TIME OPTIMAL CONTROL USING GRÖBNER BASES

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Abstract: The paper presents the solution of the time optimal control with the help of Gröbner Bases theory. First the set of polynomial equations for the triple integrator system with unknowns corresponding to the switching times is constructed and later it is solved in the computer algebra system Maple using the Groebner library. In dependence of resulting switching times the control action is designed. Finally the time optimal switching surface is compared with the variety resulted from the Gröbner Bases solution.

Keywords: time optimal control, Gröbner bases, triple integrator system

1 INTRODUCTION

The time optimal control belongs to one of the most important control strategies. It was heavily studied in 50-ties and 60-ties of the previous century but due to its high sensitivity to unmodelled dynamics, parametric variations, disturbances and noise it was later suppressed by the pole assignment control. Nowadays there exist several approaches how to cope with this problem and so the time optimal control plays an important role in the modern control theory.

Generally, the time optimal problem can be solved by computation of switching surfaces. These can be derived by using the Pontryagin's maximum principle. A different approach is offered by dynamic programming based on the Bellman's optimality principle. Another way was represented by Pavlov who solved switching surfaces from phase trajectories (Pavlov, 1966). But switching surfaces can be also expressed by the set of algebraic equations (Walther et al., 2001) that result from time solutions in the phase space. For higher order systems these can be rather complicated to find the exact solution. In this paper we will apply the Gröbner bases theory to help us to solve such a set of polynomial equations.

Gröbner bases generalize the usual Gauss reduction from linear algebra, the Euclidean algorithm for computation of univariate greatest common divisors and the simplex algorithm from linear programming. Using them it is possible to transform one set of equations to another one that can be solved more easily. We will apply this technique for the triple integrator system where it is possible to get symbolic solutions but it can be also applied for higher order systems when solving the resulted sets of equations numerically.

The paper is organized in six chapters. After introduction and problem statement chapters there is the chapter introducing the set of polynomial equations for time optimal problem. The following chapter describes the Gröbner bases theory. The fifth chapter offers the solution and comparison of switching surfaces. The paper is finished with short conclusions.

2 PROBLEM STATEMENT

Let us consider the linear system given in the state space

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}u \\ y &= \mathbf{c}^t\mathbf{x}\end{aligned} \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{c}^t = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{1}$$

that represents the triple integrator. The control input signal is saturated and can reach two values U_1 or U_2 (relay or bang-bang control)

$$u_r = \left\langle \begin{matrix} U_1 \\ \\ U_2 \end{matrix} \right. \tag{2}$$

The task is to design the time optimal controller what means to drive the system from an initial state $\mathbf{x} = [x \quad y \quad z]^t$ to the desired state $\mathbf{x}_w$ in a minimum time $t_{\min}$. Using coordinate transformation it is always possible to set the desired state equal to the origin $\mathbf{x}_w = \mathbf{0}$.

It is well known that minimum time optimal control with saturated input leads to the control action with at most *n* intervals switching between limit values where *n* represents the order of the system. Usually the control algorithm results in deriving switching surfaces as functions of states which signs determine the switching times. It can be very hard task to express these functions exactly and there is no general solution for higher order systems ($n > 3$). Bang-bang control in practice is not desirable because of chattering and noise effects but there are techniques have to cope with them (Pao and Franklin, 1993, Bistak et al. 2005).

3 SET OF EQUATIONS WITH SWITCHING TIMES

Let us denote the length of each time interval of optimal control as t_i. In the case of triple integrator ($n = 3$) we assign variables t_1, t_2, t_3 to these switching times. From the time solution in the state space one can derive the corresponding set of equations.

Starting from the vector state differential equation

$$\mathbf{x}(t) = \mathbf{\Phi}(t, \mathbf{x}_0, u_r) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}u_r(\tau)d\tau \tag{3}$$

by substituting $\mathbf{x}_0 = \mathbf{x}_w$, $t = -t$ we can get the initial states from the final states (backward integration in time). Then the points of the optimal braking trajectory called also Reference Braking Curve (RBC) that are determined by the parameter t_1 can be expressed

$$\mathbf{x}_1(t_1) = \int_0^{-t_1} e^{\mathbf{A}\tau}\mathbf{b}\, d\tau\, U_j = \begin{bmatrix} \frac{-t_1^3}{6} \\ \frac{t_1^2}{2} \\ -t_1 \end{bmatrix} U_j \ , \ j = 1,2 \tag{4}$$

Continuing the backward integration we can derive the Reference Braking Surface (RBS) that represents the switching surface. In this case the initial value $\mathbf{x}_0$ in (3) is substituted by the RBC points $\mathbf{x}_1(t_1)$ and the action value u_r reaches the opposite value U_{3-j}.

$$\mathbf{x}_{12}(t_1,t_2) = e^{\mathbf{A}(-t_2)}\mathbf{x}_1(t_1) + \int_0^{-t_2} e^{\mathbf{A}\tau}\mathbf{b}U_{3-j}\,d\tau = = \begin{bmatrix} 1 & -t_2 & \frac{t_2^{\ 2}}{2} \\ 0 & 1 & -t_2 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{x}_1(t_1) + \begin{bmatrix} \frac{-t_2^{\ 3}}{6} \\ \frac{t_2^{\ 2}}{2} \\ -t_2 \end{bmatrix} U_{3-j} \ , \ j = 1,2 \tag{5}$$

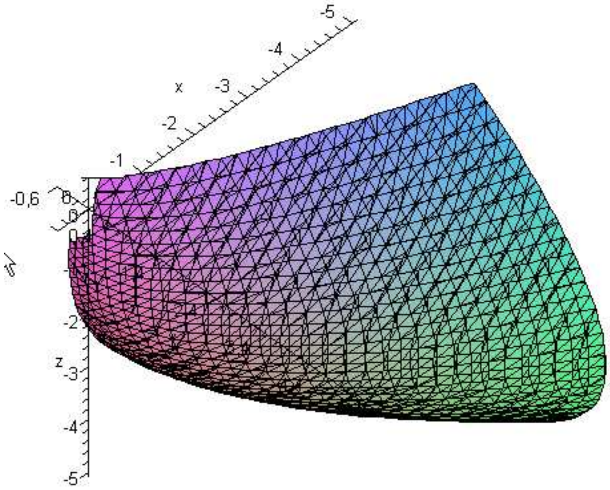

Figure 1: A part of the switching surface

An arbitrary starting point in the space can be again expressed using (3) but this time it is necessary to substitute points of RBC $\mathbf{x}_{12}(t_1,t_2)$ into the initial points of (3) and the control action u_r is opposite to the control action in the previous time interval, so again U_j.

$$\mathbf{x}_{123}(t_1,t_2,t_3) = e^{\mathbf{A}(-t_3)}\mathbf{x}_{12}(t_1,t_2) + \int_0^{-t3} e^{\mathbf{A}\tau}\mathbf{b}U_j\,d\tau = = \begin{bmatrix} 1 & -t_3 & \frac{t_3^{\ 2}}{2} \\ 0 & 1 & -t_3 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{x}_{12}(t_1,t_2) + \begin{bmatrix} \frac{-t_3^{\ 3}}{6} \\ \frac{t_3^{\ 3}}{2} \\ -t_3 \end{bmatrix} U_j \ , \ j = 1,2 \tag{6}$$

After assignment $\mathbf{x}_{123} = [x,y,z]^t$ and elimination of $\mathbf{x}_{12}$ and $\mathbf{x}_1$ from (6) using (5) and (4) we get the set of equations with three unknowns t_1, t_2, t_3 that represent the switching times.

$$\begin{aligned} x &= -\frac{1}{6}U_j t_1^3 - \frac{1}{2}U_j t_1^2 t_2 - \frac{1}{2}U_j t_1 t_2^2 - \frac{1}{6}U_{3-j} t_2^3 - \frac{1}{2}U_j t_1^2 t_3 - U_j t_1 t_2 t_3 - \frac{1}{2}U_{3-j} t_2^2 t_3 - \frac{1}{2}U_j t_1 t_3^2 - \frac{1}{2}U_{3-j} t_2 t_3^2 - \frac{1}{6}U_j t_3^3 \\ y &= \frac{1}{2}U_j t_1^2 + U_j t_1 t_2 + \frac{1}{2}U_{3-j} t_2^2 + U_j t_1 t_3 + U_{3-j} t_2 t_3 + \frac{1}{2}U_j t_3^2 \\ z &= -U_j t_1 - U_{3-j} t_2 - U_j t_3 \end{aligned} \tag{7}$$

This set of equations can be simplified when using substitution

$$t_3 = t_3\,,\ t_2 = t_2 - t_3\,,\ t_1 = t_1 - t_2 \tag{8}$$

Then (7) can be rewritten to the more readable form

$$
\begin{aligned}
x &= -\frac{1}{6}U_j t_1^3 - \frac{1}{6}(U_{3-j} - U_j)t_2^3 - \frac{1}{6}(U_j - U_{3-j})t_3^3 \\
y &= \frac{1}{2}U_j t_1^2 + \frac{1}{2}(U_{3-j} - U_j)t_2^2 + \frac{1}{2}(U_j - U_{3-j})t_3^2 \\
z &= -U_j t_1 - (U_{3-j} - U_j)t_2 - (U_j - U_{3-j})t_3
\end{aligned}
\tag{9}
$$

Continuing this procedure one can easily derive the set of equations corresponding to the chain of n integrators.

Generally, the evaluation of unknowns t_1, t_2, t_3 results from the solution of a 6th-order polynomial with one unknown and backward substitution to the quadratic equation of two unknowns and the linear equation with three unknowns. In this paper we will use Gröbner bases to solve the set of polynomial equations (9).

4 GRÖBNER BASES

The Gröbner bases theory was developed in 60-ties of the previous century. Thanks to it the algebraic geometry has practical importance. Modern computer algebra systems are based on this theory. F. Macaulay was the first who used it. B. Buchberger defined Gröbner bases and developed an algorithm for computing them.

Gröbner bases help to solve a set of polynomial equations. Using Buchberger's algorithm an original set of equations is transformed to the new one called Gröbner base. It is important that the new set has nice properties and can be solved more easily in comparison with the original one. The solutions of the new set are identical with the original one.

The elimination property of Buchberger's algorithm assures that variables will be consequently eliminated and the resulted Gröbner base will include just one univariate polynomial. The variable of this univariate polynomial is given by specified ordering of variables when computing the Gröbner base. The ordering determines the sequence in which the variables will be eliminated and the variable with lowest degree (the last one) will be included in the univariate polynomial. The univariate polynomial is then solved for this variable.

Another important property of Gröbner bases is the extension. This guarantees that the variable resulted from the univariate polynomial can be consequently substituted to the other polynomials (according to the ordering) and thus these become also univariate polynomials, i.e. solvable for one variable. The extension property gives a systematic way to find all solutions.

Gröbner bases allow us to find also the system of polynomial equations representing the variety V if this has been originally given by the set of parametric equations $x_i = g_i(t_1,...,t_m)$, $i = 1,...,n$. It is necessary to choose the lexicographic ordering of variables $t_1 > t_2 > ... > t_m > x_1 > x_2 > ... > x_n$ and from the resulted base to select only those polynomials that do not include variables t_i .

Gröbner bases have been widely used in robotics (motion planning), optimization, coding, control theory, statistics, molecular biology and many other fields.

Gröbner bases are implemented in almost each modern computer algebra system (Mathematica, Maple, Mathcad, Symbolic Math Toolbox,...). There are also noncommercial products as Macaulay2 – the system for computation in algebraic geometry and commutative algebra.

5 SOLUTION USING GRÖBNER BASES

We used the Maple computer algebra system to solve the set of polynomial equations (9). The screenshot below shows the procedure. First we simplified the set of equations by choosing $U_j = -1$ and $U_{3-j} = 1$. Then we applied the Solve command with elimination ordering t_1, t_2, t_3. The resulted set of polynomials (10) denoted GB consists of one univariate (t_3) polynomial of 4[th]-order and two polynomials of 1[st] order with respect to the variable t_2 or t_1 that include also the variable t_3.

```
> with(Groebner) :
> G := [x123s[1] − x, x123s[2] − y, x123s[3] − z] : Gs := eval(subs( U_j = −1, U_{3−j} = 1, G));
```

$$\mathrm{Gs} := \left[\frac{1}{6} t_1^3 - \frac{1}{3} t_2^3 + \frac{1}{3} t_3^3 - x,\ -\frac{1}{2} t_1^2 + t_2^2 - t_3^2 - y,\ t_1 - 2 t_2 + 2 t_3 - z \right]$$

```
> GB := Solve(Gs, [t1, t2, t3]);
```

$$\begin{aligned}\mathrm{GB} := \{[[&-72 y^3 - 72 y^2 t_3^2 + 36 y^2 z^2 - 144 y^2 t_3 z + 48 y t_3 z^3 - 288 y t_3 x + 72 y t_3^4 - 144 y t_3^3 z \\ &+ 144 y z x - 72 y t_3^2 z^2 - 6 z^4 y + 54 z^4 t_3^2 - 96 z^3 t_3^3 - 48 x z^3 + 36 z^2 t_3^4 - 432 z x t_3^2 + z^6 + 144 x t_3^3 \\ &- 72 x^2 - 12 z^5 t_3 + 288 z^2 t_3 x,\ -2 z^7 + 6 y t_3 z^4 - 36 y^2 t_3 z^2 - 156 y t_3^2 z^3 - 144 y^2 t_3^2 z - 216 t_3 z^3 x \\ &+ 36 t_3^2 x z^2 - 6 z^4 t_2 y + 72 z^2 t_3^3 y + 216 z^2 x y + 72 t_3^2 x y + 48 x t_2 z^3 - 72 t_2 y^3 - 48 y^2 z^3 + 72 y^2 t_3^3 \\ &- 144 y^2 x + 72 t_3 y^3 + 144 z x^2 - 216 t_3 x^2 + 72 t_2 x^2 - 144 y^3 z - 36 t_2 y^2 z^2 - z^6 t_2 + 15 z^6 t_3 \\ &- 42 z^5 t_3^2 + 18 z^4 t_3^3 + 96 z^4 x - 576 t_3 z x y + 144 y z t_2 x,\ -3 z^7 - 144 y^2 t_3 z^2 - 312 y t_3^2 z^3 - 288 y^2 \\ &t_3^2 z - 336 t_3 z^3 x + 72 t_3^2 x z^2 + 144 z^2 t_3^3 y + 288 z^2 x y + 144 t_3^2 x y - 60 y^2 z^3 + 144 y^2 t_3^3 - 288 y^2 x \\ &+ 216 z x^2 - 288 t_3 x^2 - 216 y^3 z + 6 y z^5 + 144 t_1 y z x + 28 z^6 t_3 - 84 z^5 t_3^2 + 36 z^4 t_3^3 + 144 z^4 x \\ &- 864 t_3 z x y - 6 t_1 z^4 y + 48 t_1 x z^3 - 36 t_1 y^2 z^2 - 72 t_1 y^3 + 72 t_1 x^2 - t_1 z^6],\ \mathrm{plex}(t_1, t_2, t_3),\ \{\ \}]\}\end{aligned} \tag{10}$$

Thus using Gröbner bases the original set of equations (9) has been converted to the other one (10). The main advantage of the new set is given by the fact that there exists one univariate polynomial (11) that can be solved and this solution can be later supplied to the other two polynomials and so the system (10) can be solved easily.

```
> GB1c := collect(GB[1, 1, 1], t3);
```

$$\begin{aligned}\mathrm{GB1c} := &(36 z^2 + 72 y) t_3^4 + (-96 z^3 + 144 x - 144 y z) t_3^3 + (-72 y^2 + 54 z^4 - 72 y z^2 - 432 z x) t_3^2 \\ &+ (288 z^2 x + 48 y z^3 - 12 z^5 - 144 y^2 z - 288 y x) t_3 - 72 y^3 - 72 x^2 + 36 y^2 z^2 + z^6 + 144 y z x \\ &- 48 x z^3 - 6 z^4 y\end{aligned} \tag{11}$$

The quartic (11) can be solved analytically or numerically. We are looking for nonnegative real solutions of time variables and because of substitution (8) the solution of (10) must fulfill the condition $t_1 \ge t_2 \ge t_3 \ge 0$. We have chosen $U_j = -1$ and $U_{3-j} = 1$. If $t_3 > 0$ then the right control action is $u_r = U_j = -1$. If $t_3 = 0$ then the control action depends on t_2. If $t_2 > 0$ then $u_r = U_{3-j} = 1$ otherwise $u_r = U_j = -1$. If the solution of (10) is not real or the condition $t_1 \ge t_2 \ge t_3 \ge 0$ is not fulfilled we have to choose opposite values for U_j and U_{3-j}, i.e. $U_j = 1$ and $U_{3-j} = -1$ and construct and solve the system (10) with these values and apply the decision procedure about the right control action similarly.

The different solution of (10) is shown in (Walther et al., 2001) that is based on the calculation of the number of positive roots using Sturm sequences.

After substitution $t_3 = 0$ the quartic (11) can also be used for calculation of the variety V

$$V = -72y^3 - 72x^2 + 36y^2z^2 + z^6 + 144yzx - 48xz^3 - 6z^4y \qquad (12)$$

that is in a certain part of the phase space identical with the switching surface derived in (Pavlov, 1966). ξ represents the Pavlov's switching surface for $sign\left(\frac{z|z|}{2} + y\right) = -1$

$$\xi = x - yz + \frac{z^3}{3} - \left(-y + \frac{z^2}{2}\right)^{\frac{3}{2}} \qquad (13)$$

and after some manipulations as shown in the following screenshot it gives the same expression as the variety V.

$$> \xi_{kvad} := -72 \cdot expand\left(\left(x - y \cdot z + \frac{z^3}{3}\right)^2 - \left(-y + \frac{z^2}{2}\right)^3\right)$$

$$\xi_{kvad} := -72\,y^3 - 72\,x^2 + 36\,y^2\,z^2 + z^6 + 144\,y\,z\,x - 48\,x\,z^3 - 6\,z^4\,y$$

The difference can be seen in the Fig. 2. In addition to the ξ the variety V includes also the "mirror part" that appeared due to quadratic operations. Because of this V could not be used as the switching surface globally. For the sectors given by inequalities $x > 0,\ y < 0,\ z > 0$ or $x < 0,\ y > 0,\ z < 0$ it is necessary to derive an additional switching condition.

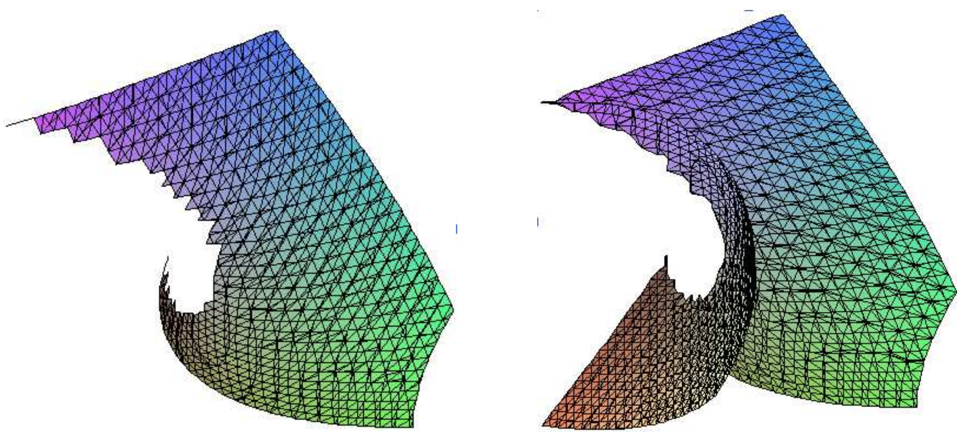

Figure 2: Pavlov's switching surface and variety V

6 CONCLUSIONS

The paper shows how it is possible to apply Gröbner Bases to the time optimal control problem. The resulted solutions of the set of polynomial equations are better than classical approach when comparing the order of the univariate polynomial. Gröbner Bases are not only suitable for analytical solutions but they are also used in many numerical procedures to find solutions of higher order systems. There exists an automatic generator of Gröbner Bases solvers (see the link in the References) that can solve a set of polynomial equations even in the real time.

There exist also limitations of using Gröbner Bases for the time optimal control as it was demonstrated on the derivation of the switching surface. The Gröbner Bases have an algebraic nature and working with polynomials they could not reach for specific problems such effective results as were presented by Pavlov.

But the use of Gröbner Bases is general because many scientific problems can be formulated by the set of polynomial equations. In the future we plan to use them also for problems of the time suboptimal control.

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REFERENCES

ATHANS, M., FALB, P. (1996): *Optimal Control: An Introduction to the Theory and its Applications.* McGraw-Hill, New York

PAVLOV,A.A. (1966): *Sintez relejnych sistem optimaľnych po bystrodejstviju*. Nauka, Moskva.

BUCHBERGER, B. (2001): Gröbner Bases: A Short Introduction for Systems Theorists. http://www.risc.uni-linz.ac.at/people/buchberg/papers/2001-02-19-A.pdf (accessed on November 11, 2009)

BUCHBERGER, B. (2001): Gröbner Bases and Systems Theory, *Multidimensional Systems and Signal Processing*, 12, 223– 251

COX, D., LITTLE, J., O'SHEA, D. (2007): *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Springer-Verlag, New York

WALTHER, U., GEORGIOU, T. T., TANNENBAUM, A. (2001): On the Computation of Switching Surfaces in Optimal Control: A Gröbner Basis Approach, *IEEE Transactions on Automatic Control*, Vol. 46, No. 4

Automatic generator of Groebner basis solvers: http://cmp.felk.cvut.cz/minimal/automatic_generator.php (accessed on November 3, 2009)

BISTÁK, P., ŤAPÁK, P., HUBA, M.(2005): Constrained Pole Assignment Control of Double and Triple Integrator. In: *CASYS 05, Seveth International Conference on Com-puting Anticipatory Systems*, HEC-Ulg, LIEGE, Belgium.

PAO, L. Y., FRANKLIN, G. F. (1993): Proximate Time-Optimal Control of Third-Order Servomechanisms, *IEEE Transactions on Automatic Control*, Vol. 38, No. 4