Analysis of Critical Phenomenon on Gossip Protocol using Back-Ultradiscretization

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Abstract

Critical probability of a cellular automaton is investigated via a novel approach of back-ultradiscretization. Specifically, back-ultradiscretization of a gossip protocol provides a conservative yet analytical lower bound, which is usually hard to be evaluated in the form of cellular automata. Comparison of theoretical and numerical values are provided for several representative grids to evaluate efficacy of the proposed approach.

1. Introduction

Cellular automaton is suitable for representing a phenomenon where a multitude of particles/agents interact with each other. In the framework of cellular automaton, spatial field is divided into numerous cells and the state of each cell evolves according to governing dynamics. There are two types of cellular automaton; deterministic cellular automaton and stochastic cellular automaton. The framework of stochastic cellular automaton is widely applied in practice to modeling propagation of forest fires, information transmission, disease propagation, etc. Stochastic cellular automata are typically not tractable in evaluating analytical properties and very few theoretical results are concluded if the structure of the partitioned cells are intricately interrelated.

In the past decade, the method of ultradiscretization has been developed in the field of applied physics. Ultradiscretization brings algebraic equations which are described with summation and multiplication to a class where the transformed algebraic equations are written with summation and max operation.

In this paper, we apply the idea of (back-)ultradiscretization to dynamical systems. Specifically, we consider a stochastic cellular automaton that models a diffusive property of information propagation. We focus on a special type of gossip protocol which can be described as a stochastic cellular automaton with summation and max operation.

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n\times 1$ real column vectors, and $\mathbb{N}_0$ denotes the set of nonnegative integers. Furthermore, we write $(\cdot)^{\mathrm{T}}$ for transpose.

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2. Literature Review

2.1. Cellular Automaton

In the field of epidemiology, Harris introduces a contact process which is described as a stochastic cellular automaton [11] and the infection probability was a major topic to determine if the disease transmission takes place to the infinitely large area. Furthermore, it is also known that in some cases this model gives rise to spatial oscillation [6, 12]. In recent years, analysis of complex networks attracts much attention and several practical results concerning the effective prevention of disease proliferation are reported [5, 7].

In the field of information science, stochastic cellular automata are often used to model information propagation among a large number of agents. One of the notable examples is a gossip protocol proposed by Haas *et al.*[10]. The gossip protocol is a modified version of a fundamental algorithm of flooding [25, 26, 29] such that information transmission is more efficiently achieved. Mathematically, the gossip protocol can be formulated as a sort of stochastic cellular automaton. Even though computational and some extended results of the gossip protocol are presented in the literature [1, 3, 4, 8, 13, 16, 20–22], there are very few theoretical results. Some of the exceptions are given in [14,15] where the authors make theoretical connections between gossip protocols and percolation theory [2, 9].

There are also purely mathematical results on the cellular automaton. Wolfram[27, 28] investigated elementally cellular automata and categorize them into several classes. It is important to note that many of cellular automata can be expressed with summation and max operations to describe time evolution of the state of the cells.

2.2. Ultradiscretization

Ultradiscretization is under extensive investigation in the field of applied physics. The procedure of ultradiscretization brings algebraic equation written with summation and multiplication into a class of algebraic equations described by summation and max operations. This transformation can be viewed as a method to transform a class of arithmetic algebra to another class of max-plus algebra (or, tropical algebra). Ultradiscretization is sometimes called tropicalization because it transforms regular algebraic equations into the equations with tropical algebra. Transforming equations with summation and multiplication into equations with summation and max operations implies discretizing independent variables involved in the equations. This operation can be understood as a class of discretization.

The procedure of ultradiscretization is proposed by Tokihiro *et al.*[24] and it has been successfully applied to several partial differential equations, such as Lotka-Volterra equation, Burgers equation, and KdV equation. In particular, KdV equation is shown to be ultradiscretized into a box an ball system which forms a cellular automaton and both pre- and post-transformed systems exhibit a similar qualitative phenomenon which is preserved over the transformation.

The inverse operation of ultradiscretization is called the back-ultradiscretization. In general, the operation of (forward) ultradiscretization is unique whereas that of back-ultradiscretization is not. Even though the research on back-ultradiscretization is virtually nonexistent in the literature, the references [18, 19] try to give some analysis on the classification of elementary cellular automata from viewpoint of back-ultradiscretization.

3. Ultradiscretization and Inverse Ultradiscretization

As mentioned in the Introduction, ultradiscretization describes the transformation from an algebraic equation that is written with sums and products to an equation represented by max and plus via changes of variables and taking limits. In this section we present a fundamental procedure of ultradiscretization. Readers are advised to see [17, 23, 24] for more details.

3.1. Ultradiscrete Limit

We begin by presenting a most fundamental equation in ultradiscretization given by

$$\lim_{\varepsilon\to 0} \varepsilon \ln\left(\exp\frac{\alpha}{\varepsilon} + \exp\frac{\beta}{\varepsilon}\right) = \max(\alpha, \beta), \quad (1)$$

where α, β, and ε are positive scalars. This quantity is called the *ultradiscrete limit* and the equation is key in discretizing dynamics under consideration. The proof of the relationship follows from the fact that

$$\varepsilon \ln\left(\exp\frac{\alpha}{\varepsilon} + \exp\frac{\beta}{\varepsilon}\right) = \beta + \varepsilon \ln\left(\exp\frac{\alpha-\beta}{\varepsilon} + 1\right), \quad (2)$$

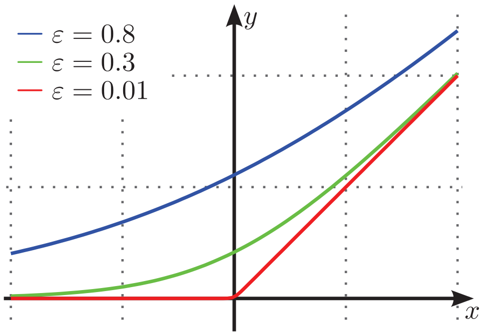

Figure 3.1: Graph of $y = \varepsilon \ln(\exp(x/\varepsilon) + 1)$ for $\varepsilon =$ 0.8, 0.3, 0.01

whose second term on the right-hand side converges to $\alpha - \beta$ (resp., 0) when $\alpha - \beta \geq 0$ (resp., $\alpha - \beta < 0$), and hence,

$$\varepsilon \ln\left(\exp\frac{\alpha-\beta}{\varepsilon} + 1\right) \xrightarrow{\varepsilon\to 0} \max(\alpha - \beta, 0). \quad (3)$$

This limit is also understood from Fig. 3.1, where x represents $\alpha - \beta$.

3.2. Procedure of Ultradiscretization

Consider the algebraic equation given by

$$a + b = c, \quad (4)$$

where a, b, c are positive scalars, and, with a positive scalar ε, consider the change of variables

$$a = \exp\frac{A}{\varepsilon}, \quad b = \exp\frac{B}{\varepsilon}, \quad c = \exp\frac{C}{\varepsilon}. \quad (5)$$

Substituting (5) into (4) yields $\exp(A/\varepsilon) + \exp(B/\varepsilon) = \exp(C/\varepsilon)$. Hence, taking log operation and multiplying ε to the both side of the equation, it follows that

$$\varepsilon \ln\left(\exp\frac{A}{\varepsilon} + \exp\frac{B}{\varepsilon}\right) = C. \quad (6)$$

Consequently, in the limit of $\varepsilon \to 0$, it follows from (1) that

$$\max(A, B) = C. \quad (7)$$

This result suggests that the algebraic equation (4) can be transformed into (7) under the limit of $\varepsilon \to 0$. The sequence of change of variables and taking limits is called *scale transformation*.

Likewise, the equation

$$d \cdot e = f, \quad (8)$$

can be transformed into

$$D + E = F, \quad (9)$$

through the change of variables $d = \exp(D/\varepsilon)$, $e = \exp(E/\varepsilon)$, $f = \exp(F/\varepsilon)$, and taking the limit of $\varepsilon \to 0$.

Table 3.1: Representative examples of ultradiscretization

algebraic equation	ultradiscretized equation
$a+b=c$	$\max(A,B)=C$
$ab=c$	$A+B=C$
$ab+cd=e$	$\max(A+B,C+D)=E$
$\dfrac{a+b}{c+d}=e$	$\max(A,B)-\max(C,D)=E$

In other words, the operations of summation (4) and multiplication (8) are transformed to the operations of maximization (7) and summation (9), respectively, through the scale transformation. The discretization through the scale transformation is called the *ultradiscretization*. Furthermore, a dynamic equation derived though the ultradiscretization is called a ultradiscretized equation. Note that the mapping of the variables (e.g., from a to A) through the ultradiscretization is not one to one because of the taking limit $\varepsilon \to 0$. Some of the representative examples of ultradiscretization is shown in Table 3.1.

3.3. Inverse Ultradiscretization

Inverse ultradiscretization denotes the inverse operation of ultradiscretization. For example, (4) describes the relation of (7) through back-ultradiscretization. It is important to note that because of the lack of one-to-one property for the mapping of ultradiscretization, back-ultradiscretization is not a unique transformation. This is understood by the fact that two different algebraic equations can be transformed into the same ultradiscretized equation. For example, ultradiscretization of (4) yields (7) and so does $a^2+b^2+ab=c^2$, because

$$\begin{aligned} 2C &= \lim_{\varepsilon\to 0} \varepsilon \ln\left(\exp\frac{2A}{\varepsilon}+\exp\frac{2B}{\varepsilon}+\exp\frac{A+B}{\varepsilon}\right) \\ &= \max(2A,2B,A+B) \\ &= \max(2A,2B). \end{aligned} \tag{10}$$

4. Characterization of SIS-Type Gossip Protocol via Inverse Ultradiscretization

In this section we begin by introducing a SIS-type gossip protocol for a model of information propagation over a network. Specifically, let a graph $G=(V,E)$ represent the network, where V and E denote set of the vertices (nodes) and the edges of the graph and represent the communicating agents and the communication routes, respectively. For example, in the case of sensor networks, the vertices represent the sensors and the edges represent the wired/wireless communication channels.

In this paper, we assume that the SIS-tipe gossip protocol is prescribed by the sequence of following rules:

Rule 1: There is only one node that initially possess a message. (We call this node a source node.)

Rule 2: The source node sends the message at time 0 to its neighboring nodes.

Rule 3: Every node receives the message when its neighboring nodes send the message to it.

Rule 4: When a node receives the message, it sends the message to the neighboring nodes with the common probability q. (We call this probability the *gossip probability*.)

Note that the SIS-type gossip protocol as defined above represents a probabilistic model of describing the way of message propagation.

It is intuitive that the higher the gossip probability is, the more likely the message transmission continues. In the case where the gossip probability is one, every node broadcasts the message whenever it receives the message. Of course, in this case message transmission keeps on going. On the other hand, if the gossip probability is zero, then none of the nodes ever transmit the message when they receive the message from the source node at the initial time, and hence, there is no more message transmission at all after the next time step. Note that the persistent activity of message transmission depends not only on the gossip probability, but also the structure of the graph G.

After all, the SIS-type gossip protocol prescribed by the above rules are described in a dynamic equation as follows.

Definition 4.1. Consider the finite graph $G = (V,E)$ and let $\mathcal{N}_v$ be the set of neighboring nodes of $v \in V$. Furthermore, consider the stochastic difference equation given by

$$x_{k+1}^v = \max_{u\in\mathcal{N}_v}(x_k^u, w_k^v) - w_k^v, \quad x_0^v = \begin{cases} 1, & v=0, \\ 0, & \text{else}, \end{cases} \tag{11}$$

where $x_k^v \in \{0,1\}$, $v\in V$, $k\in\mathbb{N}_0$, denotes the state representing if node v is broadcasting the message at time k ($x_k^v=1$) or if it is not ($x_k^v=0$), $w_k^v\in\{0,1\}$, $v\in V$, $k\in\mathbb{N}$, denotes the random variable determining in Rule 4 above if the message is broadcast at node v at time k ($w_k^v=1$) or not ($w_k^v=0$), and $\dot{v_0}$ denotes the source node. It is assumed that w_v^k, $k\in\mathbb{N}_0$, are i.i.d. processes and mutually independent with respect to $v\in V$ such that $\mathrm{P}\left(w_k^v=1\right)=1-q$ and $\mathrm{P}\left(w_k^v=0\right)=q$. We denote this protocol by $\mathcal{S}(G,q)$ and call it the SIS-type gossip protocol.

Remark 4.1. Even though the gossip protocol described in Rules 1–4 requires a random variable only when a node v receives the message at that time instant, the gossip protocol defined in (11) requires w_k^v for all time k. This fact, however, does not produce any mathematical inconsistency in defining the SIS-type gossip protocol and, henceforth, we provide analysis for the gossip protocol as defined in (11) in this paper.

As explained above, steady-state characteristics of the gossip protocol for $q=0$ and $q=1$ are qualitatively different in message transmission activities. For the discussion of the results in this paper, we define the survival

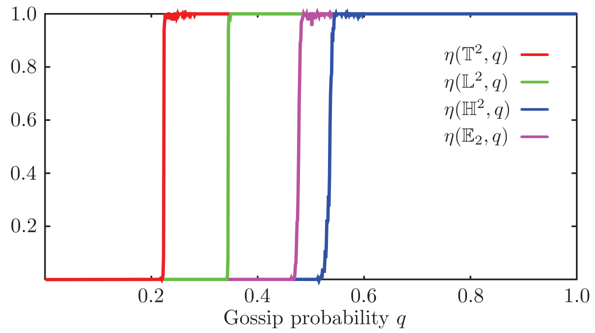

Figure 4.1: Survival probability versus the gossip probability for the cases of two-dimensional triangle lattice, square lattice, hexagonal lattice, and the regular tree with degree 3.

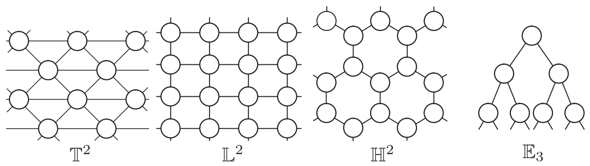

Figure 4.2: Two-dimensional triangle lattice $\mathbb{T}^2$, square lattice $\mathbb{L}^2$, hexagonal lattice $\mathbb{H}^2$, and the uniform tree $\mathbb{E}_3$ with degree 3

probability describing the likelihood of continuing message transmission infinitely many times as below.

Definition 4.2. Consider the gossip protocol given in Definition 4.1 for a graph G with the gossip probability $q \in [0, 1]$. The *survival probability* $\eta(G, q)$ is defined as

$$\eta(G, q) \triangleq \mathrm{P}\left[\lim_{k\to\infty} z_k > 0\right], \quad z_k \triangleq \sum_{v\in V} x_k^v. \tag{12}$$

Fig. 4.1 shows the survival probability versus the gossip probability for the cases of two-dimensional triangle lattice $\mathbb{T}^2$, square lattice $\mathbb{L}^2$, hexagonal lattice $\mathbb{H}^2$, and the uniform tree $\mathbb{E}_3$ with degree 3, that are represented in Fig. 4.2. Note that the values are numerically calculated through Monte-Carlo simulation. It can be seen from Fig. 4.1 that the survival probability $\eta(\cdot, q)$ takes values either 0 or 1 for almost all values of q and, in the case of square lattice $\mathbb{L}^2$, for example, the survival probability $\eta(\mathbb{L}^2, q)$ suddenly changes from 0 to 1 around $q = 0.35$. This critical value of q is called the critical gossip probability around which qualitative behavior of the message transition essentially changes. As seen from Table 4.1, the critical probability depends on the structure of the graph as well as the gossip probability. Mathematically, the critical probability is defined as below.

Definition 4.3. Consider the gossip protocol $\mathcal{S}(G, q)$ given in Definition 4.1 for a graph G with the gossip probability $q \in [0, 1]$ and let $\eta(G, q)$ be the survival probability for $\mathcal{S}(G, q)$. Then the *critical probability* $q_c(G)$ is

Table 4.1: Numerically calculated critical probability

Type of graphs	Critical probability
Triangle lattice ($\mathbb{T}^2$)	0.22
Square lattice ($\mathbb{L}^2$)	0.35
Hexagonal lattice ($\mathbb{H}^2$)	0.52
Uniform tree with degree 3 ($\mathbb{E}_3$)	0.47

defined as

$$q_c(G) \triangleq \inf\{q \in [0, 1] : \eta(G, q) > 0\}. \tag{13}$$

4.1. Inverse Ultradiscretization of SIS-Type Gossip Protocol

As described in the preceding sections, the critical probability $q_c(G)$ represents a characteristic of the survival probability $\eta(G, q)$. In this section we consider the problem of characterizing the critical probability $q_c(G)$ for the SIS-type gossip protocol $\mathcal{S}(G, q)$. Specifically, we establish an equivalent problem to the problem of finding the critical probability for the gossip protocol by applying back-ultradiscretization. It turns out that the derived problem is expected to have better prospects because the transformed dynamics are affine. Predicated on the affine property of the dynamics, we show that the transformed system has better tractability in calculating expectation.

Theorem 4.1. Consider the dynamics $\mathcal{S}_{\mathrm{bud}}(G, q)$ given by

$$X_{k+1}^v = 1 + R_k^v \sum_{u\in\mathcal{N}_v} X_k^u, \quad X_0^v = 1, \quad v \in V, \tag{14}$$

where R_k^v, $k \in \mathbb{N}_0$, denote i.i.d. processes and mutually independent with respect to $v \in V$ such that $\mathrm{P}(R_k^v = 1) = 1 - q$ and $\mathrm{P}(R_k^v = 0) = q$. Then the back-ultradiscretization of $\mathcal{S}_{\mathrm{bud}}(G, q)$ is given by $\mathcal{S}(G, q)$.

Next, we present back-ultradiscretization concerning the survival probability for the gossip protocol.

Theorem 4.2. Consider the function

$$\eta_{\mathrm{bud}}(G, q) \triangleq \mathrm{P}\left[\lim_{k\to\infty} Z_k = \infty\right], \quad Z_k \triangleq \prod_{v\in V} X_k^v. \tag{15}$$

Then it follows that

$$\eta_{\mathrm{bud}}(G, q) = \eta(G, q). \tag{16}$$

Theorem 4.2 indicates that the survival probability (12) of the SIS-type gossip protocol is equivalent to that in the transformed world. Hence, the equivalence on the critical probabilities is also immediate.

Corollary 4.1. The critical probability $q_c(G)$ of the SIS-type gossip protocol $\mathcal{S}(G, q)$ is equivalent to the critical value of the survival probability $\eta_{\mathrm{bud}}(G, q)$ in the inversely ultradiscretized protocol $\mathcal{S}_{\mathrm{bud}}(G, q)$ given by

$$q_{\mathrm{bud}}^{\mathrm{c}}(G) \triangleq \inf\{q \in [0, 1] : \eta_{\mathrm{bud}}(G, q) > 0\}.$$

Table 4.2: Comparison of critical probabilities obtained from Theorem 4.4 and those obtained numerically.

Type of graphs	Lower bound obtained from Theorem 4.4	Numerically obtained critical probability
$\mathbb{T}^2$	0.16	0.22
$\mathbb{L}^2$	0.25	0.35
$\mathbb{H}^2$	0.33	0.52
$\mathbb{E}_3$	0.36	0.47

It is important to note that the dynamics of the protocol $\mathcal{S}(G,q)$ are nonlinear involving the max operation, whereas the dynamics of the back-ultradiscretized protocol $\mathcal{S}_{\rm bud}(G,q)$ is affine in the state variable. For example, it becomes easier to calculate expectation for the back-ultradiscretized gossip protocol. The following result presents a characterization of the expectation.

Theorem 4.3. Consider the back-ultradiscretized gossip protocol $\mathcal{S}_{\rm bud}(G,q)$ and let $\overline{X}_k^v$ denote the expectation $\mathbb{E}[X_k^v]$ of the state variable X_k^v. Then it follows that

$$\overline{X}_{k+1}^v = 1 + q \sum_{u\in\mathcal{N}_v} \overline{X}_k^u, \quad X_0^v = 1, \quad v \in V. \tag{17}$$

Equivalently, in the vector form, (17) is described as

$$\overline{X}_{k+1} = e + qA_G\overline{X}_k, \quad \overline{X}_0 = e, \tag{18}$$

where A_G denotes the adjacency matrix of the graph G, $\overline{X}_k \triangleq [\overline{X}_k^1, \ldots, \overline{X}_k^N]^{\rm T}$, and $e \triangleq [1,\ldots,1]^{\rm T} \in \mathbb{R}^N$.

4.2. Characterization of Survival Probability

In this section we evaluate the survival probability of the SIS-type gossip protocol. Specifically, instead of dealing with the original gossip protocol $\mathcal{S}(G,q)$, we consider the inversely ultradiscretized gossip protocol $\eta_{\rm bud}(G,q)$ and derive the lower bound of the critical probability.

Theorem 4.4. Let $\lambda_{\max}(A_G)$ denote the maximum (real) eigenvalue of the adjacency matrix A_G of the graph G. Then the critical probability $q_{\rm c}(G)$ of the SIS-type gossip protocol satisfies $q_{\rm c}(G) \geq 1/\lambda_{\max}(A_G)$.

Table 4.2 compares the numerically obtained critical probabilities and its estimate obtained as a lower bound from Theorem 4.4 for different graphs. It can be seen from the table that Theorem 4.4 indeed provides lower bounds for each graph, but the bounds are conservative especially for the hexagonal lattice and the uniform tree with degree 3. This conservativity stems from the nonequivalence between the fact that the probability of diverging X_k is positive and the fact that its expected value $\overline{X}_k$ diverges.

5. Conclusion

In this paper we proposed the back-ultradiscretization framework for modeling the stochastic propagation of a message. Specifically, we define the survival probability for the SIS-type gossip protocol and further defined the critical probability for this protocol. In addition, we presented a novel characterization of transformation from the cellular automaton to another dynamic model, where expectation calculation is more tractable.

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