Technische Universiteit Eindhoven University of Technology

Efficient Simulation of Large-Scale Dynamical Systems using Tensor Decompositions

Femke van Belzen
Siep Weiland
P.O. Box 513
5600 MB Eindhoven
The Netherlands
{f.v.belzen,s.weiland}@tue.nl

Background

- Most model reduction techniques are projection based methods.
- Proper Orthogonal Decompositions reduce the complexity of systems with both space and time as independent variables: described by PDEs.
- The projection spaces are computed via singular value decompositions of matrices that have dimension # of finite elements times # of physical variables.
- We propose a more efficient way of computing bases from multi-dimensional arrays.
- We demonstrate the procedure by applying it to a heat diffusion process. However, the underlying mathematics are generic.

Step 1: Snapshot generation

$$0 = -\rho c_p \frac{\partial w}{\partial t} + \kappa_x \frac{\partial^2 w}{\partial x^2} + \kappa_y \frac{\partial^2 w}{\partial y^2}. \tag{1}$$

Here, $w(x, y, t)$ denotes temperature on position (x, y) and time t. Using spatial and temporal discretization a Finite Element (FE) solution of this process is computed. The solution data is stored in a three-dimensional array $[[w]] \in \mathbb{R}^{L_1 \times L_2 \times L_3}$, where L_1, L_2, L_3 are the number of spatial and temporal grid points.

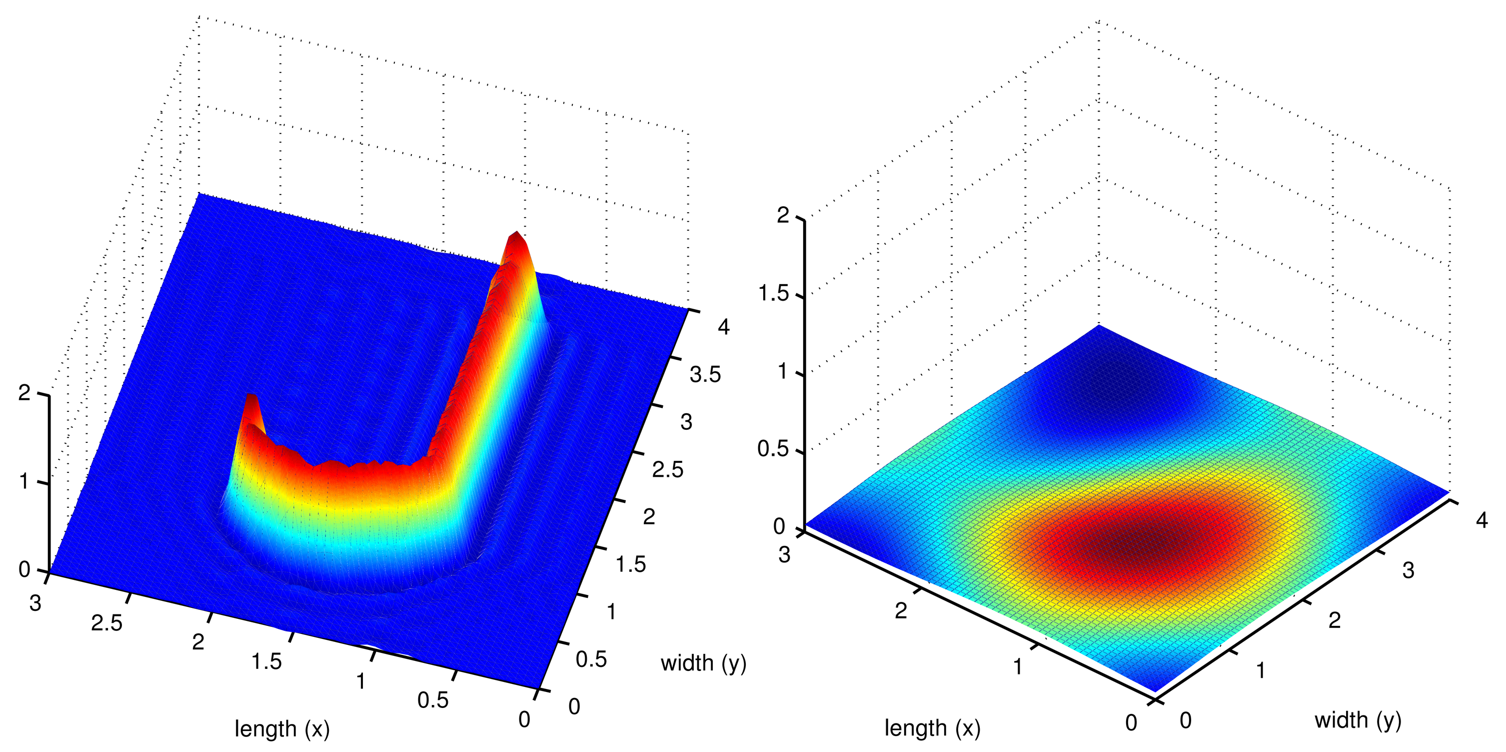

Figure 1: Snapshots of original data t_1 (left) and t_{L_3}

Step 2: Computation of Projection Spaces

Projection spaces can be computed by computing a SVD of the three-dimensional array $[[w]]$ [1,2]. The SVD gives a three-dimensional array of coefficients $[[\hat{w}]]$ and three matrices U_1, U_2, U_3. The columns of these matrices contain orthonormal bases for the projection spaces.

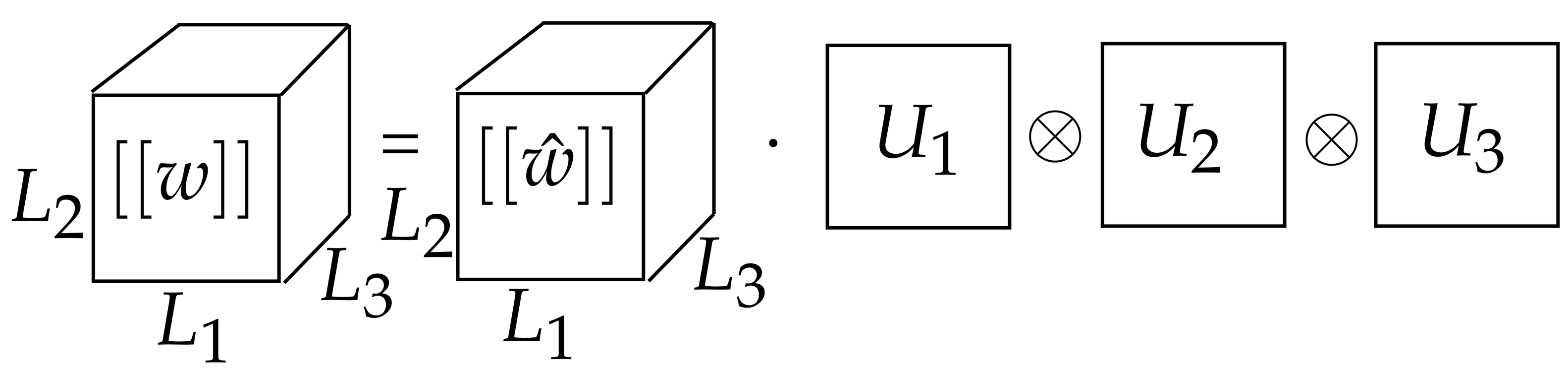

Figure 2: Visualization of the SVD of a 3-way array

The columns of U_1 and U_2 contain the basis functions that will be used in the model reduction step.

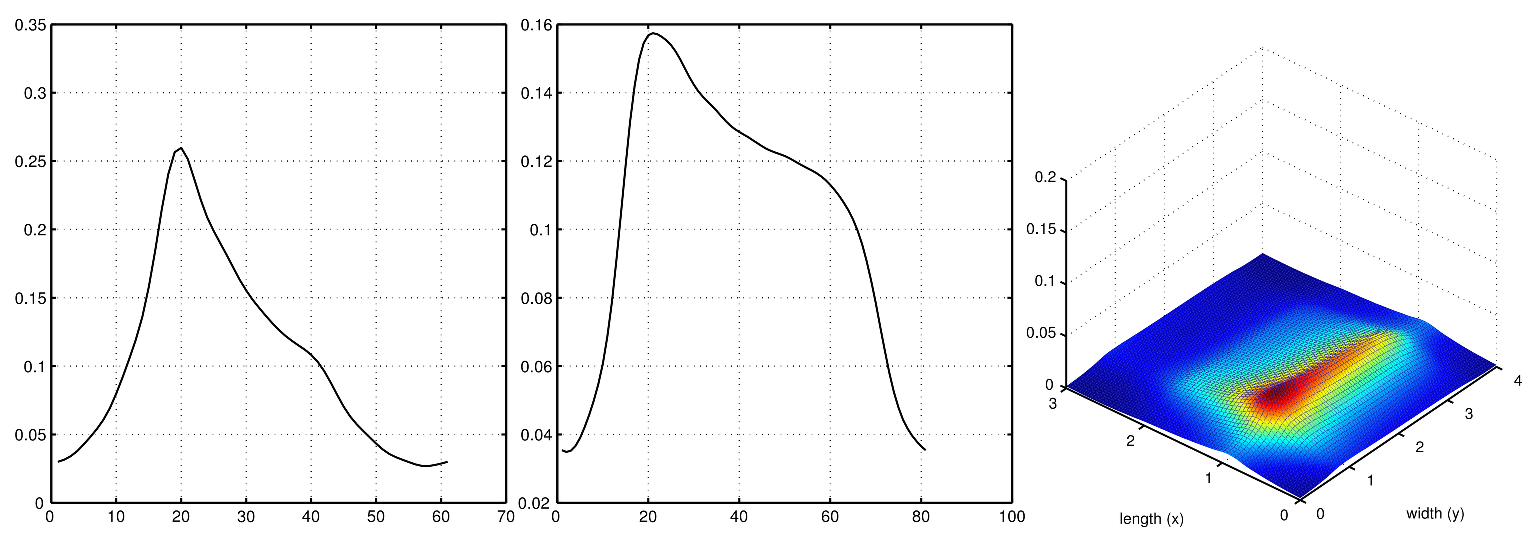

Figure 3: $u_1^{(1)}$ (left), $u_2^{(1)}$ (middle) and $u_1^{(1)} \otimes u_2^{(1)}$ (right)

Step 3: Model Reduction

Model reduction is carried out by Galerkin projections[3]. The spectral expansion of the signal is truncated to $w_r(x, y, t) = \sum_{ij}^{r_1, r_2} a_{ij}(t) u_1^{(i)}(x) u_2^{(j)}(y)$, and a Galerkin projection defines the time trajectory of the coefficients $a_{ij}(t) = [A(t)]_{ij}$ as a solution of the Ordinary Differential Equation (ODE)

$$0 = -\rho c_p \dot{A} + \kappa_x F A + \kappa_y A P. \tag{2}$$

Simulations of the reduced order model can be compared with the original model:

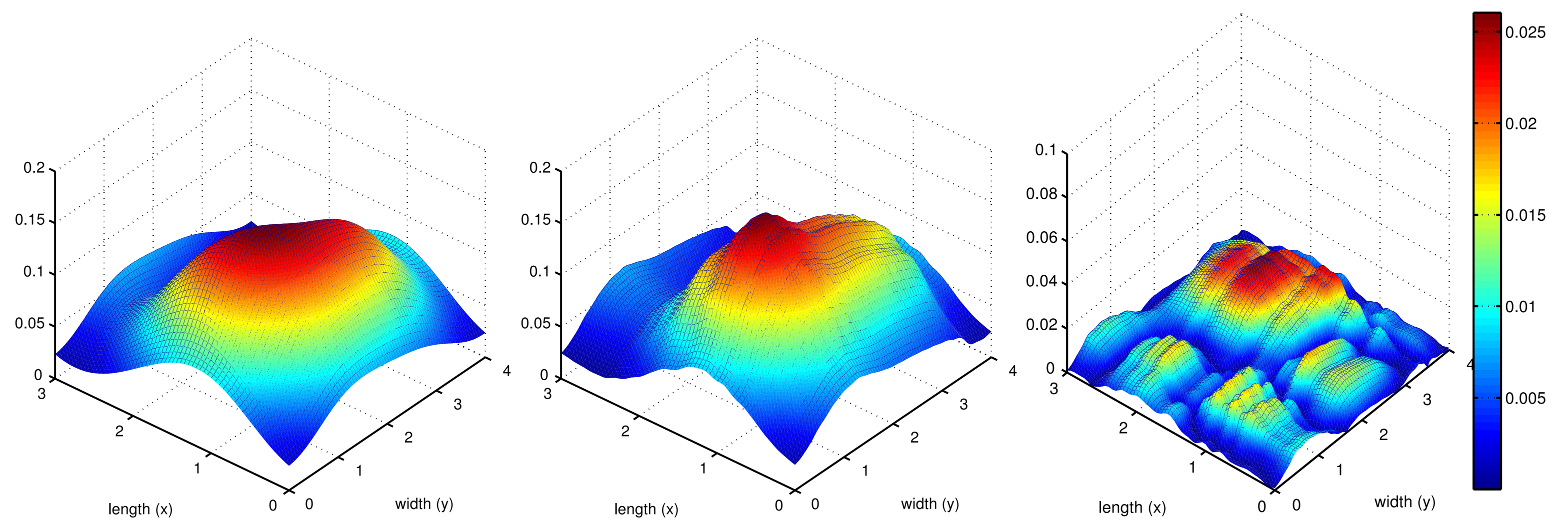

Figure 4: Time slice of original data at time t_{40}(left), time slice of reduced model of order $(7, 7)$ at time t_{40} (middle) and time slice of absolute error at time t_{40} (right).

Conclusions

- Considered model reduction for multidimensional systems
- Proposed a method for the computation of empirical projection spaces using tensor decompositions.
- Generic method, applied to heat diffusion process
- Future work: test the method on more complex examples and compare different tensorial decompositions to assess accuracy, computational effort and reliability.

References

[1] L. de Lathauwer et al., *A Multilinear Singular Value Decomposition*,SIAM J. Matrix Anal. Appl., Vol. 21 (4), 2000.

[2] F. van Belzen, S. Weiland and J. de Graaf, *Singular value decompositions and low rank approximations of multi-linear functionals*, Proc. 46th IEEE Conf. on Decision and Control, 2007.

[3] S. Volkwein and S. Weiland, *An Algorithm for Galerkin Projections in both Time and Spatial coordinates*, Proc. 17th MTNS, 2006.