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Stability of the Observer-Based Pole Placement for Discrete Time-Varying Non-Lexicographically-Fixed Systems

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***Abstract*—This paper concerns the observer-based pole placement control for MIMO time varying non-lexicographically-fixed discrete systems. If both of the reachability indices and the observability indices are non-lexicographically-fixed, augmented plant equation and augmented observer are needed. Design procedure of this control system is proposed and the stability and the separation principle of the total closed loop system is also shown.**

Keywords - Pole Placement; Observer; Time Varying System; Descrete System; Non-Lexicographically-Fixed System

I. Introduction

It is well known that the pole placement control can be designed for linear time-varying system by using the controllability canonical form as in the time-invariant case [5], [6]. The linear time-varying multivariable system whose controllability indices or observability indices are not constant is called the non-lexicographically-fixed system. Valasec et. al. [7] proposed the pole placement design method for such a system by augmenting the system equation so that the augmented system is lexicographically-fixed. This paper concerns the pole placement and the observer design method for linear time-varying discrete non-lexicographically-fixed system. Using the Valasec's idea, the procedure to extend a discrete non-lexicographically-fixed system to a lexicographically-fixed augmented system will be presented. Then, the simple pole placement technique can be applied to the augmented system without transforming the system into any canonical form [12]. Further, using the property of the anti-causal dual system, it will be shown that the same design method can be used for the augmented observer for non-lexicographically-fixed systems. Finally, as for the time-invariant case, the stability and the separation principle of the total closed loop system are also shown for the case where both of the augmented pole placement controller and the augmented observer are used.

II. Preliminaries

Consider the following linear time-varying m-input p-output MIMO discrete system.

$$x(k+1) = A(k)x(k) + B(k)u(k) \tag{1}$$
$$y(k) = C(k)x(k) \tag{2}$$

where $x \in R^n$, $u \in R^m$, and $y \in R^p$ are the state variable, the input and output, respectively. $A(k) \in R^{n\times n}$, $B(k) \in R^{n\times m}$ and $C(k) \in R^{p\times n}$ are time-varying coefficient matrices.

Definition 1: System (1) is called "completely reachable in n steps" if for any $x_1 \in R^n$ there exists a bounded input $u(j)$ $(j = k, \cdots, k+n-1)$ such that $x(k) = 0$ and $x(k+n) = x_1$ for all k.

The reachability matrix, $R(k)$, of this system is defined by

$$\begin{aligned} R(k) \;=\; & \big[\; b_0^1(k) \;\; \cdots \;\; b_0^m(k) \,\big|\, \cdots \\ & \cdots \,\big|\, b_{n-1}^1(k) \;\; \cdots \;\; b_{n-1}^m(k) \;\big] \end{aligned} \tag{3}$$

Here, $b_i^l(k) \in R^n$ is calculated by the following recurrence equations.

$$\begin{aligned} b_0^l(k) &= b^l(k+n-1) \\ b_{i+1}^l(k) &= A(k+n-1)b_i^l(k-1) \\ & \quad (i = 0, \cdots, n-2, \quad l = 1, \cdots, m\;) \end{aligned} \tag{4}$$

where, $b^l(k) \in R^n$ is the l-th column of $B(k)$.

If the system (1) is completely reachable in n steps, the rank of $R(k)$ is n, from which we can define the nonsingular $n \times n$ matrix $\bar{R}(k)$ using the reachability indices $\mu_i (i = 1, 2, \cdots, m)$ as follows.

$$\begin{aligned} \bar{R}(k) \;=\; & \big[\; b_0^1(k), \cdots, b_{\mu_1-1}^1(k) | \cdots \\ & \cdots | b_0^m(k), \cdots, b_{\mu_m-1}^m(k) \;\big] \end{aligned} \tag{5}$$

The reachability indices satisfy that $\sum_{i=1}^m \mu_i = n$ and are assumed that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ without loss of generallity.

Here, we state the definition of the observability in n steps as a dual concept of the reachability in n steps.

Definition 2: The system (1), (2) is said to be completely observable in n steps if for any k, $x(k)$ is determined uniquely from $y(k)$, $y(k+1)$, $\cdots$, $y(k+n-1)$.

The following steps are the pole placement control design procedure proposed by the authors in [11] without using a transformation into any canonical form.

STEP 1 Check the reachability of the system (1) and obtain $\bar{R}(k)$ and μ_i $(i=1,\cdots,m)$.

STEP 2

Calculate the new output signal, $\tilde{y}(k)$, by the following equation, so that the relative degree from $u(k)$ to $\tilde{y}(k)$ is the system degree, n.

$$\tilde{y}(k) = \tilde{C}(k)x(k) = W\bar{R}^{-1}(k-n)x(k) \tag{6}$$

where W is the matrix defined by the following.

$$\begin{aligned} W &= diag(w_1, w_2, \cdots, w_m) \\ w_i &= \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \in R^{1\times\mu_i} \end{aligned} \tag{7}$$

STEP 3 Let $q^i(z)$ be the ideal and stable characteristic polynomial for the closed-loop system of degree μ_i, i.e.,

$$q^i(z) = z^{\mu_i} + \alpha^i_{\mu_i-1}z^{\mu_i-1} + \cdots + \alpha^i_1 z + \alpha^i_0 \tag{8}$$

Here, z is the shift operator. Then, we have the following equation [11].

$$\begin{bmatrix} q_1(z) & & \\ & \ddots & \\ & & q_m(z) \end{bmatrix} \tilde{y}(k) = F(k)x(k) + \Lambda(k)u(k) \tag{9}$$

where $\Lambda(k) \in R^{m\times m}$ is nonsingular. (See Appendix.)

STEP 4 From (9), the state feedback

$$u(k) = D(k)x(k) = -\Lambda^{-1}(k)F(k)x(k) \tag{10}$$

makes the closed loop system

$$\begin{bmatrix} q_1(z) & & \\ & \ddots & \\ & & q_m(z) \end{bmatrix} \tilde{y}(k) = 0 \tag{11}$$

This implies that the closed loop state equation

$$x(k+1) = \{A(k) + B(k)D(k)\}x(k) \tag{12}$$

is equivalent to the time invariant system with desired poles, i.e., there exists some transformation matrix, $P(k)$, that satisfies the following equation.

$$P(k+1)\{A(k) + B(k)D(k)\}P^{-1}(k) = A^* \tag{13}$$

where

$$\det(zI - A^*) = \prod_{i=1}^{m} q^i(z) \tag{14}$$

Then, if the matrix $P(k)$ is the Lyapunov transformation, the closed loop system is stable and equivalent to some time-invariant system that has the ideal and stable eigen values.

III. Pole Placement of Non-Lexicographically-Fixed Systems

In the previous section, the reachability indices are supposed to be fixed. Such indices are said to be lexicographically-fixed. However, since the system has time-varying parameters, the reachability indices might be variable as well. Such indices are said to be non-lexicographically-fixed. In this section, we consider the pole placement control designing procedure for a system with non-lexicographically-fixed indices. Valasek et. al. proposed the pole placement design method for non-lexicographically-fixed multivariable continuous systems in [8]. In this paper, we apply this idea to the discrete system together with the new pole placement technique stated in the previous section.

Suppose that the system (1) is completely reachable in n steps, and has non-lexicographically-fixed reachability indices. It is assumed that the maximum value of each reachability index μ_i is known, i.e.,

$$v_i = \max_k \mu_i(k) \qquad (i=1,\cdots,m) \tag{15}$$

Using v_i, we define n_g by

$$n_g = \sum_{i=1}^{m} v_i \tag{16}$$

Define the augmented system by

$$x_g(k+1) = A_g(k)x_g(k) + B_g(k)u(k) \tag{17}$$

$$\begin{cases} x_g(k) = \begin{bmatrix} x(k) \\ x_e(k) \end{bmatrix} \\ A_g(k) = \begin{bmatrix} A(k) & 0 \\ A_2(k) & A_1(k) \end{bmatrix}, B_g(k) = \begin{bmatrix} B(k) \\ B_e(k) \end{bmatrix} \end{cases}$$

where $x_g \in R^{n_g}$ and $x_e \in R^{n_g-n}$. $A_1(k) \in R^{(n_g-n)\times(n_g-n)}$, $A_2(k) \in R^{(n_g-n)\times n}$, and $B_e(k) \in R^{(n_g-n)\times m}$ are design parameter matrices so that the above augmented system has lexicographically-fixed reachability indices, v_i $(i=1,\cdots,m)$.

The reachability matrix $R_g(k)$ of this augmented system is

$$\begin{aligned} R_g(k) = \big[\ & {b_g}^1_0(k) \quad \cdots \quad {b_g}^m_0(k) \ \big| \cdots \\ & \cdots \big| \ {b_g}^1_{n_g-1}(k) \quad \cdots \quad {b_g}^m_{n_g-1}(k) \ \big] \end{aligned} \tag{18}$$

where ${b_g}^l_i(k) \in R^{n_g}$ is defined by the following recurrence equations.

$$\begin{aligned} {b_g}^l_0(k) &= {b_g}^l(k+n_g-1) \\ {b_g}^l_{i+1}(k) &= A_g(k+n_g-1){b_g}^l_i(k-1) \\ & \qquad (i=0,\cdots,n_g-2, \quad l=1,\cdots,m\ ) \end{aligned} \tag{19}$$

Here, ${b_g}^l(k) \in R^{n_g}$ is the l-th column of $B_g(k)$.

For the augmented system to have lexicographically-fixed reachability indices, v_i, the following $n_g \times n_g$ matrix $\bar{R}_g(k)$ should be nonsingular for all k.

$$\begin{aligned} \bar{R}_g(k) &= \big[\ {b_g}^1_0(k), \cdots, {b_g}^1_{v_1-1}(k) | \cdots \\ & \qquad \cdots | {b_g}^m_0(k), \cdots, {b_g}^m_{v_m-1}(k)\ \big] \end{aligned} \tag{20}$$

On the other hand, $\bar{R}_g(k)$ can be written as

$$\bar{R}_g(k) = \begin{bmatrix} \bar{R}_v(k) \\ \bar{R}_e(k) \end{bmatrix} \tag{21}$$

where $\bar{R}_v(k) \in R^{n \times n_g}$ and $\bar{R}_e(k) \in R^{(n_g - n) \times n_g}$ are

$$\begin{aligned} \bar{R}_v(k) &= [\ b_0^1(k), \cdots, b_{v_1-1}^1(k) | \cdots \\ &\qquad \cdots | b_0^m(k), \cdots, b_{v_m-1}^m\ ] \end{aligned} \tag{22}$$

$$\begin{aligned} \bar{R}_e(k) &= [\ r_e{}_0^1(k), \cdots, r_e{}_{v_1-1}^1(k) | \cdots \\ &\qquad \cdots | r_e{}_0^m(k), \cdots, r_e{}_{v_m-1}^m(k)\ ] \end{aligned} \tag{23}$$

Since, from the assumption, the rank of $\bar{R}_v(k)$ is n, there exists a matrix, $\bar{R}_e(k)$, such that $\bar{R}_g(k)$ is nonsingular for all k. The problem is to find $A_1(k)$, $A_2(k)$, and $B_e(k)$ that give such $r_e{}_i^l(k) \in R^{n_g - n}$.

From (20)-(23), we have

$$b_g{}_0^l(k) = \begin{bmatrix} b_0^l(k) \\ r_e{}_0^l(k) \end{bmatrix} \tag{24}$$

then, using (17) and (24), the recurrence equation (19) can be modified as follows.

$$b_g{}_0^l(k) = \begin{bmatrix} b_0^l(k) \\ r_e{}_0^l(k) \end{bmatrix} = \begin{bmatrix} b^l(k + n_g - 1) \\ b_e{}^l(k + n_g - 1) \end{bmatrix}$$

$$\begin{aligned} b_g{}_{i+1}^l(k) &= \begin{bmatrix} b_{i+1}^l(k) \\ r_e{}_{i+1}^l(k) \end{bmatrix} \\ &= \begin{bmatrix} A(k+n_g-1) & 0 \\ A_2(k+n_g-1) & A_1(k+n_g-1) \end{bmatrix} b_g{}_i^l(k-1) \\ &= \begin{bmatrix} A(k+n_g-1) b_i^l(k-1) \\ [\ A_2(k+n_g-1) \quad A_1(k+n_g-1)\ ]\, b_g{}_i^l(k-1) \end{bmatrix} \\ &\qquad (i = 0, 1, \cdots, \quad l = 1, \cdots, m\ ) \end{aligned} \tag{25}$$

Here, $b_e{}^l(k)$ is the l-th column of $B_e(k)$. From (25), the relation between $r_e{}_i^l(k)$ and $A_1(k)$, $A_2(k)$, and $B_e(k)$ is obtained as follows.

$$\begin{aligned} B_e(k + n_g - 1) &= [\ r_e{}_0^1(k) \quad \cdots, \quad r_e{}_0^m(k)\ ] \\ [\ A_2(k+n_g-1) \quad & A_1(k+n_g-1)\ ]\, \bar{R}_g(k-1) \\ &= \bar{R}_{e+}(k) \end{aligned} \tag{26}$$

where $\bar{R}_{e+}(k)$ is defined by

$$\begin{aligned} \bar{R}_{e+}(k) = [\ r_e{}_1^1(k) \quad \cdots \quad r_e{}_{v_1}^1(k) | \cdots \\ \cdots\ | r_e{}_1^m(k) \quad \cdots \quad r_e{}_{v_m}^m(k)\ ] \end{aligned} \tag{27}$$

From the above, design parameter matrices such that the augmented plant (17) has lexicographically-fixed reachability indices, $v_i (i = 1, \cdots, m)$, can be calculated as follows. First, determine $\bar{R}_e(k)$ so that $\bar{R}_g(k)$ is nonsingular for all k. Then, using arbitrarily determined parameters $r_e{}_{v_1}^1(k), \cdots, r_e{}_{v_m}^m(k)$ in (26) and (27), and then, $A_1(k)$, $A_2(k)$ and $B_e(k)$ are obtained by

$$\begin{aligned} B_e(k) = [\ r_e{}_0^1(k - n_g + 1) \quad \cdots, \quad r_e{}_0^m(k - n_g + 1)\ ] \\ [\ A_2(k) \quad A_1(k)\ ] = \bar{R}_{e+}(k - n_g + 1) \bar{R}_g^{-1}(k - n_g) \end{aligned} \tag{28}$$

The state feedback for the pole placement can be obtained as the following form by applying the pole placement design procedure stated in the previous section to this augmented system.

$$u(k) = [D_x(k),\ D_e(k)] \begin{bmatrix} x(k) \\ x_e(k) \end{bmatrix} = D_g(k) x_g(k) \tag{29}$$

This implies that there exists the time-varying transformation matrix $P_g(k) \in R^{n_g \times n_g}$ that satisfies

$$P_g(k+1)\{A_g(k) + B_g(k) D_g(k)\} P_g^{-1}(k) = A_g^* \tag{30}$$

Hence, if the transformation matrix $P_g(k)$ is the Lyapunov transformation, the closed loop system is stable and equivalent to some time-invariant system that has desired and stable constant eigenvalues.

IV. Observer of Non-Lexicographically-Fixed Systems

In this section, we consider the design of the observer for the system that has non-lexicographically-fixed observability indices. Suppose that the system (1),(2) is completely observable in n steps and has observability indices, $\nu_i (i = 1, \cdots, p)$, which are non-lexicographically-fixed. Further, it is assumed that the following d_i are known.

$$d_i = \max_k \nu_i(k) \qquad (i = 1, \cdots, p) \tag{31}$$

Using these d_i, we define n_s by

$$n_s = \sum_{i=1}^{m} d_i \tag{32}$$

If the system has lexicographically-fixed observability indices, its observer can be written as follows.

$$\begin{aligned} \hat{x}(k+1) &= A(k)\hat{x}(k) + B(k)u(k) \\ &\quad - H(k)(y(k) - C(k)\hat{x}(k)) \end{aligned} \tag{33}$$

where $\hat{x}(k) \in R^n$ is the state estimation of $x(t)$. Then, the problem is to find the observer gain matrix $H(k) \in R^{n \times p}$. But, since the observability indices are non-lexicographically-fixed, we augment the observer system as follows.

$$\begin{aligned} \hat{x}(k+1) &= A(k)\hat{x}(k) + B(k)u(k) \\ &\quad - H(k)(y(k) - C(k)\hat{x}(k)) \\ &\quad - (A_4(k) + H(k)C_e(k))\epsilon(k) \end{aligned} \tag{34}$$

$$\begin{aligned} \epsilon(k+1) &= A_3(k)\epsilon(k) + H_e(k)(y(k) - C(k)\hat{x}(k)) \\ &\quad + H_e(k)C_e(k)\epsilon(k) \end{aligned} \tag{35}$$

Here, $\epsilon(k) \in R^{n_s - n}$ is an auxiliary signal and $A_3(k) \in R^{(n_s-n)\times(n_s-n)}$, $A_4(k) \in R^{n \times (n_s - n)}$, and $C_e(k) \in R^{p \times (n_s - n)}$ are design parameter matrices determined later. Using the state estimation error, $e(k) = x(k) - \hat{x}(k)$ the following state error equation is obtained from (1), (2), (34), and (35).

$$e_s(k+1) = A_s(k)e_s(k) + H_s(k)C_s(k)e_s(k) \tag{36}$$

where

$$\begin{cases} e_s(k) = \begin{bmatrix} e(k) \\ \epsilon(k) \end{bmatrix}, A_s(k) = \begin{bmatrix} A(k) & A_4(k) \\ 0 & A_3(k) \end{bmatrix} \\ H_s(k) = \begin{bmatrix} H(k) \\ H_e(k) \end{bmatrix}, C_s(k) = \begin{bmatrix} C(k) & C_e(k) \end{bmatrix} \end{cases} \tag{37}$$

From this, the observer design problem is to find $H_s(k)$ so that $A_s(k) + H_s(k)C_s(k)$ is equivalent to some constant matrix which has desired constant eigenvalues.

For this purpose, consider the following anti-causal system as a dual system of the system $(A_s(k),\ C_s(k))$.

$$\begin{aligned} \xi_s(k-1) &= A_s^T(k)\xi_s(k) + C_s^T(k)v(k) \\ A_s^T &= \begin{bmatrix} A^T(k) & 0 \\ A_4^T(k) & A_3^T(k) \end{bmatrix},\ C_s^T(k) = \begin{bmatrix} C^T(k) \\ C_e^T(k) \end{bmatrix} \end{aligned} \tag{38}$$

Since the system $(A(k),C(k))$ is completely observable in n steps and has the observability indices, $\nu_i (i = 1, \cdots, p)$, its dual system is completely reachable in n steps and has reachability indices, ν_i.

The reachability matrix, $R_s(k)$, of the augmented dual system (38) can be written as

$$\begin{aligned} R_s(k) = [\ & {c_s}_0^1(k) \quad \cdots \quad {c_s}_0^m(k) \mid \cdots \\ & \cdots \mid {c_s}_{n_s-1}^1(k) \quad \cdots \quad {c_s}_{n_s-1}^m(k)\ ] \end{aligned} \tag{39}$$

where ${c_s}_i^l(k) \in R^{n_s}$ is defined by the following recurrence equations.

$$\begin{aligned} {c_s}_0^l(k) &= {c_s}^l(k - n_s + 1) \\ {c_s}_{i+1}^l(k) &= A_s^T(k - n_s + 1){c_s}_i^l(k+1) \\ & \quad (i = 0, \cdots, n_s - 2, \quad l = 1, \cdots, m\ ) \end{aligned} \tag{40}$$

Here, ${c_s}^l(k) \in R^{n_s}$ is the l-th column of $C_s^T(k)$.

For the augmented system to have lexicographically-fixed reachability indices, d_i, the following $n_s \times n_s$ matrix, $\bar{R}_s(k)$, should be nonsingular for all k.

$$\begin{aligned} \bar{R}_s(k) &= [\ {c_s}_0^1(k), \cdots, {c_s}_{d_1-1}^1(k) | \cdots \\ & \qquad \cdots | {c_s}_0^m(k), \cdots, {c_s}_{d_m-1}^m(k)\ ] \end{aligned} \tag{41}$$

$\bar{R}_s(k)$ can be written as

$$\bar{R}_s(k) = \begin{bmatrix} \bar{R}_d(k) \\ \bar{R}_h(k) \end{bmatrix} \tag{42}$$

where, $\bar{R}_d(k) \in R^{n \times n_s}$ and $\bar{R}_h(k) \in R^{(n_s - n) \times n_s}$ are defined by

$$\begin{aligned} \bar{R}_d(k) &= [\ c_0^1(k), \cdots, c_{v_1-1}^1(k) | \cdots \\ & \qquad \cdots | c_0^m(k), \cdots, c_{d_m-1}^m\ ] \end{aligned} \tag{43}$$

$$\begin{aligned} \bar{R}_h(k) &= [\ {r_h}_0^1(k), \cdots, {r_h}_{d_1-1}^1(k) | \cdots \\ & \qquad \cdots | {r_h}_0^m(k), \cdots, {r_h}_{d_m-1}^m(k)\ ] \end{aligned} \tag{44}$$

Since, the anti-causal dual system is reachable in n steps, the rank of $\bar{R}_d(k)$ is n, and, hence, there always exists the matrix, $\bar{R}_h(k)$, such that the rank of $\bar{R}_s(k)$ is n_s for all k. Thus, as the previous section, $C_e^T(k)$, $A_3^T(k)$, and $A_4^T(k)$ can be obtained by

$$\begin{aligned} C_e^T(k) &= [\ {r_h}_1^0(k + n_s - 1) \quad \cdots \quad {r_h}_m^0(k + n_s - 1)\ ] \\ [\ A_4^T(k) \quad & A_3^T(k)\ ] = \bar{R}_{h+}(k + n_s - 1)\bar{R}_s^{-1}(k + n_s). \end{aligned} \tag{45}$$

Here, $\bar{R}_{h+}(k)$ is defined by

$$\begin{aligned} \bar{R}_{h+}(k) = [\ & {r_h}_1^1(k) \quad \cdots \quad {r_h}_{d_1}^1(k) | \cdots \\ & \cdots | {r_h}_1^m(k) \quad \cdots \quad {r_h}_{d_m}^m(k)\ ] \end{aligned} \tag{46}$$

where ${r_h}_{d_1}^1(k), \cdots, {r_h}_{d_m}^m(k)$ are arbitrarily determined parameters.

From the above, the anti-causal augmented dual system, (38), has lexicographically-fixed reachability indices, d_i. Thus, using the pole placement technique stated in the section 2, the matrix $H_s^T(k)$ can be obtained so that $A_s^T(k) + C_s^T(k)H_s^T(k)$ is equivalent to some constant matrix A_o^{*T} which has desired constant eigenvalues. i.e., there exists some transformation matrix, $Q_s(k) \in R^{n_s \times n_s}$, such that

$$Q_s(k+1)\{A_s(k) + C_s(k)H_s(k)\}Q_s^{-1}(k) = A_o^* \tag{47}$$

Hence, if $Q_s(k)$ is the Lyapunov transformation, (34) and (35) becomes the augmented observer, and $e(k)$ and $\epsilon(k)$ converge to 0.

V. Stability of the Total Closed Loop

If the system (1), (2) has both of non-lexicographically-fixed reachability indices and non-lexicographically-fixed observability indices, the augmented plant and the augmented observer are needed for the observer based pole placement. In this section, for such a system, the stability of the total closed loop system and the separation principle are considered.

The augmented plant is (17), and the augmented observer is (34), (35). Then, for the observer based pole placement, the state feedback (29) is modified to

$$u(k) = \begin{bmatrix} D_x(k) & D_e(k) \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{x}_e(k) \end{bmatrix} \tag{48}$$

where $\hat{x}(k)$ is the state estimation. In this state feedback, $\hat{x}_e(k)$ is used instead of $x_e(k)$, because, in the second equation of (17), $x(t)$ should be replaced by $\hat{x}(k)$.

Hence, the total closed loop system for this case becomes as follows.

$$\begin{bmatrix} x(k+1) \\ \hat{x}_e(k+1) \\ \hat{x}(k+1) \\ \epsilon(k+1) \end{bmatrix} = \begin{bmatrix} A & BD_e & BD_x & 0 \\ 0 & A_1 + B_eD_e & A_2 + B_eD_x & 0 \\ -HC & BD_e & A + BD_x + HC & -A_4 - HC_e \\ H_eC & 0 & -H_eC & A_3 + H_eC_e \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}_e(k) \\ \hat{x}(k) \\ \epsilon(k) \end{bmatrix} \tag{49}$$

Using the transformation matrix

$$T = \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline I & 0 & -I & 0 \\ 0 & 0 & 0 & I \end{array}\right] \tag{50}$$

the total system (49) is transformed into

$$\begin{bmatrix} x(k+1) \\ \hat{x}_e(k+1) \\ e(k+1) \\ \epsilon(k+1) \end{bmatrix} = \begin{bmatrix} A+BD_x & BD_e & -BD_x & 0 \\ A_2+B_eD_x & A_1+B_eD_e & -A_2-B_eD_x & 0 \\ 0 & 0 & A+HC & A_4+HC_e \\ 0 & 0 & H_eC & A_3+H_eC_e \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}_e(k) \\ e(k) \\ \epsilon(k) \end{bmatrix}$$

$$= \begin{bmatrix} A_g+B_gD_g & E \\ 0 & A_s+C_sH_s \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}_e(k) \\ e(k) \\ \epsilon(k) \end{bmatrix} \tag{51}$$

where

$$E(k) = \begin{bmatrix} -B(k)D_x(k) & 0 \\ -A_2(k)+B_e(k)D_x(k) & 0 \end{bmatrix}. \tag{52}$$

In (49) and (51), the symbol "(k)" is omitted because of the small space.

From the above, using the transformation matrix

$$\Phi(k) = \begin{bmatrix} P_g(k) & 0 \\ 0 & Q_s(k) \end{bmatrix} \tag{53}$$

the following relation is obtained.

$$\Phi(k+1)\begin{bmatrix} A_g(k) & E(k) \\ 0 & A_s(k) \end{bmatrix}\Phi^{-1}(k) = \begin{bmatrix} A^* & P_g(k+1)E(k)Q_s^{-1}(k) \\ 0 & A_o^* \end{bmatrix} \tag{54}$$

Thus, since the system matrix of (49) is equivalent to the right hand side of (54), if. $P_g(k)$ and $Q_s(k)$ are the Lyapunov transformation matrices, the total closed system is stable and has a property of the separation principle.

VI. Numerical Example

Consider the system (1), (2) with

$$A(k) = \begin{bmatrix} 2\cos(1.5k) & 0 & 0 \\ 2\sin(1.5(k-1)) & 0 & -2 \\ 2\sin(1.5k) & 2\cos(1.5k) & 0 \end{bmatrix} \tag{55}$$

$$B(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \sin(1.5k) & 0 \end{bmatrix} \tag{56}$$

$$C(k) = \begin{bmatrix} \frac{1}{2}\cos(1.5(k-1)) & 0 & \frac{1}{2}\cos(1.5(k-1)) \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \tag{57}$$

This system has non-lexicographically-fixed reachability indices and non-lexicographically-fixed observability indices. We design the observer based pole placement for this system. Because of the small space, we use the following symbols, i.e., $S = \sin(1.5k), S_1 = \sin(1.5(k-1)), C = \cos(1.5k), C_1 = \cos(1.5(k-1))$

The reachability indices of this system is $\mu_1 = 2$, $\mu_2 = 1$ or $\mu_1 = 1$, $\mu_2 = 2$. From this $v_1 = v_2 = 2$. The Augmented plant equation (17) becomes

$$x_g(k+1) = A_g(k)x_g(k) + B_g(k)u(k) \tag{58}$$

$$\begin{cases} x_g(k) = \begin{bmatrix} x(k) \\ x_e(k) \end{bmatrix} \\ A_g(k) = \begin{bmatrix} A(k) & 0 \\ A_2(k) & A_1(k) \end{bmatrix}, B_g(k) = \begin{bmatrix} B(k) \\ B_e(k) \end{bmatrix} \end{cases}$$

where $x_g(k) \in R^4$ and $x_e(k) \in R^1$, and

$$B_e(k) = \begin{bmatrix} -1 & 0 \end{bmatrix} \tag{59}$$

$$\begin{bmatrix} A_2(k) & A_1(k) \end{bmatrix} = \begin{bmatrix} 2S^2S_1^2(C_1+1) & -2S & -2C_1S^2S_1 & -2C_1^2S^2 \end{bmatrix}. \tag{60}$$

On the other hand, the observability indices of this system is also $\nu_1 = 2$, $\nu_2 = 1$ or $\nu_1 = 1$, $\nu_2 = 2$. Then, the augmented observer becomes

$$\begin{aligned} \hat{x}(k+1) &= A(k)\hat{x}(k) + B(k)u(k) \\ &\quad -H(k)(y(k) - C(k)\hat{x}(k)) \\ &\quad -(A_4(k) + H(k)C_e(k))\epsilon(k) \end{aligned} \tag{61}$$

$$\begin{aligned} \epsilon(k+1) &= A_3(k)\epsilon(k) + H_e(k)(y(k) - C(k)\hat{x}(k)) \\ &\quad +H_e(k)C_e(k)\epsilon(k). \end{aligned} \tag{62}$$

Here, $\epsilon(k) \in R^1$ and

$$C_e^T(k) = \begin{bmatrix} 2 & 0 \end{bmatrix} \tag{63}$$

$$\begin{bmatrix} A_4^T(k) & A_3^T(k) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}. \tag{64}$$

Using the following desired stable characteristic polynomial for both of the pole placement and the observer

$$q^1(z) = \alpha_2^1 z^2 + \alpha_1^1 z + \alpha_0^1 = z^2 + 0.4z - 0.05 \tag{65}$$

$$q^2(z) = \alpha_2^2 z^2 + \alpha_1^2 z + \alpha_0^2 = z^2 + 0.4z - 0.05 \tag{66}$$

the simulation results are shown in Fig.1 and Fig.2. Fig.1 shows the response of the augmented system, $[x(k), \hat{x}_e(k)] = [x_1(k), x_2(k), x_3(k), \hat{x}_e(k)]$, and Fig.2 shows the response of augmented state estimation error, $[e(k), \epsilon(k)] = [e_1(k), e_2(k), e_3(k), \epsilon(k)]$.

VII. Conclutions

In this paper, the design procedure of the observer-based pole placement for linear time-varying MIMO systems is proposed. Especially, the system is supposed to have the non-lexicographically-fixed reachability indices and non-lexicographically-fixed observability indices. The total closed loop stability and the separation principle are also established.

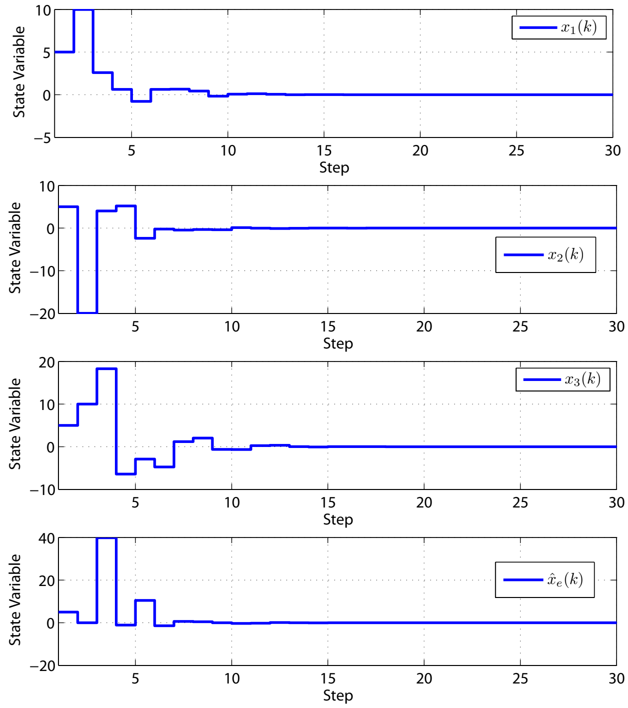

Fig. 1. Response of the Augmented Plant of the Observer-Based Pole Placement Control for the non-lexicographically-fixed System

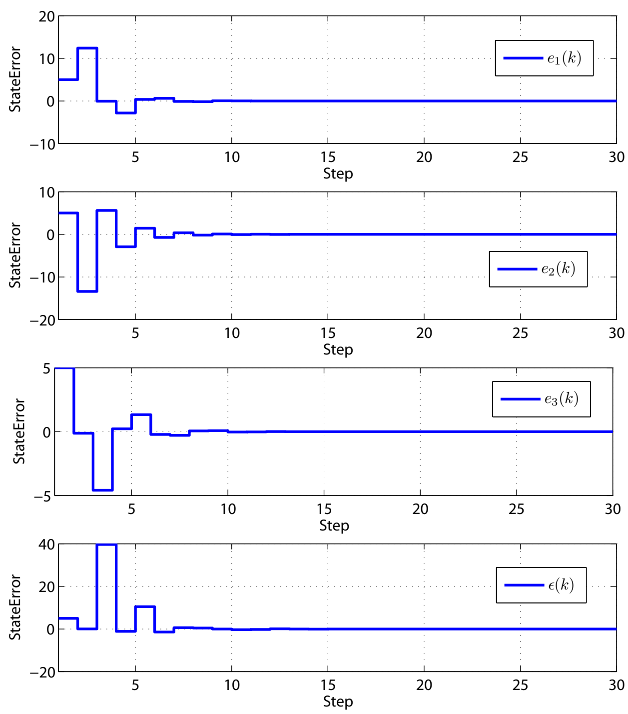

Fig. 2. Response of the Augmented State Estimation Error of the Observer-Based Pole Placement Control for the non-lexicographically-fixed System

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Appendix

Let $\tilde{c}_i^T(k)$ be the i-th row of $\tilde{C}(k)$. Then, we define $\tilde{c}_i^{lT}(k)$ by the following for $i = 1, \cdots, m$.

$$\begin{aligned} \tilde{c}_i^{0T}(k) &= \tilde{c}_i^T(k) \\ \tilde{c}_i^{(l+1)T}(k) &= \tilde{c}_i^{lT}(k+1)A(k) \end{aligned}$$

Using this, $F(k)$ is calculated by

$$F(k) = \begin{bmatrix} F_1^T(k) \\ \vdots \\ F_m^T(k) \end{bmatrix}$$

where

$$F_i^T(k) = [\alpha_0^i, \alpha_1^i, \cdots, \alpha_{\mu_i - 1}^i, 1] \begin{bmatrix} \tilde{c}_i^{0T}(k) \\ \tilde{c}_i^{1T}(k) \\ \vdots \\ \tilde{c}_i^{\mu_i T}(k) \end{bmatrix}.$$

$\Lambda(k)$ is calculated as

$$\Lambda(k) = \begin{bmatrix} 1 & \gamma_{12}(k) & \gamma_{13}(k) & \cdots & \gamma_{1m}(k) \\ 0 & 1 & \gamma_{23}(k) & \cdots & \gamma_{2m}(k) \\ 0 & 0 & 1 & \cdots & \gamma_{3m}(k) \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \end{bmatrix}$$

where $\gamma_{ij} = \tilde{c}_i^{(\mu_i - 1)T} b^j(k) \quad (j \geq i+1)$.