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A New Discrete Slug-Flow Controller for Production Pipeline Risers ⋆

Stasiak, M. E. * **Pagano, D. J.** * **Plucenio, A.** *

* *Departament of Automation and Systems, Federal University of Santa Catarina, 88040-900 Florianopolis - SC, Brazil*
e-mail: (m-stasiak, daniel, plucenio)@ das.ufsc.br

Abstract: In this paper we propose a new discrete slug-flow control to stabilize severe slugging oscillations in submarine oil-risers. Since the practical objetive is to stabilize the flow keeping the surface choke with a minimum pressure drop, the idea of pressure set-point looses significance. One could say that the control problem is well solved if the pressures and flow-rates do not oscillate while the surface choke is kept opened well above the opening which characterizes the beginning of the limit cycle. The idea is to develop a control strategy which supres the oscillation while keeping the choke opening operating around a desired opening value. If the oscillations are suppressed the resultant pressures will be a consequence of the input mass flow-rate, fluid characteristics and the system geometry. Simulation results were obtained for a case study using a comercial software in order to validate the proposed control system.

Keywords: Production oil system, Production riser, Control of oscillations, Slug-flow control.

1. INTRODUCTION

Severe slugging is a well-known dynamic phenomenon which occurs in offshore pipeline production oil-risers. This dynamic behavior is characterized by intermittent axial distribution of gas and liquid. The pressure and flow-rate oscillations induced by the slug-flow can provoke several undesired effects on the surface equipments. Indeed these types of disturbances can cause serious problems in the input of the multiphase flow separator, deteriorating the separation quality and causing level overflow (Godhavn et al. (2005)).

The suppression of this type of oscillations can be achieved by means of feedback automatic control methods. Two main objectives should be achieved acting on the surface production valve: (i) to stabilize the flow in risers minimizing the problems on the separator; (ii) to increase oil production with valve openings larger than in open-loop. At the same time, two other benefits can be obtained: (i) in cases where the oil is pumped from sea bottom, energy consumption is minimized; (ii) in cases of risers connected to wells with natural or artificial lift flows, higher production is obtained by minimizing the back pressure in front of the well perforated zones.

Several control methods to prevent slug-flow oscillations in pipeline risers have been proposed in different works (see for instance Storkaas and Skogestad (2007), Storkaas (2005), Godhavn et al. (2005), Pagano et al. (2009), Di Meglio et al. (2009, 2010), Ogazi et al. (2009, 2010), Sivertsen et al. (2009)).

Most of these control approaches have been derived from a model of the system operating at an equilibrium point on the unstable manifold of the pressure-valve opening curve. In this way, pressure controller set-points are defined according to this operating point. Since the practical objective is to stabilize the flow keeping the surface choke with a minimum pressure drop, the idea of pressure set-point looses significance. Moreover, these approaches have not been efficient in the presence of flow-rate disturbances entering the riser. On the one hand, it is very difficult to estimate the steady state value of the pressure at the bottom of the riser. On the other hand, using an infeasible set-point for the riser bottom pressure does not help stabilization.

One could say that the control problem is well solved if the pressures and flow-rates do not oscillate while the surface choke is kept opened well above the opening which characterizes the beginning of the limit cycle. Our idea is to develop a control strategy based on an oscillatory model behavior of the process and without fixing any operational set-point. This approach allows to suppress system oscillations while keeping the valve opening operating around a desired opening value. If the oscillations are suppressed the resultant pressures will be a consequence of the input mass flow-rate, fluid characteristics and the system geometry.

The paper is organized as follows. In Section 2, our control strategy to suppress oscillations in non-linear systems is presented. An illustrative example applying the proposed control technique to eliminate oscillations on the classical Van der Pol system is shown in Section 3. In Section 4,

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a case study simulated with OLGA [1] software allow us to evaluate the proposed controller performance in order to stabilize slugging oscillations in risers.

2. PROPOSED CONTROL LAW

The main idea is to design a control law based on a simple model which represents a generic oscillatory process behavior. The basic idea was first presented in Ganzaroli (2011). During an oscillatory phenomena we can observe the existence of a periodic behavior with a fundamental frequency ω_n. The proposed discrete model that represents this behavior is given by

$$x(k) = a_1 x(k-1) - a_2 x(k-2) + b\,u(k-1), \quad (1)$$
$$y(k) = x(k) + C.$$

where x is the state of the system, y is the output's system, a_1, a_2 are parameters, u is the input's system and C is a constant. For an oscillatory signal, can be easily proven (see Appendix A) that $a_1 = 2 - b$, $a_2 = 1$ and $b = \omega_n^2 T_s^2$. From (1), we have

$$x(k+1) = a_1 x(k) - a_2 x(k-1) + b\,u(k), \quad (2)$$
$$y(k+1) = x(k+1) + C.$$

The control purpose is to suppress oscillations ensuring stable system operation and, at the same time, to keep the control at a desired value. We propose stabilize output system oscillations by making $\frac{\partial y}{\partial t} = 0$. In the discrete time, this objective can be rewriting as $\Delta y = e(k+1) = y(k+1) - y(k) = 0$. From (1) and (2) we can be express the error between two output samples as

$$\begin{aligned} e(k+1) &= y(k+1) - y(k) \\ e(k+1) &= a_1 e(k) - a_2 e(k-1) + b\,\Delta u(k) \end{aligned} \quad (3)$$

where

$$\Delta u(k) = u(k) - u(k-1). \quad (4)$$

The controller design relies on the Lyapunov theory. We propose the following candidate Lyapunov function

$$L(t) = \frac{1}{2} e(t)^2 \quad (5)$$

which is positive definite since $L(0) = 0$ and $L(e(t)) > 0$, $\forall e(t) \neq 0$. For the closed loop to be stable,

$$\frac{dL(e(t))}{dt} \leq 0. \quad (6)$$

Using the discrete form to represent the equation (6) , we have

$$\Delta L(e(k)) \leq 0, \quad (7)$$

whereas

$$\Delta L(e(k)) = e(k)\left(e(k) - e(k-1)\right). \quad (8)$$

[1] Multiphase flow software simulation commercialized from Scandpower.

The control application must ensure that $\Delta L(e(k)) \leq 0$ or $e(k)(e(k) - e(k-1)) \leq 0$. Thus, if we apply a control action that ensures $e(k+1) = Ge(k)$ with $0 < G < 1$,then

$$\begin{aligned} \Delta L(e(k)) &= Ge(k-1)\left(Ge(k-1) - e(k-1)\right) \\ \Delta L(e(k)) &= e(k-1)^2 \left(G^2 - G\right) \end{aligned} \quad (9)$$

with $e(k-1)^2 > 0$ and $\left(G^2 - G\right) < 0$. Replacing $e(k+1) = Ge(k)$ in equation (3) we achieve

$$Ge(k) = a_1 e(k) - a_2 e(k-1) + b\,\Delta u(k). \quad (10)$$

From (10) it is possible to obtain the following equivalent discrete PI controller

$$\begin{aligned} \Delta u(k) &= \left(\frac{G - a_1}{b}\right) e(k) + \left(\frac{a_2}{b}\right) e(k-1) \\ u(k) &= u(k-1) + \frac{1}{b}(G - a_1) e(k) + \left(\frac{a_2}{b}\right) e(k-1). \end{aligned} \quad (11)$$

The equation for a standard PI controller, discretized using backward difference method, is given by

$$\begin{aligned} u(k) &= u(k-1) + \gamma_0 e(k) + \gamma_1 e(k-1) \\ \gamma_0 &= K_c(1 + \frac{T_s}{T_i}) \\ \gamma_1 &= -K_c \end{aligned} \quad (12)$$

Now, comparing the similar terms in equations (11) and (12), the gains of the new controller are

$$\begin{aligned} K_c &= -\frac{1}{b} \\ T_i &= \frac{T_s}{1 - G - b} \end{aligned} \quad (13)$$

where $b = \omega_n^2 T_s^2$ and, as previously mentioned, T_s is the sampling time; G is the only design parameter, with $0 < G < 1$. The control law presented in (11) is modified in order to lead the control for the desired operating point. Then, the new control law is given by

$$u(k) = u(k-1) + K_c\left(1 + \frac{T_s}{T_i}\right) e(k) - K_c e(k-1) + \beta\left(u_d - u(k-1)\right) \quad (14)$$

where u_d is the desired operating point and β is a parameter that adjusts how fast the control action reaches the desired operating point.

3. A DEMONSTRATIVE EXAMPLE

In this Section, we present an example to illustrate our proposed control technique to suppress oscillations. Let the system that represents the controlled Van der Pol oscillator given by

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\omega_n^2 x_1 - \mu\left(x_1^2 - 1\right) x_2 + \omega_n^2 u(t) \end{cases} \quad (15)$$

where $u(t)$ is the control action and $y(t) = x_1(t)$.

The equilibrium points of the system (15) are $(\bar{x_1}, \bar{x_2}) = (u, 0)$ where the equilibrium $\bar{x_1}$ depends on control u. For $\mu > 0$ this equilibrium point is an unstable focus and around it there exists a stable limit cycle.

The control purpose is to reduce or suppress the amplitude of limit cycles of the system (15) stabilizing the oscillations. We assume that $\mu = 1$ and the oscillation frequency is approximately $\omega_n = \frac{2\pi}{T} = 0.91$. The oscillatory state-responses x_1 and x_2 are shown in Figure(1).

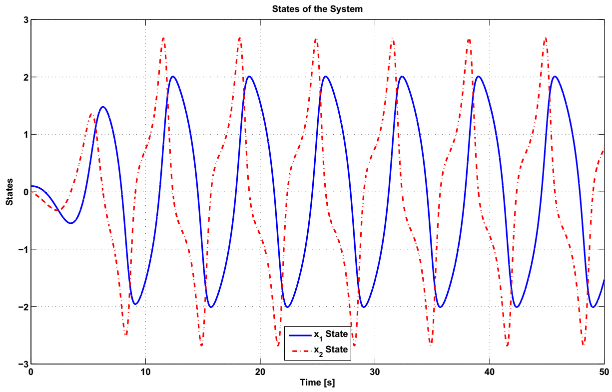

Fig. 1. Van der Pol system state-responses without control for $\mu = 1$.

In figure (2), it can be seen the system's state-responses when the proposed control law given by (14) is applied to system (15). At $t = 47s$ automatic control is turned ON and at $t = 250s$ the control is again turned OFF and the system comes back to the oscillatory behavior. The parameter values used for tuning the controller are shown in Table (1).

Table 1. Control law parameters

Parameter	Value
ω_n	0.91
T_s	0.01
G	0.9
K_c	-12076
T_i	0.1001
β	0.3
u_d	2

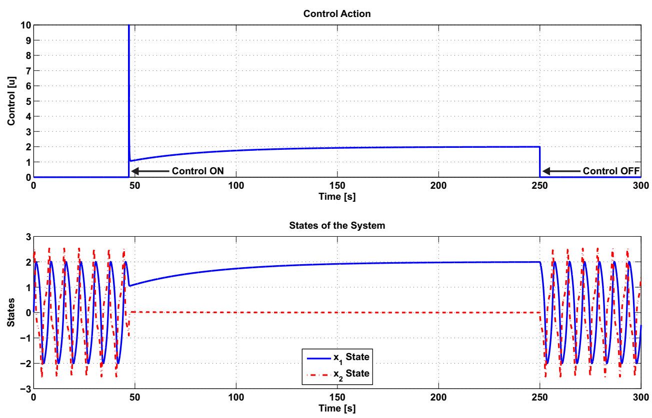

Fig. 2. Controlled Van der Pol system. At $t = 47s$ automatic control is turned ON and at $t = 250s$ the control is again turned OFF.

As shown in figure (2), the proposed control law stabilizes the system at one feasible equilibrium point and also achieve the desired operating point at $u_d = 2$.

4. SEVERE SLUGGING CONTROL IN PRODUCTION RISERS

A schematic diagram of a riser used in an oil production off-shore system is shown in Fig. 3 with parameters shown in Table 2. This system was simulated in OLGA.

In figure 3, bottom and top riser pressures P_1 and P_2, respectively, are measured in [Pa] units and the control action is applied on the production choke. Modeling this system is quite complex since it involves partial differential equations. In Storkaas (2005), a simplified third order dynamical model developed in ordinary differential equations. Other attempts to capture the slug-flow behavior in a model can be found in Di Meglio et al. (2009).

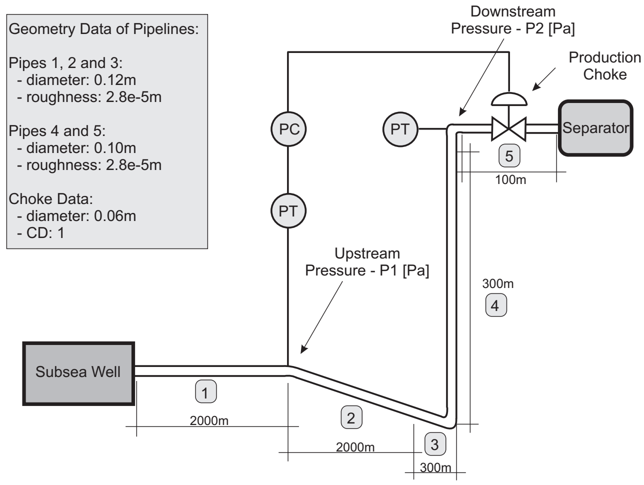

Fig. 3. Oil-riser system setup simulated in OLGA.

Table 2. Parameter values for OLGA simulation

Parameter	Value	Unit
Mass flow-rate in riser input	5	$Kg.s^{-1}$
Separator pressure	5.10^6	Pa
Temperature in riser output	22	^oC
Temperature in well	62	^oC

The bifurcation diagram considering the choke opening as a bifurcation parameter (see figure 4), was obtained based on OLGA data simulations for a mass flow-rate entering the riser equal to $5Kg.s^{-1}$ and a separator pressure of $5.10^6 Pa$. The bifurcation diagram of figure 4 is qualitatively similar to the diagram shown in Storkaas (2005). The stable and unstable equilibria manifold are depicted in figure 5 where the curves corresponding to maximal and minimum values of the limit cycle can be observed. As can be seen in figure 5, a supercritical Hopf Bifurcation takes place, at the point HB_{sup} in the diagram, giving rise to a stable limit cycle. Thus, without active feedback control it is necessary to operate the system with choke opening below 10% in order to avoid output system oscillations. Hence, the control objective is to stabilize the oscillations caused by severe slugging phenomena and to keep the choke with a maximum aperture. For it, we choose the

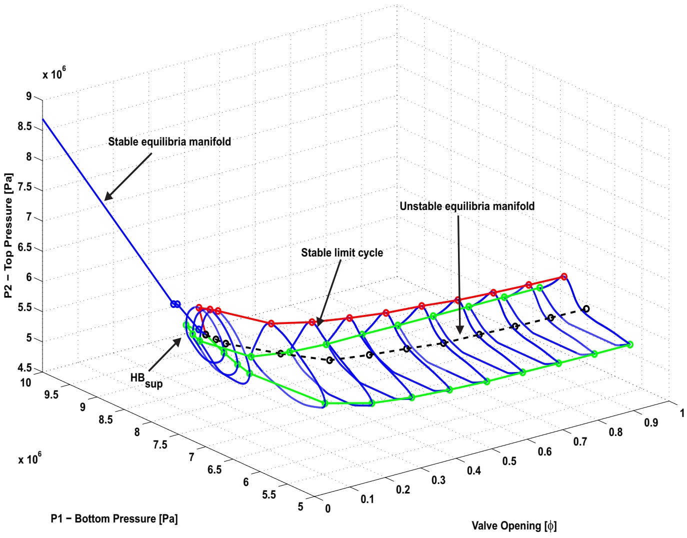

Fig. 4. Bifurcation diagram considering the choke opening as the bifurcation parameter. A stable limit cycle undergoes from a supercritical Hopf Bifurcation (HB_{sup}).

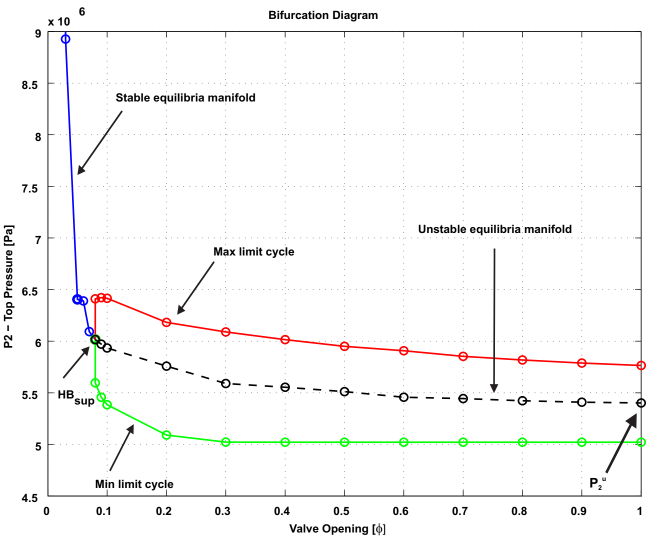

Fig. 5. Bifurcation diagram in (ϕ, P_2)-plane.

production choke opening as the manipulated variable and $\frac{\partial P_1}{\partial t}$ as the controlled variable. Using the theory proposed in Section 2, we obtained the simplified discrete model to represent the oscillatory system behavior. This model is given by

$$P_1 = \bar{P}_1 + \tilde{P}_1 \tag{16}$$
$$\tilde{P}_1(k+1) = a_1\tilde{P}_1(k) - a_2\tilde{P}_1(k-1) + b\, u_{P_2}(k)$$

where P_1 is the bottom pressure, $\bar{P}_1$ is the offset value of the pressure signal which includes the separator pressure P_s, $\tilde{P}_1$ represents its zero average oscillatory component, u_{P_2} stands for the pressure drop in the production choke. The dynamics behavior of $\tilde{P}_1$ is captured by means of a discrete model given by (16), with the sampling time T_s and where $a_1 = (2 - \omega_n^2 T_s^2)$, $a_2 = 1$ and $b = \omega_n^2 T_s^2$. In this way, $\tilde{P}_1$ is approximated by a sinusoidal signal with the frequency $\omega_n = \frac{2\pi}{T}$, where T is the oscillation period of $\tilde{P}_1$. Now defining

$$e(k+1) = \tilde{P}_1(k+1) - \tilde{P}_1(k), \tag{17}$$

the following equation is obtained

$$e(k+1) = a_1 e(k) + a_2 e(k-1) + b\Delta u_{P_2}(k). \tag{18}$$

One approximate model to represents the pressure drop in the production choke u_{P_2} is given by

$$u_{P_2} = \frac{B}{f^2(\phi)}, \quad B = \frac{q^2}{C_{vn}^2 \rho_m}, \tag{19}$$

where q is the mass flow-rate through the production choke, C_{vn} is the C_v for valve operating at 100%, ρ_m is the average fluid density and $f^2(\phi)$ is the function which represents the valve opening characteristic. Since q and ρ_m are not usually measured, a simplified expression is adopted to u_{P_2} where B is obtained from the characteristic (ϕ, P_2) curve, see figure 5. More precisely, this value is defined as $B = P_2^u - P_s$, where P_2^u is the pressure value estimated on the unstable equilibria manifold at the maximum choke opening, i.e. $\phi = 100\%$. Considering a linear characteristic valve, $f(\phi) = \phi$, equation (19) is rewriting as

$$u_{P_2} = \frac{B}{\phi^2}, \tag{20}$$

Substituting (20) in (16) we obtain

$$\tilde{P}_1(k+1) = a_1\tilde{P}_1(k) - a_2\tilde{P}_1(k-1) + b\left(\frac{B}{\phi^2}\right), \tag{21}$$

Note that, in the former equation, the actual control variable is the valve opening ϕ that indirectly controls the pressure drop u_{P_2}.

The proposed control law is given as follows

$$u_{P_2}(k) = u_{P_2}(k-1) + K_c\left(1 + \frac{T_s}{T_i}\right)e(k) - K_c e(k-1) + \beta\left(u_{P_2}^d - u_{P_2}(k-1)\right),$$

with $u_{P_2}^d = \frac{B}{\phi_d^2}$, where ϕ_d represents the desired choke opening and β is a parameter that determines the choke opening velocity. Since the control action u_{P_2} is a function of the valve opening, its saturation limits are computed according to the valve opening lower and upper bounds,

$$\begin{aligned} &\text{if } u_{P_2}(k) \le u_{P_{2min}} = \frac{B}{\phi_{max}^2} \text{ then } u_{P_2}(k) = u_{P_{2min}}, \\ &\text{if } u_{P_2}(k) \ge u_{P_{2max}} = \frac{B}{\phi_{min}^2} \text{ then } u_{P_2}(k) = u_{P_{2max}}, \end{aligned} \tag{22}$$

being $\phi_{min} = 0.01$ and $\phi_{max} = 1$. In practice, the actual control variable is the choke opening (ϕ), then, the pressure drop in the production choke (u_{P_2}) is converted in choke opening by

$$\phi(k) = \sqrt{\frac{B}{u_{P_2}(k)}}. \tag{23}$$

4.1 Simulation results

Table (3) shows the parameter values used to adjust the controller described in equation (22). Simulation results of the proposed slug-flow control law operating with $\phi_d = 0.7$ are shown in figure (6). At $t = 4h$ the control was switched ON and a disturbance in the input riser flow-rate from $5Kgs^{-1}$ to $4Kgs^{-1}$ is applied between $t = 24h$ and $t = 25h$. Between $t = 40h$ and $t = 41h$, another input flow disturbance from $5Kgs^{-1}$ to $6Kgs^{-1}$ is applied and at $t = 56h$ the control was switched OFF and the system comes back to the oscillatory regime.

Table 3. Control law parameters

Parameter	Value
ω_n	$0.0024[rad/s]$
T_s	$15[s]$
G	0.5
K_c	-771.60
T_i	20.03
B	$0.5\,10^6[Pa]$
β	0.03
ϕ_d	0.7

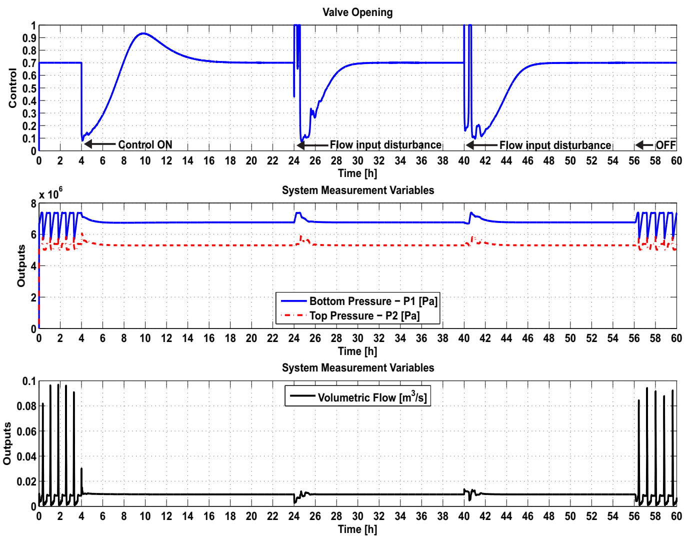

Fig. 6. Control and output system responses with the proposed control law and $\phi_d = 0.7$. At $t = 4h$ the control was switched ON. A disturbance in the input riser flow-rate from $5Kgs^{-1}$ to $4Kgs^{-1}$ is applied at $t = 24h$ and from $5Kgs^{-1}$ to $6Kgs^{-1}$ at $t = 40h$. Again at $t = 56h$ the control was switched OFF.

After several simulations, it was verified that the proposed control kills the oscillations and maintain the choke openings up to $\phi_d = 0.95$. Simulation results obtained with choke openings of $\phi_d = 0.9$ are shown in Fig. (7).

5. CONCLUSION

In this paper we propose a new slug-flow control to stabilize severe slugging oscillations in production risers. This control action reveals itself as a robust way of suppressing limit cycles when the mathematical model of the process is not available in practice and the operating point is unknown. This situation is manifested in the presence of mass flow-rate input riser disturbances. It is important to mention that the control algorithm proposed in this paper does not compute the choke opening but the pressure needed across the choke to stabilize the riser. The choke opening is updated using the control action and equation (20).

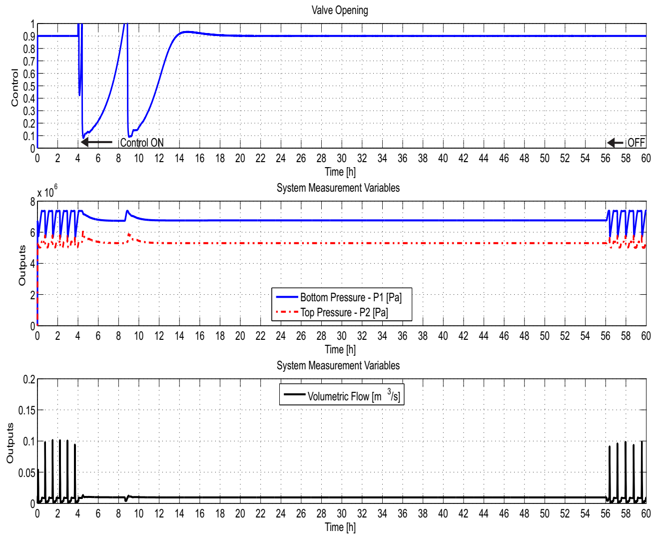

Fig. 7. Control and output system responses with the proposed control law and $\phi_d = 0.9$. At $t = 4h$ the control was switched ON and at $t = 56h$ the control was switched OFF.

Besides, the proposed control algorithm works suppressing process oscillations without forcing an operational bottom-pressure set-point. One drawback of the proposed oscillations control technique is the use of the output signal derivative (bottom pressure derivative for the slug control case, presented in equation (17)) since noise measurement is always present and it may create difficulties. The same technique has been applied with success in the control of heading and density wave in gas-lift wells and the results are presented in another publication. Finally, we believe that the control strategy here presented to suppress system oscillations can be applied to a more general class of non-linear systems.

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Appendix A

The controller design is based on an oscillatory model behavior. We use a modified Van der Pol system as a model that exhibits an oscillatory behavior. We adapt this model in order to include an input control variable as follows

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\omega_n^2 x_1 - \mu\left(x_1^2 - 1\right) x_2 + \omega_n^2 u\left(t\right) \end{cases} \tag{A.1}$$

with ω_n being the oscillation frequency of the output variable $y(t) = x_1(t)$. The equilibrium point of the system is $(\bar{x_1}, \bar{x_2}) = (0, 0)$, for $u\left(t\right) = 0$. Applying the standard local stability analysis to the system A.1, we obtain

$$J\left(x_1, x_2\right) = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 - 2\mu x_2 & -\mu\left(x_1^2 - 1\right) \end{pmatrix} \tag{A.2}$$

Solving the Jacobian matrix for the equilibrium point $(0, 0)$, we get

$$J\left(0, 0\right) = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & \mu \end{pmatrix} \tag{A.3}$$

being the determinant $D\left(J\right) = \omega_n^2$ and the trace $T\left(J\right) = \mu$. Hence, since $D\left(J\right) > 0$, $(0, 0)$ is a stable equilibrium point if $\mu < 0$ and it is an unstable equilibrium point if $\mu > 0$. At $\mu = 0$, we have a focus-center-limit-cycle bifurcation (F-C-L)) (see Freire et al. (1999)). The limit cycle corresponds with the outermost linear periodic orbit of the center configuration that exists for $\mu = 0$. Note also that at $\mu = 0$, the amplitude behavior evolves having a jump, different from the one in the case of the classical Hopf bifurcation. The corresponding bifurcation diagram is shown in figure A.1. The changes in the dynamical behavior of the system for different values of the parameter μ can be observed in figure A.2. As can be seen, for $\mu = 0$, we have a center but for $\mu > 0$, appears a stable limit cycle.

For enough small values of the parameter μ, for instance $\mu = 0.01$, the system's behavior can be approximated by the behavior of its linear part (A.4) given by

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\omega_n^2 x_1 + \omega_n^2 u\left(t\right) \end{cases} \tag{A.4}$$

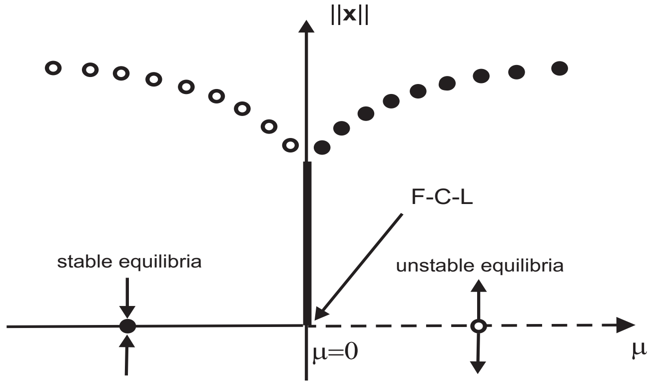

Fig. A.1. Bifurcation diagram for the μ parameter showing a Focus-Center-Limit cycle bifurcation that appears in $\mu = 0$. The vertical line stands for the existence of a center. (–) stable equilibria, (- -) unstable equilibria, (●●) stable limit cycle, (○○) unstable limit cycle.

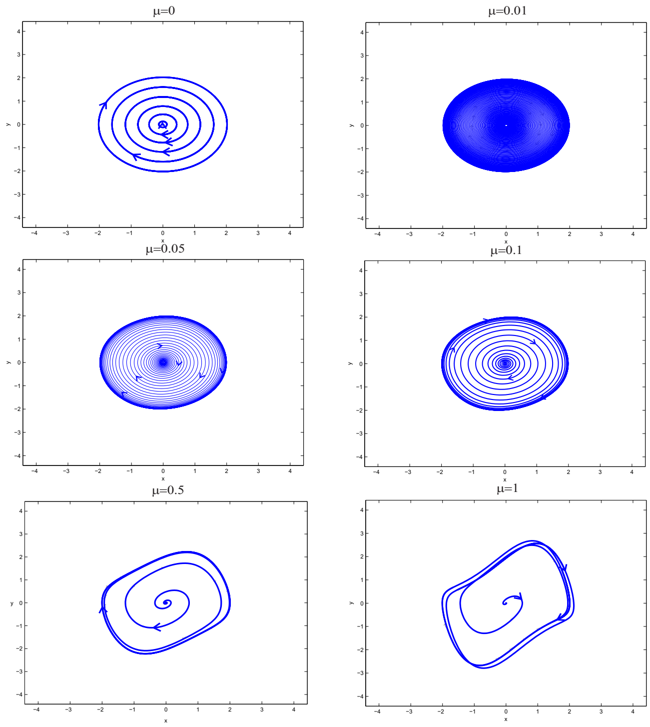

Fig. A.2. System dynamical behavior for different values of the bifurcation parameter μ showing the abrupt appearing of a stable limit cycle due to F-C-L bifurcation.

with $y\left(t\right) = x_1\left(t\right)$. This linear model can be also represented by the following model

$$\ddot{y}\left(t\right) + w_0^2 y\left(t\right) = w_0^2 u\left(t\right). \tag{A.5}$$

Using backward approximation in (A.5), where $\dot{y}\left(t\right) \cong \frac{y(k)-y(k-1)}{T_s}$ and $\ddot{y}\left(t\right) \cong \frac{y(k+1)-2y(k)+y(k-1)}{T_s^2}$, we get the following discrete model

$$y\left(k+1\right) - 2y\left(k\right) + y\left(k-1\right) + \omega_n^2 T_s^2 y\left(k\right) = \omega_n^2 T_s^2 u\left(k\right). \tag{A.6}$$

Rearranging the terms of the equation (A.6), we have

$$y\left(k+1\right) = a_1 y\left(k\right) - a_2 y\left(k-1\right) + bu\left(k\right) \tag{A.7}$$

where $a_1 = (2 - \omega_n^2 T_s^2)$, $a_2 = 1$ and $b = \omega_n^2 T_s^2$.