PERSISTENTLY EXCITING MODEL PREDICTIVE CONTROL USING FIR MODELS

G. Marafioti*, R.R Bitmead, and M. Hovd***

**Department of Engineering Cybernetics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway.*
e-mail: {giancarlo.marafioti,morten.hovd}@itk.ntnu.no
***Department of Mechanical and Aerospace Engineering, University of California, San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0411, USA.*
e-mail: rbitmead@ucsd.edu

Abstract. Model Predictive Control (MPC) is a well known and widely used advanced optimal control technique. It is a common practice to use a process model to predict the future behavior of the plant. The model/plant mismatch may have direct consequences on the quality of the prediction, causing potential controller performance degradation. An approach to tackle this problem may be to use closed loop system identification such that the model parameters are estimated and updated online, while the feedback controller is running. This adaptation requires that the plant input is persistently exciting, however a standard MPC controller is not able to provide sufficient input frequency content to obtain reliable parameter estimates. In this article a Persistently Exciting Model Predictive Control (PE-MPC) formulation is given. A Finite Impulse Response (FIR) model is adopted for prediction, and Recursive Least Square (RLS) is used for parameter estimation. Moreover, it is shown how to derive a persistently exciting constraint, suitable for implementation with MPC. Finally, it is explained how to implement the optimization problem such that, every sample time, only two Quadratic Programming (QP) problems are solved, and the optimal solution is applied in a receding horizon fashion. In the final part of the work, a simulation based example is given to show the effectiveness of the approach.

Keywords: Model Predictive Control, FIR model, Persistent Excitation Condition, Recursive Least Square.

1 INTRODUCTION

It is well known that when closed loop system identification is performed, issues due to the frequency content of the input and output plant signals arise. To obtain a reliable parameter estimation, those signals have to fulfill the persistent excitation condition defined in Goodwin and Sin (1984). Due to the stabilizing nature of the controller very often the closed loop signals do not satisfy the persistent excitation condition, then several method are used to obtain persistently exciting signals. For example, a solution may be to add some external signal, often called dithering signal Sotomayor et al. (2009). However, in a MPC framework, an unfortunate choice of dithering signal may cause constraint violation. For example, after the manipulated variable is calculated, if the magnitude of a dithering signal is large enough a possible input or output constraint may be violated. Another approach is to use the MPCI framework provided by Genceli and Nikolaou (1996) where the input is forced to be persistently exciting by imposing a PE constraint. However, the MPCI has the drawback that due to the new constraint, the resulting optimization problem is non-convex. In this work, a similar approach to Genceli and Nikolaou (1996) is taken, however a particular PE constraint is imposed only on the first MPC manipulated variable. Moreover, the PE constrain structure is such that, the resulting optimization problem is equivalent to the solution of maximum two QP problems, without loss of convexity. The model implemented in the PE-MPC is a FIR model, such as the one in Prasath and Jorgensen (2008).

2 FIR MODEL PREDICTIVE CONTROL

The plant is assumed to be linear and represented in a state space form

$$x_{k+1} = Ax_k + Bu_k, \tag{1a}$$

$$z_k = Cx_k, \tag{1b}$$

$$y_k = z_k + v_k, \tag{1c}$$

with x as plant state, u as plant input or manipulated variable, z as plant output or controlled variable, y as measured output, and v as Gaussian distributed white measurement noise.

It is well know that stable processes can be represented by a FIR model, such as

$$z_k = \sum_{i=1}^{n} \theta_i u_{k-i}, \tag{2}$$

where $\{\theta_i\}_{i=1}^{n}$ are the impulse coefficients or Markov parameters calculated as

$$\theta_i = CA^{i-1}B, \qquad i = 1, \ldots, n, \tag{3}$$

and $\{u_{k-i}\}_{i=1}^{n}$ are the past n inputs to the process, finally n is the number of Markov parameters.

By using the FIR model (2), a MPC regulation problem with input constraints may be formulated as

$$\min_{u} J_k = \frac{1}{2} \sum_{j=0}^{N_p-1} \parallel z_{j+1} \parallel_Q^2 + \parallel u_j \parallel_R^2, \tag{4a}$$

$$s.t. \quad z_j = \sum_{i=1}^{n} \theta_i u_{j-i}, \qquad j = 1, \ldots, N_p, \tag{4b}$$

$$u_{\min} \leq u_j \leq u_{\max}, \qquad j = 0, \ldots, N_p - 1, \tag{4c}$$

where N_p is the prediction horizon, identical to the control horizon, in this formulation. Finally, R and Q are input and output weights, respectively.

Clearly, the solution of (4) may be found converting the problem into an equivalent Quadratic Programming (QP) problem. QP problems are known to be convex, and thus have a unique unique global optimum. Moreover, efficient and reliable algorithms are available to solve them. Standard manipulations may yield the QP formulation

$$\min_{U} \quad \frac{1}{2}U'HU + g'U \tag{5a}$$

$$s.t. \quad U_{min} \leq U \leq U_{max} \tag{5b}$$

where H is the Hessian, and g the gradient.

As in any receding horizon control law, only the first element u_0^*, of the optimal solution vector $U^* = \left[u_0^*, u_1^* \ldots, u_{N_p-1}^*\right]$ is implemented on the plant. At the subsequent sample time, the open loop optimization is reformulated, using new information due to new measurements, and solved.

3 PARAMETER ESTIMATION

There is a large academic literature on parameter estimation, and many algorithms have been adopted with success by the industry. An interesting and well known book on the topic is Ljung (1998).
In this article the Recursive Least Square (RLS) algorithm is used to estimate parameters in (2).

3.1. **Recursive Least Square.** Given

$$z_k = \Phi_{k-1}' \Theta_k, \tag{6}$$

where Φ_{k-1} is known as the regressor and Θ_k is known as the parameter vector. Given also an initial parameter estimate $\hat{\Theta}_0$, and its covariance matrix P_0, the RLS algorithm is, for $k = 1, \ldots, \infty$

$$L_k = \frac{\lambda^{-1} P_{k-1} \Phi_k}{1 + \lambda^{-1} \Phi_k' P_{k-1} \Phi_k}, \tag{7a}$$

$$\hat{\Theta}_k = \hat{\Theta}_{k-1} + L_k \left(z_k - \Phi_k' \hat{\Theta}_{k-1} \right), \tag{7b}$$

$$P_k = \lambda^{-1} P_{k-1} - \lambda^{-1} L_k \Phi_k' P_{k-1}, \tag{7c}$$

where λ is the forgetting factor, and P is a measure of the parameter estimation accuracy. $P \simeq 0$ means that $\hat{\theta}$ converges to the actual parameter values. Reminding the reader to Goodwin and Sin (1984), or Ljung (1998), for more details on the RLS algorithm, here we recall that the parameter estimate converge to the real value if

$$\lim_{k \to \infty} \lambda_{min} P_{k-1}^{-1} = \infty \tag{8}$$

which is true if

$$\lim_{k \to \infty} \lambda_{min} \sum_{i=0}^{k-1} \Phi_i \Phi_i' = \infty. \tag{9}$$

It is important to notice that (9) is a general condition on Φ, moreover it is not straightforward to implement it in a MPC framework.
For FIR models the general condition (9) on Φ may be easily related to the input u. It is sufficient to notice how Φ is composed, that is a vector of the previous n inputs. Then, to apply the RLS algorithm to (2) it is sufficient to define

$$\Theta_k = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix}, \qquad \Phi_{k-1} = \begin{bmatrix} u_{k-1} \\ u_{k-2} \\ \vdots \\ u_{k-n} \end{bmatrix}. \tag{10}$$

3.2. **Persistent excitation.** Before deriving the Persistently Exciting (PE) constraint for MPC, some definitions on excitation of input signals are given.

Definition 3.1. *A scalar input signal u is strongly persistently exciting of order n if for all k there exists an integer T such that*

$$\rho_1 I > \sum_{i=k}^{k+T} \begin{bmatrix} u_{i+n} \\ u_{i+n-1} \\ \vdots \\ u_{i+1} \end{bmatrix} \begin{bmatrix} u_{i+n} \\ u_{i+n-1} \\ \vdots \\ u_{i+1} \end{bmatrix}' > \rho_0 I \tag{11}$$

where $\rho_1, \rho_0 > 0$.

Definition 3.2. *A scalar input signal u is weakly persistently exciting of order n if*

$$\rho_1 I \geq \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \begin{bmatrix} u_{i+n} \\ u_{i+n-1} \\ \vdots \\ u_{i+1} \end{bmatrix} \begin{bmatrix} u_{i+n} \\ u_{i+n-1} \\ \vdots \\ u_{i+1} \end{bmatrix}' \geq \rho_0 I \tag{12}$$

where $\rho_1, \rho_0 > 0$. The last condition may be interpreted in the frequency domain, and it is equivalent to the following one.

Definition 3.3. *A stationary input u is weakly persistently exciting of order n if its two sided spectrum is non zero at n points or more.*

In Goodwin and Sin (1984) it is shown that for FIR models the RLS algorithm converges provided that the system input is weakly persistently exciting of order n. In particular, a stationary input whose spectral distribution is nonzero at n points or more provides parameter convergence.
In the next section a PE constraint which is more appropriate for the implementation in a MPC framework is derived.

4 PERSISTENTLY EXCITING MODEL PREDICTIVE CONTROL

4.1. **PE constraint derivation.** In this section, by using definition (3.1), a constraint suitable for implementation with MPC is derived. The basic idea of the PE constraint implementation is that, due to the receding horizon property, the PE constraint is dependent only on the first manipulated variable. By defining k as the sample time step, m as the backward looking input horizon, it is possible then to write (11) as

$$\rho_1 I > \Omega_{k-1} = \sum_{j=0}^{m-1} \begin{bmatrix} u_{k-1-j} \\ u_{k-2-j} \\ \vdots \\ u_{k-n-j} \end{bmatrix} \begin{bmatrix} u_{k-1-j} \\ u_{k-2-j} \\ \vdots \\ u_{k-n-j} \end{bmatrix}' > \rho_0 I. \tag{13}$$

It is also possible to show that if u is bounded, as in (4c), there exist always a positive scalar ρ_1. Now, by shifting one step ahead in time such that u_k, the first MPC manipulated variable, appears in the equation (13), Ω_k may be written as

$$\Omega_k = \sum_{j=0}^{m-1} \begin{bmatrix} u_{k-j} \\ u_{k-1-j} \\ \vdots \\ u_{k-n-1-j} \end{bmatrix} \begin{bmatrix} u_{k-j} \\ u_{k-1-j} \\ \vdots \\ u_{k-n-1-j} \end{bmatrix}' > \rho_0 I, \tag{14}$$

or in a more compact form

$$\Omega_k = \sum_{j=0}^{m-1} \Phi_{k-j}\Phi'_{k-j} > \rho_0 I. \tag{15}$$

As mentioned in Bitmead (1984), it is necessary to have $m \geq n$, for the summation in (15) to yield a positive definite matrix. Moreover, for a given ρ_0 a larger m implies a slower parameter convergence rate.
By applying some simple arithmetics, it is possible to rewrite (15) as

$$\begin{aligned}
\Omega_k &= \sum_{j=0}^{m-1} \Phi_{k-j}\Phi'_{k-j} && \text{(16a)}\\
&= \Phi_k\Phi'_k + \sum_{j=1}^{m-1} \Phi_{k-j}\Phi'_{k-j} && \text{(16b)}\\
&= \Phi_k\Phi'_k + \sum_{j=1}^{m} \Phi_{k-j}\Phi'_{k-j} - \Phi_{k-m}\Phi'_{k-m} && \text{(16c)}\\
&= \Phi_k\Phi'_k + \Omega_{k-1} - \Phi_{k-m}\Phi'_{k-m}. && \text{(16d)}
\end{aligned}$$

Notice the recursion, that is useful for MPC implementation. Assuming $\Omega_{k-1} > 0$ (15) is satisfied if

$$\tilde{\Omega}_k = \Phi_k\Phi'_k + \Omega_{k-1} - \Phi_{k-m}\Phi'_{k-m} - \rho_0 I > 0 \tag{17}$$

where only Φ_k is dependent on the first manipulated variable u_k, whereas the remaining parts are all known at the time instant k.
For $m = n = 1$ (17) becomes

$$u_k^2 > \rho_0 \tag{18}$$

which is a quadratic scalar inequality constraint on u_k.
For $m \geq n > 1$ (17) becomes a $n \times n$ dimensional symmetric matrix, partitioned as

$$\tilde{\Omega}_k = \begin{bmatrix} \tilde{\Omega}_{k,11} & \tilde{\Omega}_{k,12} \\ \tilde{\Omega}_{k,21} & \tilde{\Omega}_{k,22} \end{bmatrix} > 0, \tag{19}$$

where $\tilde{\Omega}_{k,12} = \tilde{\Omega}'_{k,21}$ and

$$\tilde{\Omega}_{k,11} = u_k^2 + \sum_{j=1}^{m-1} u_{k-j}^2 - \rho_0, \tag{20}$$

$$\tilde{\Omega}_{k,21} = \begin{bmatrix} u_k u_{k-1} \\ u_k u_{k-2} \\ \vdots \\ u_k u_{k-n+1} \end{bmatrix} + \sum_{j=1}^{m-1} \begin{bmatrix} u_{k-j} u_{k-j-1} \\ u_{k-j} u_{k-j-2} \\ \vdots \\ u_{k-j} u_{k-j-n+1} \end{bmatrix}, \tag{21}$$

$$\begin{aligned}\tilde{\Omega}_{k,22} = &\begin{bmatrix} u_{k-1}^2 & u_{k-1}u_{k-2} & \dots & u_{k-1}u_{k-n+1} \\ u_{k-2}u_{k-1} & u_{k-2}^2 & \dots & u_{k-2}u_{k-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{k-n+1}u_{k-1} & u_{k-n+1}u_{k-2} & \dots & u_{k-n+1}^2 \end{bmatrix} \\ &+ \sum_{j=1}^{m-1} \begin{bmatrix} u_{k-j-1}^2 & u_{k-j-1}u_{k-j-2} & \dots & u_{k-j-1}u_{k-j-n+1} \\ u_{k-j-2}u_{k-j-1} & u_{k-j-2}^2 & \dots & u_{k-j-2}u_{k-j-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{k-j-n+1}u_{k-j-1} & u_{k-j-n+1}u_{k-j-2} & \dots & u_{k-j-n+1}^2 \end{bmatrix} - \rho_0 I.\end{aligned} \tag{22}$$

It can be noticed that only $\tilde{\Omega}_{11}$ and $\tilde{\Omega}_{k,12} = \tilde{\Omega}'_{k,21}$ depend on u_k. By applying Theorem A.2 to (19)

$$\begin{aligned}\tilde{\Omega}_k \succ 0 \;&\Leftrightarrow\; \tilde{\Omega}_{k,22} \succ 0 \quad \text{and} \quad \tilde{\Omega}_k / \tilde{\Omega}_{k,22} \succ 0 \\ &\Leftrightarrow\; \tilde{\Omega}_{k,22} \succ 0 \quad \text{and} \quad \tilde{\Omega}_{k,11} - \tilde{\Omega}_{k,21}\tilde{\Omega}_{k,22}^{-1}\tilde{\Omega}'_{k,21} > 0.\end{aligned} \tag{23}$$

Considering that $\tilde{\Omega}_{k,22}$ is equal to $\tilde{\Omega}_{k-1}$ without its last row and last column, it is easy to see that $\tilde{\Omega}_{k,22} \succ 0$ it is always satisfied. Thus, the candidate PE constraint for the MPC is

$$\tilde{\Omega}_{k,11} - \tilde{\Omega}_{k,21}\tilde{\Omega}_{k,22}^{-1}\tilde{\Omega}'_{k,21} > 0. \tag{24}$$

Due to the Schur complement transformation, the PE constraint (24) is still a quadratic scalar inequality, for SISO systems, obviously. This is an important result, as it can be seen on the next subsection, the PE-MPC optimization problem with FIR model it consists on the solution of maximum two QP problems every time step.

4.2. **Formulation.** Combining (4) and (24) the following persistently exciting model predictive control formulation is obtained.

$$\min_u J_k = \frac{1}{2} \sum_{j=0}^{N_p-1} \parallel z_{j+1} \parallel_Q^2 + \parallel u_j \parallel_R^2, \tag{25a}$$

$$s.t. \quad z_j = \sum_{i=1}^{n} \theta_i u_{j-i}, \qquad j = 1, \dots, N_p, \tag{25b}$$

$$u_{\min} \le u_j \le u_{\max}, \quad j = 0, \dots, N_p - 1, \tag{25c}$$

$$\tilde{\Omega}_{j,11} - \tilde{\Omega}_{j,21}\tilde{\Omega}_{j,22}^{-1}\tilde{\Omega}'_{j,21} > 0, \qquad j = 1. \tag{25d}$$

Notice how the PE constraint is applied only to the first manipulated variable u_k, ($j = 1$).

4.3. **Optimization: two QP problems.** The standard FIR MPC problem is equivalent to a convex QP problem. Constraint (25d) is non-convex, but it is possible to split it into two linear scalar inequalities. Thus, it is shown that the PE-MPC solution is given by choosing the best optimum between two convex QP problems.
For example, consider $m = n = 2$, then (25d) becomes

$$u_k - \rho_0 + u_{k-1}^2 - \frac{(u_k u_{k-1} + u_{k-1} u_{k-2})^2}{u_{k-1}^2 + u_{k-2}^2 - \rho_0} > 0. \tag{26}$$

The two solutions of the associated equation are

$$u_k = \frac{u_{k-1}^2}{u_{k-2} - \sqrt{\rho_0}} + \sqrt{\rho_0}, \qquad u_k = \frac{u_{k-1}^2}{u_{k-2} + \sqrt{\rho_0}} - \sqrt{\rho_0}, \tag{27}$$

where there is always one solution that is not infinite. Depending on the sign of the determinant of the associated equation, the two new linear PE constraints are, for instance

$$u_k < \frac{u_{k-1}^2}{u_{k-2} + \sqrt{\rho_0}} - \sqrt{\rho_0}, \qquad u_k > \frac{u_{k-1}^2}{u_{k-2} - \sqrt{\rho_0}} + \sqrt{\rho_0}. \tag{28}$$

In a general case $m \geq n > 0$ there will be always two inequalities

$$u_k > \gamma_1, \qquad u_k < \gamma_2 \tag{29}$$

where γ_1 and γ_2 are functions of the previous inputs and ρ_0. For instance, in the case where $\gamma_1 > \gamma_2$, the set of inequality constraints (25c)-(25d) may be written as two separate sets of constraints

$$u_{min} \leq u_k < \gamma_2, \qquad \gamma_1 < u_k \leq u_{max} \tag{30}$$

yielding two FIR MPC problems of the form (4), which are solvable as QP problems. Every time step k, the best solution vector between the two QP problems is chosen and, the first element of the vector is applied to the plant.

5 AN EXAMPLE

In this section, an example is given to show how the PE-MPC in Section **??** produces a persistently exciting input. A plant is regulated by a constrained FIR model based predictive controller, a RLS algorithm is used to constantly estimated the Markov parameters of the plant, however the corresponding FIR coefficient in the MPC model are updated with a frequency of 50 time steps.
The plant is a discrete time stable system (1), with

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0.3 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.7 & 0 & 0.1 \end{bmatrix}, \tag{31}$$

and Gaussian measurement noise with covariance $\sigma_v = 0.0625$. The sampling time is $T = 1$ *sec*. Using (3), for $n = 3$ the corresponding Markov parameters are

$$\Theta = \begin{bmatrix} 0.03 \\ 0.28 \\ 0.63 \end{bmatrix} \tag{32}$$

which are considered to be the 'true' system parameters. Notice that A is a nilpotent matrix such that $A^p = 0$ for $p \geq n$. This is just a simple strategy taken to avoid the introduction of a steady state bias estimator.

The controller has a prediction horizon $N_p = 10$, input constraints $-2 \leq u \leq 2$, cost function output weight $Q = 5$, and input weight $R = 0.3$. The model in the controller is initialized with the following FIR coefficients

$$\Theta_0 = \begin{bmatrix} 0.13 \\ 0 \\ 1.26 \end{bmatrix}. \tag{33}$$

Regarding the persistently excitation constraint, the looking backward horizon is $m = 6$ and the design parameter $\rho_0 = 2.5$.
Finally, for the RLS algorithm (7) the forgetting factor is $\lambda = 1$, and the initial conditions are

$$\hat{\Theta}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad P_0 = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 1000 & 0 \\ 0 & 0 & 1000 \end{bmatrix}. \tag{34}$$

5.1. **Simulation result.** The simulation is run in MATLAB 7.9 and the solver *e04nf* from NAG library is used to efficiently and reliably solve the two QP problems every time step. In the first $n + m - 1$ time steps the standard FIR MPC is used such that all the needed vectors, for the PE constraint formulation, are properly initialized. Figure 1 shows

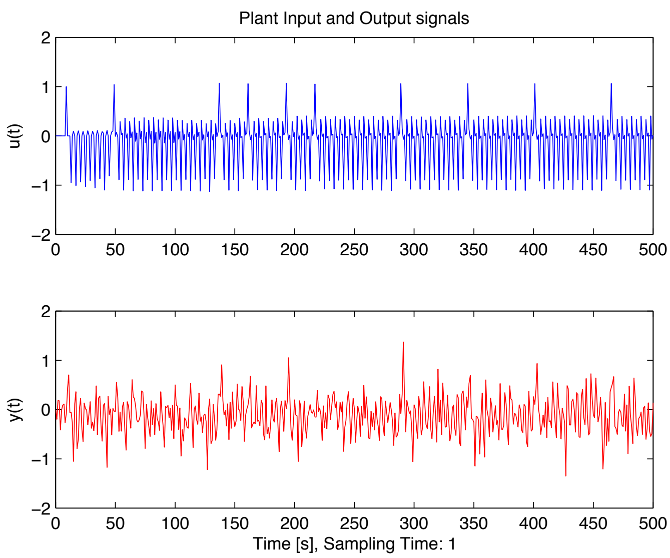

FIGURE 1. Input and output to the plant.

the plant input and output. Input constraints are not hit, and the apparently noisy signal is needed for having a correct parameter estimation. In Figure 2 the FIR coefficients are shown, the dashed lines represent the real parameter values (32), the solid lines are the RLS estimates, and the stars are the old values used in the MPC model. For example, the stars at time step 50 are the FIR coefficients (33), they are used in the controller for the first 50 steps, then the estimate from RLS is used to update the MPC model, which is again kept constant until the next update occurs (50 steps later).
Finally, Figure 3 shows the plant input in time domain, and more importantly the magnitude of its Fourier transform. The three peaks confirm that the input is persistently exciting of order three, which is the minimum order needed to estimate three parameters.

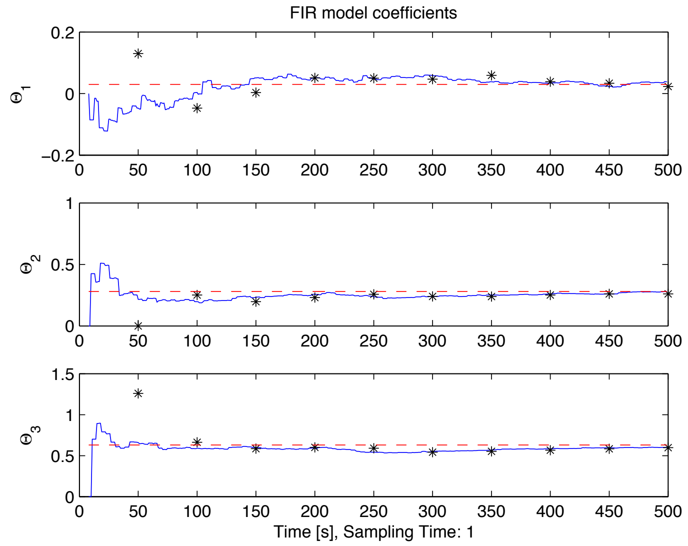

FIGURE 2. FIR parameters: RLS estimates (solid line), 'real' values (dashed line), MPC model parameter (stars).

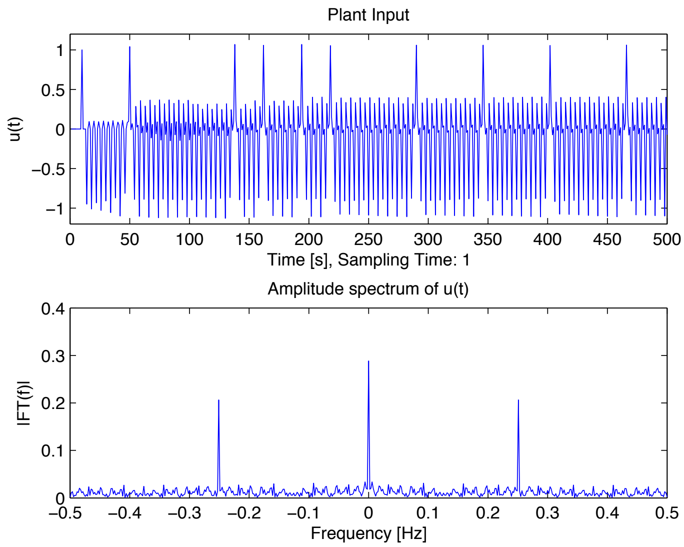

FIGURE 3. Persistently exciting input and its spectrum. Third order of excitation.

6 CONCLUSIONS

A persistently exciting model predictive control formulation is given, and its effectiveness is shown. Using a FIR model for prediction, and implementing a persistently exciting constraint, it is possible to correctly estimate and update the MPC model parameters. Moreover, due to the structure of the PE constraint, the PE-MPC problem is expressed as two QP problems. Every time step, its optimal solution is given by the best solution between the QPs. Ongoing research is likely to extend the result to state space model and ARMAX based MPC.

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APPENDIX

Schur complement for positive definite matrices. Given a matrix

$$M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}, \tag{35}$$

the Schur complement is defided as

$$M/P = S - RP^{-1}Q \tag{36}$$

where the nonsingular matrix P is the leading submatrix of M. The notation M/P indicates the Schur complement of P in M. Analogously, the Schur complement of S in M is:

$$M/S = P - QS^{-1}R. \tag{37}$$

Definition A.1. *Hermitian Matrix*
A matrix H is a Hermitian matrix if $H^* = H$, where the superscript * denotes the conjugate transpose if $H \in \mathcal{C}^{m\times n}$, or just the transpose for real matrices $H \in \mathcal{R}^{m\times n}$.

Definition A.2. *Inertia of Hermitian Matrices*
The inertia of an $n \times n$ Hermitian matrix H is the ordered triple

$$In(H) := (p(H), q(H), z(H)) \tag{38}$$

where $p(H)$, $q(H)$, and $z(H)$ are the numbers of positive, negative, and zero eigenvalues of H including multiplicity, respectively. The rank is $\mathrm{rank}(H) = p(H) + q(H)$.

Theorem A.1. *Let H be a Hermitian matrix, S a nonsingular principal submatrix of H. Then*

$$In(H) = In(S) + In(H/S). \tag{39}$$

Proof. See Zhang (2005) □

Definition A.3. *Positive semidefinite matrices*
A Hermitian matrix N is positive definite ($N \succ 0$) if and only if all its eigenvalues are positive.

Theorem A.2. *Let H be a Hermitian matrix partitioned as*

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} \tag{40}$$

where H_{22} is square and nonsingular. Thus

- *$H \succ 0$ if and only if both $H_{22} \succ 0$ and $H/H_{22} \succ 0$.*

Proof. Proof as consequence of Theorem A.1. □