achieve the required performance, the feedback gains must be increased, resulting in high-gain feedback. On the other hand, there are always the constrain scope of control input in the practical engineering systems. To avoid the high-gain feedback, an Lyapunov-based adaptive control approach was proposed in [21]. However, the synthesis conditions were formulated as Bilinear Matrix Inequities(BMIs), the computation problem of BMIs is still a common challenge.

In this paper, we study the adaptive control design problem of PWL systems with matching parametric uncertainties. The basic idea is to construct a piecewise control law and a piecewise parameter adaptation law in such a way that a piecewise quadric Lyapunov function[PQLF] can be used to establish the global stability. Comparing with [21], all of the synthesis conditions are formulated as Linear Matrix Inequities(LMIs) instead of BMIs, thus are much more simple to solve with existing software such as MATLAB. Moreover, we consider the possibility of sliding motion at the boundary of polytopic regions, which guarantees the rigidity of the proposed approach.

II. Problem Formulation and Preliminary

Consider the following uncertain PWL systems,

$$\dot{x} = \begin{cases} A_1 x + B_1 u + \varphi_1(x)\theta & \text{if } x \in R_1, \\ A_2 x + B_2 u + \varphi_2(x)\theta & \text{if } x \in R_2, \\ \cdots \\ A_m x + B_m u + \varphi_m(x)\theta & \text{if } x \in \mathcal{R}_m, \end{cases} \tag{1}$$

where $\mathbb{R}^n = \cup_{i\in\{1,2,\cdots,m\}} R_i$ denotes a partition of the state space into a number of closed polytopic regions; $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$ and $\theta \in \mathbb{R}^{n_\theta}$, denotes the state, input and unknown parameter vector, respectively. Moreover, A_i, B_i are all constant matrices with appropriate dimensions; $\varphi_i(x) : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x \times n_\theta}$ is a continuous linear or nonlinear function with $\varphi_i(0) = 0$.

As shown in [6], we can construct matrices E_i, F_i for each region R_i such that

$$\begin{cases} E_i x \geq 0, & x \in R_i \\ F_i x = F_j x, & x \in R_i \cap R_j \end{cases} \tag{2}$$

Assumption 1: The proposed PWL systems satisfy the matching condition, i.e., there exist function $\psi_i(x)$ for each region, such that

$$B_i \psi_i(x) = \varphi_i(x), \quad x \in R_i. \tag{3}$$

Let $\hat{\theta}$ be the estimation of θ, our objectives are to design a piecewise control law $u(t) = u_i(x, \hat{\theta}), x \in R_i$ and a piecewise parameter adaptation law $\dot{\hat{\theta}} = \upsilon_i(x, \hat{\theta}), x \in \mathcal{R}_i$, such that the closed-loop PWL system is asymptotically stable, i.e, all the possible state trajectories will converge to origin.

III. Main Results

In this section, we provide the main contribution of the paper. An LMI-based adaptive control approach will be developed for the PWL system (1) based on the PQLF theory [6] and the following two lemmas.

Lemma 1: [22] For all positive definite matrices P, the following inequality holds:

$$G^T P^{-1} G \geq G^T + G - P. \tag{4}$$

Lemma 2: [23] Let Φ be any given positive definite matrix. The following statements are equivalent:

$$\begin{array}{l} (1).\ \Psi + \Xi + \Xi^T < 0; \\ (2).\ \text{There exists a matrix } W \text{ such that} \\ \begin{bmatrix} \Psi + \Phi - (W + W^T) & \Xi^T + W^T \\ * & -\Phi \end{bmatrix} < 0. \end{array} \tag{5}$$

Now, we give the main result of this paper.

Theorem 1: If there exist symmetric matrices T, U_i, W_i and general matrix V_i, R_i, such that U_i, W_i have nonnegative entries and the following LMIs are satisfied with $P_i = F_i^T T F_i$,

$$P_i - E_i^T W_i E_i > 0, \tag{6}$$

$$\begin{bmatrix} -(V_i + V_i^T) & V_i^T A_i^T + R_i^T B_i^T + P_i & V_i^T \\ * & -P_i & 0 \\ * & * & E_i^T U_i E_i - P_i \end{bmatrix} < 0. \tag{7}$$

Then, using the piecewise control law

$$\begin{cases} u = -\psi_i(x)\hat{\theta} + K_i x \\ K_i = -R_i V_i^{-1} \end{cases} \quad x \in \mathcal{R}_i \tag{8}$$

and the piecewise parameter adaptation law

$$\dot{\hat{\theta}} = \varphi_i(x)^T P_i x, \quad x \in \mathcal{R}_i, \tag{9}$$

the resulting closed-loop PWL system is asymptotically stable.

Proof: We choose a piecewise quadratic Lyapunov function

$$V(x, \hat{\theta}) = \sum_i \beta_i [x^T P_i x + \tilde{\theta}^T \tilde{\theta}],\ \beta_i = \begin{cases} 1 & x \in R_i \\ 0 & \text{others} \end{cases} \tag{10}$$

where $\tilde{\theta} = \hat{\theta} - \theta$ denotes the estimation error of unknown parameter θ.

Then, the sufficient conditions for the asymptotical stability are

$$\begin{cases} V(x, \hat{\theta}) \geq 0 \\ \dot{V}(x, \hat{\theta}) < 0 \end{cases} \tag{11}$$

That is because if considering the augmented system (1),(8),(9), and let

$$E = \{(x, \hat{\theta}) \in (X, \mathbb{R}^{n_\theta}) | \dot{V}(x, \hat{\theta}) = 0\} \tag{12}$$

then

$$E = \{(0, \theta_1) | \theta_1 \in \mathbb{R}^{n_\theta}\} \tag{13}$$

and E is a invariant set. Therefore,using LaSalle's theorem[25], the solution of the augmented system $(x(t), \hat{\theta}(t))$ converges to the set E, i.e.,

$$\lim_{t \to \infty} x(t) = 0 \tag{14}$$

Therefore, we just need to show the conditions (11) hold. With the help of (10), we learn that the conditions (11) can be implied by

$$\begin{cases} x^T P_i x > 0 \\ 2\dot{x}^T P_i x + 2\tilde{\theta}^T \dot{\hat{\theta}} < 0 \end{cases} \quad x \in R_i \setminus 0\ . \tag{15}$$

Replacing the piecewise control law and adaptation law by (8–9), the sufficient conditions become

$$\begin{cases} x^T P_i x > 0 \\ 2(\bar{A}_i x + \varphi_i(x)\theta - B_i \psi_i(x)\hat{\theta})^T P_i x + 2\tilde{\theta}^T \varphi_i(x)^T P_i x < 0 \end{cases} \tag{16}$$

where $\bar{A}_i = A_i + B_i K_i$.

By eliminating the terms containing $\tilde{\theta}$, the sufficient conditions can be reformulated as the following inequalities

$$\begin{cases} x^T P_i x > 0 \\ x^T (\bar{A}_i^T P_i + P_i \bar{A}_i) x < 0 \end{cases} \quad x \in R_i\ . \tag{17}$$

These conditions can be relaxed using S-procedure and the polyopic region description (2), yielding the sufficient conditions as

$$\begin{cases} P_i - E_i^T W_i E_i > 0 \\ \bar{A}_i^T P_i + P_i \bar{A}_i + E_i^T U_i E_i < 0 \end{cases} \quad x \in R_i\ . \tag{18}$$

It is noted that the first inequality of the above conditions is same with (6), so our rest work is to show the proof of second inequality using condition (7).

Let $Q_i = P_i^{-1}$, $S_i = E_i^T U_i E_i$, then the second inequality can be written as

$$Q_i \bar{A}_i^T + \bar{A}_i Q_i + Q_i S_i Q_i < 0 \quad x \in R_i. \tag{19}$$

The use of Lemma 2 with $\Psi = Q_i S_i Q_i$ and $\Xi = Q_i \bar{A}_i^T$ yields,

$$\begin{bmatrix} Q_i S_i Q_i + \Phi_i - (W_i + W_i^T) & \bar{A}_i Q_i + W_i^T \\ * & -\Phi_i \end{bmatrix} < 0. \tag{20}$$

By the congruence transformation $\begin{bmatrix} V_i & 0 \\ * & P_i \end{bmatrix}$ with $V_i = W_i^{-1}$, the inequality (20) becomes

$$\begin{bmatrix} V_i^T (P_i^{-1} S_i P_i^{-1} + \Phi_i) V_i - (V_i + V_i^T) & V_i^T \bar{A}_i + P_i \\ * & -P_i \Phi_i P_i \end{bmatrix} < 0. \tag{21}$$

A Schur complement argument on the term $V_i^T (P_i^{-1} S_i P_i^{-1} + \Phi_i) V_i$ shows that, the sufficient condition is equivalent to the following inequality

$$\begin{bmatrix} -(V_i + V_i^T) & V_i^T \bar{A}_i + P_i & V_i^T \\ * & -P_i \Phi_i P_i & 0 \\ * & * & -(P_i^{-1} S_i P_i^{-1} + \Phi_i)^{-1} \end{bmatrix} < 0. \tag{22}$$

The use of Lemma 1 with $\Phi_i = P_i^{-1}$ yields,

$$(P_i^{-1} S_i P_i^{-1} + \Phi_i)^{-1} = P_i (S_i + P_i)^{-1} P_i \geq P_i - S_i. \tag{23}$$

This inequality show that the following condition implies the inequality (22),

$$\begin{bmatrix} -(V_i + V_i^T) & V_i^T \bar{A}_i + P_i & V_i^T \\ * & -P_i & 0 \\ * & * & E_i^T U_i E_i - P_i \end{bmatrix} < 0. \tag{24}$$

Replacing $\bar{A}_i$ by $\bar{A}_i^T$, we can obtain the dual of (25) as

$$\begin{bmatrix} -(V_i+V_i^T) & V_i^T\bar{A}_i^T+P_i & V_i^T \\ * & -P_i & 0 \\ * & * & E_i^TU_iE_i-P_i \end{bmatrix} < 0. \quad (25)$$

substituting $\bar{A}_i = \bar{A}_i + B_iK_i$, $R_i = K_iV_i$ into (25), we obtain the inequality 7. In summary, (6–7) are the sufficient conditions for the the asymptotical stability of the resulting closed-loop PWL SYSTEM, which completes the proof.

Remark 1: Although the proposed piecewise adaption law can only ensure the boundedness of $\hat{\theta}$ rather than asymptotical convergence to the true parameter value θ, the state trajectory will still converge to the origin as the disturbance converges to zero.

IV. Simulation Results

In this section, we consider an epoxy core linear motor drive system with negligible electrical dynamics described in the following [24].

$$\dot{x_1} = x_2 \quad (26)$$
$$M\dot{x_2} = u - \theta x_2 + [f_c - f_c e^{-|\dot{x_1}|}]sgn(\dot{x_1}) \quad (27)$$

where x_1, x_2 are the position and speed of the inertia load, respectively. The control input is u, which ensures the system stable. All the parameter values and their physical meanings are illustrated in Table 1, where $\theta \in [0.2, 0.28]$ is the uncertain parameter.

TABLE I: Simulation parameters

M=0.55	Motor mass (Kg)
f_c=1	the Coulomb friction force
$\theta \in [0.2, 0.28]$	an equivalent viscous friction coefficient

The objective is to design a state feedback controller with the control gain constrain $\|K\|_\infty \leq 2$ that forces the motor to stop at a certain point.

Given the possible initial velocity $x_2 \in [-2, 2]$. The nonlinear function (27) – (28) can be approximated by PWA functions yielding a PWL system with four regions as below.

$$R_1 = \{x|x \in \mathbb{R}^2|x_2 \in [-2,-1]\},$$
$$R_2 = \{x|x \in \mathbb{R}^2|x_2 \in [-1,0]\},$$
$$R_3 = \{x|x \in \mathbb{R}^2|x_2 \in [0,1]\},$$
$$R_4 = \{x|x \in \mathbb{R}^2|x_2 \in [1,2]\}.$$

Moreover, the system matrix A_i is obtained by its PWA approximation, and

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0.6645 & -0.2690 \\ 0 & 0 & 0 \end{bmatrix} \quad (28)$$
$$A_2 = A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1.8182 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$A_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0.6645 & 0.2690 \\ 0 & 0 & 0 \end{bmatrix}$$
$$B_i = \begin{bmatrix} 0 \\ 1.8182 \\ 0 \end{bmatrix}, \quad i \in \{1,2,3,4\}$$

such as

$$\dot{\overline{x}} = A_i\overline{x} + B_iu + \varphi_i\theta \quad (29)$$
$$where\ \overline{x} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \ and\ \varphi_i(x) = \begin{bmatrix} 0 \\ -x_2 \\ 0 \end{bmatrix}. \quad (30)$$

Thus, the Assumption 1 is satisfied, where $\psi_i(x)$=$-0.55x_2$. Therefore, the following piecewise control law and parameter adaptation law are employed,

$$\begin{cases} u = 2\overline{x}_2\hat{\theta} + K_i\overline{x} \\ \dot{\hat{\theta}} = \varphi_i(\overline{x})^TP_i\overline{x} \end{cases} \quad x \in R_i\ , \quad (31)$$

where K_i and P_i can be obtained by solving the LMIs proposed in Theorem 1. The solutions are obtained by MATLAB as below.

$$K_1 = \begin{bmatrix} -0.7003 & -1.1917 & 0.2382 \end{bmatrix} \quad (32)$$
$$K_2 = \begin{bmatrix} -0.6106 & -1.9195 & 0.0118 \end{bmatrix}$$

$$K_3 = \begin{bmatrix} -0.5024 & -1.8053 & -0.0158 \end{bmatrix} \quad (33)$$
$$K_4 = \begin{bmatrix} -0.4098 & -0.8887 & -0.1969 \end{bmatrix}$$

Simulations have been carried out with the initial condition $[x_1, x_2]$=$[-2, -1.5]$ and $\hat{\theta}(0)$=0.24. Fig.1 shows that, the state trajectory of closed-loop PWL system converges to the origin, which illustrates the efficacy of the proposed LMI-based adaptive control method.

For comparison, we try to design the state feedback controller using the BMI-based adaptive control method [21] for the same PWL system, no feasible solution can be found using YALMIP, which illustrates the advantage of the proposed LMI-based adaptive control approach.

V. A Remark on Sliding Motion

Actually, by extending the idea of [6], if there exists a quadratic Lyapunov function $V_l(x, \tilde{\theta}) \geq 0$ for each boundary

$l = R_i \cap R_j$, such that for all $x \in l$,

$$V_l(x, \tilde{\theta}) \geq 0 \tag{34}$$

$$\frac{\partial V_l}{\partial x} \bar{co}\{A_{\tau k}x + B_\tau u + \varphi_\tau(x)\theta\} + \frac{\partial V_l}{\partial \tilde{\theta}}\dot{\tilde{\theta}} < 0, \tau = i, j \tag{35}$$

the asymptotical stability can be guaranteed, although the sliding motion may happen. Note that for all $x \in l$, $E_l x = 0$, then using similar procedure in the proof of Theorem 1, the constraints (34-35) can also be expressed as LMIs.

VI. Conclusion

In this paper, a LMI-based adaptive control method is developed for uncertain PWL systems. By constructing proper piecewise model compensator and online parameter adaptation law, the adaptive control design problem can be converted to the control design problem for a nominal PWL system without parameter uncertainties. Furthermore, the stabilization problem for the obtained nominal PWL system are formulated as optimization problems subject to a set of LMIs, which are numerical solvable by the existing software. However, the proposed method still has a limitation that the parametric uncertainty of the PWL systems must satisfy matching condition, which will restrict the application of the proposed synthesis method. Therefore, our future work will focus on how to get rid of this assumption to achieve wider applications.

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