Optimality of passivity-based controls for distributed port-Hamiltonian systems

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Abstract: This paper discusses the (inverse) optimality and practical usage of passivity-based controls for distributed port-Hamiltonian systems. We first clarify that passivity-based controls, damping assignment and potential shaping can be derived from a linear quadratic type optimal control problem. Next, we describe the limitation of passivity-based boundary controls and propose a practical usage of the methods in terms of discretization. Finally, we illustrate numerical results having a similar property to the strain feedback methods derived from semigroup theory for stabilizing and stiffness controlling flexible beams.

Keywords: distributed parameter systems, passivity, optimal control, variational calculus

1. INTRODUCTION

In recent developments of advanced control systems, nonlinear PDEs have become important as numerical models of complex systems to reduce experimental productions. We only have to find numerical solutions for such analysis. On the other hand, many traditional approaches to controlling systems of PDEs are based on analytical solutions [1]. Control theory for systems from which it is difficult to derive analytical solutions seems not to be sufficiently discussed. The development of control theory that can be applied in the process of numerical analysis is practically significant.

This paper discusses the optimality of passivity-based controls for distributed port-Hamiltonian (DPH) systems [2] and a practical usage of the controls in numerical schemes. Passivity-based controls are conventional techniques for nonlinear systems; they are simple, versatile, robust in regard to disturbance, and we can apply them without information on analytical solutions of control systems [3, 4, 5, 6, 7]. Passivity-based controls follow two major strategies. One is to connect a control system with a compensating energy to a system for changing the energy of the original system through particular pairs of inputs/outputs, called port variables. The other strategy is to assign dissipative elements to increase the stability of the connected system at the global minimum of the shaped energy. Thus, passivity-based controls can globally stabilize nonlinear systems, which are difficult to analyze. However, this advantage is realized by trading off tight control performances based on analytical solutions, e.g., frequency response and orbits explicitly described by equations. Thus, they are, so to say, geometrical methods. Therefore, to bring out the best in passivity-based controls, we focus attention on enhancing the versatility of port representations.

On the other hand, a DPH system is a standard representation for partial differential equations (PDEs) for passivity based controls extended as boundary energy controls [3]. DPH systems satisfy a power balance that means an internal energy flow defined on the domain balances an energy flowing across the boundary. The power balance is derived from boundary integrability in the sense of Stokes theorem [12] and it is expressed by using boundary port variables. Hence, we can discuss boundary controls and observations with respect to the power balance by using DPH systems. However, system representations should be discretized when we apply the boundary controls to PDEs. Moreover, distributed controls have been expected as smart controller, e.g., light-weight, low power, and multi-degree of freedom with the developments in sensors and actuators made from soft materials [8].

This paper is constructed as follows. The second section is devoted to recall basic definitions on optimal problems and DPH systems. In the third section, we show that passivity-based control methods, i.e., damping assignment and potential shaping can be derived from optimal problems on DPH systems. This means that passivity-based controls can be considered as a solution of inverse optimal controls. In the final section, we illustrate applications of passivity-based controls for the DPH system of the flexible beam with large deformations [9, 10, 8]. The flexible beam model is nonlinear and it can be solved in terms of numerical calculations; however, the analytical solution of the model has not been derived. Furthermore, we clarify that passivity-based controls have a similar feature of the strain feedback methods [1, 11] derived from semigroup theory for stabilization and stiffness controls of Euler-Bernoulli beams. The strain feedback method is based on analytical solutions of PDEs; however, the methods derived from DPH systems do not depend on them.

2. PRELIMINARIES

This section is devoted to briefly summarize the basic definitions of variational problems and DPH systems.

2.1 Variational Problems

Variational problems with constraints are expressed as the problem of determining the minima of the functional

$$\mathcal{J} = P(t_0, t_1, x, \dot{x}) + \int_{t_0}^{t_1} F(t, x, \dot{x}, \lambda)\, dt, \tag{1}$$

$$F(t, x, \dot{x}, \lambda) = L(t, x, \dot{x}) + \lambda^\top(t) C(t, x, \dot{x}), \tag{2}$$

where $x(t) = [x_1(t), \cdots, x_n(t)]^\top$ is the vector of independent state variables, we denoted $\dot{x} = dx/dt$, $L(t, x, \dot{x})$ is the objective function, $P(t_0, t_1, x, \dot{x})$ is the boundary condition that free end-points must satisfy, $\lambda(t) = [\lambda_1(t), \cdots, \lambda_m(t)]^\top$ is the vector of Lagrange multipliers, and $C(t, x, \dot{x}) = [C_1(t, x, \dot{x}), \cdots, C_m(t, x, \dot{x})]^\top$ is the vector of constraints that a local minimum of L is subject to.

Let us consider control inputs $u(t) = [u_1(t), \cdots, u_n(t)]^\top$ as independent variables in (1), and consider the system of differential equations $\dot{x}_i = f_i(t, x, u)$ as the constraint $C_i(t, x, u, \dot{x}) = \dot{x}_i - f_i(t, x, u) = 0$ for $1 \leq i \leq m$. Then, the integrand (2) can be rewritten by the Hamiltonian $H(t, x, u, \lambda) = L(t, x, u) + \lambda^\top(t) f(t, x, u)$. Accordingly, the stationary condition of the variational derivative of H is given by the equations:

$$\dot{\lambda} = -\left(\frac{\partial H}{\partial x}\right)^\top, \quad \frac{\partial H}{\partial u} = 0, \quad \dot{x} = \left(\frac{\partial H}{\partial \lambda}\right)^\top. \tag{3}$$

The linear optimal regulator problem can be defined by (1) with $F = H$, where $f = A(t)x + B(t)u$, $L(t, x, u) = x^\top Q(t) x + u^\top R(t) u$, and $P(t_1, x) = x^\top(t_1) S x(t_1)$, where A and B are $(n \times n)$-matrices, Q and S are symmetric non-negative definite $(n \times n)$-matrices, and R is a symmetric positive definite $(n \times n)$-matrix. Then, the solution (3) of the problem is given as follows:

$$\dot{\lambda} = -Qx - A^\top \lambda, \; u = -R^{-1} B^\top \lambda, \; \dot{x} = Ax + Bu, \tag{4}$$

where $\lambda(t_1) = Sx(t_1)$. To solve the two boundary value problem determined by (4), we consider $S(t)$ such that $\lambda(t) = S(t)x(t)$. Indeed, we can calculate $S(t)$ by solving the Riccati equation. In this case, the control input can be described by $u = -R^{-1} B^\top S x$.

2.2 Distributed port-Hamiltonian systems

A DPH system is a standard control system representation for passivity-based boundary controls. We first define a system domain for DPH systems.

Definition 1. Consider the product space $Y = T \times X$ consisting of a 1-dimensional Euclidian space $T \simeq \mathbb{R}$ and an n-dimensional smooth Riemannian manifold X, where we denote the local coordinates of T and X, respectively, by the time coordinate t and the vector x of spatial coordinates, where x is the vector of functions of time: $x(t) = (x_1(t), \cdots, x_n(t))$. Then, we define the subspace $I \times Z \subset Y$ with the boundary $\partial I \times \partial Z$, where I means a time interval, and a connected n-dimensional submanifold $Z \subset X$ means the spatial domain at each time $t \in I$.

Next, we define state variables used in DPH systems.

Definition 2. Let

$$\begin{cases} (f^p, f^q) \in \Omega^p(Z) \times \Omega^q(Z), \\ (e^p, e^q) \in \Omega^{n-p}(Z) \times \Omega^{n-q}(Z), \\ (f^b, e^b) \in \Omega^{n-p}(\partial Z) \times \Omega^{n-q}(\partial Z), \end{cases} \tag{5}$$

where (f^i, e^i) for $i \in \{p, q\}$ and (f^b, e^b) are the pairs in the sense of the pairings,

$$\langle\, f^i, e^i \,\rangle = \int_Z e^i \wedge f^i, \quad \langle\, f^b, e^b \,\rangle = \int_{\partial Z} e^b \wedge f^b, \tag{6}$$

$\wedge$ is the wedge product, and we denoted the spaces of differential i-forms on a manifold $A \in \{Z, \partial Z\}$ by $\Omega^i(A)$.

Definition 3. Consider the following Hamiltonian density functional $\mathscr{H}$ with the Hamiltonian density $\mathcal{H}$:

$$\mathscr{H} = \int_Z \mathcal{H}(\alpha^p, \alpha^q), \tag{7}$$

$$\mathcal{H} = \alpha^p \wedge e^p + \alpha^q \wedge e^q \in \Omega^n(Z). \tag{8}$$

We define the variational derivative of (8) with respect to α^p and α^q as follows:

$$\begin{aligned} \delta\mathcal{H} &= e^p \wedge \delta\alpha^p + e^q \wedge \delta\alpha^q \\ &:= \frac{\partial \mathcal{H}}{\partial \alpha^p} \wedge \delta\alpha^p + \frac{\partial \mathcal{H}}{\partial \alpha^q} \wedge \delta\alpha^q, \end{aligned} \tag{9}$$

where $\delta\alpha^i \in \Omega^i(Z)$ means the infinitesimal variation with respect to α^i for $i \in \{p, q\}$. Then, the variational derivative (9) restricted to the time space T can be described as follows:

$$\delta\mathcal{H}|_T = \frac{\partial \mathcal{H}}{\partial \alpha^p} \wedge \frac{\partial \alpha^p}{\partial t} + \frac{\partial \mathcal{H}}{\partial \alpha^q} \wedge \frac{\partial \alpha^q}{\partial t}. \tag{10}$$

Thus, we define the variables (5) as

$$f^p = -\frac{\partial \alpha^p}{\partial t}, \; f^q = -\frac{\partial \alpha^q}{\partial t}, \; e^p = \frac{\partial \mathcal{H}}{\partial \alpha^p}, \; e^q = \frac{\partial \mathcal{H}}{\partial \alpha^q} \tag{11}$$

according to (6).

The variables f^i and e^i for $i \in \{p, q\}$ are called *port variables*, and they correspond to collocated control inputs and outputs for passivity-based distributed controls. Moreover, the variables f^b and e^b are called *boundary port variables* that are collocated control inputs and outputs for passivity-based boundary controls. DPH systems on the system domain Z can be defined as follows.

Theorem 4. ([2]). Consider (11). A *DPH system* is defined by substituting (11) into the following *Stokes-Dirac structure* with distributed terms:

$$\begin{cases} \begin{bmatrix} f^p \\ f^q \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} e^p \\ e^q \end{bmatrix} + f^d, \\ \begin{bmatrix} f^b \\ e^b \end{bmatrix} = \begin{bmatrix} e^p|_{\partial Z} \\ (-1)^p e^q|_{\partial Z} \end{bmatrix}, \; f^d = \begin{bmatrix} f^p_d \\ f^q_d \end{bmatrix}, \; e^d = \begin{bmatrix} e^p \\ e^q \end{bmatrix}, \end{cases} \tag{12}$$

where $d\colon \Omega^i(Z) \to \Omega^{i+1}(Z)$ is the exterior differentiation, $|_{\partial Z}$ is the restriction of differential forms to ∂Z, $r = pq + 1$, and $p + q = n + 1$. Here, we have defined the distributed energy variables f^d and e^d that are used for describing boundary non-integrable subsystems, where $f^p_d \in \Omega^p(Z)$ and $f^q_d \in \Omega^p(Z)$.

DPH systems satisfy the following boundary power balance.

Proposition 5. ([2]). A DPH system satisfies the following power balance:

$$\int_Z \left(e^p \wedge f^p + e^q \wedge f^q + e^d \wedge f^d\right) + \int_{\partial Z} e^b \wedge f^b = 0, \tag{13}$$

where each term $e^i \wedge f^i$ for $i \in \{p, q, b\}$ has the dimension of power. We call $e^b \wedge f^b$ a *boundary energy flow* and call $e^d \wedge f^d$ a *distributed energy flow*.

The power balance (13) is the essential relation used for passivity-based boundary controls and boundary observations. Because, in (13), the first integral is equivalent to the change in the Hamiltonian of the system defined on Z, and the second integral means the energy flow on ∂Z. Thus, the change in the energy defined on Z can be transformed into that on ∂Z.

3. OPTIMALITY OF PASSIVITY-BASED CONTROLS

This section shows that passivity-based controls for DPH systems, i.e., damping assignment and potential shaping are naturally derived from optimal problems. We first recall the dual structure of vector fields on oriented Riemannian manifolds.

3.1 Duals on Riemannian manifolds

Let M be an n-dimensional oriented Riemannian manifold. Differential k-forms ω are equivalent to antisymmetric contravariant k-tensors that define an alternating multilinear map $\omega_p(v_1, \cdots, v_k) \colon (T_pM)^k \to \mathbb{R}$ at any point $p \in M$, where T_pM is the tangent space of M at p, and we introduced the natural pairing $\langle df, v\rangle$ between $v \in T_pM$ and $df \in T_p^*M$. The space $\Omega^k(M)$ of differential k-forms on M is defined as all of smooth sections of the kth exterior power $\Lambda^k T^*M$ of the cotangent bundle of M.

The dimensions of $\Lambda^k T_p^*M$ and $\Lambda^{n-k}T_p^*M$ at $p \in M$ coincide with each other, i.e., $\dim \Lambda^k T_p^*M = \dim \Lambda^{n-k}T_p^*M = \binom{n}{k}$. Therefore, $\Lambda^k T_p^*M$ is isomorphic to $\Lambda^{n-k}T_p^*M$ as a vector space. Since M is oriented Riemannian, we can define the natural isomorphism by the linear map $*_p \colon \Lambda^k T_p^*M \to \Lambda^{n-k}T_p^*M; \theta_1 \wedge \cdots \theta_k \mapsto \theta_{k+1} \wedge \cdots \theta_n$, where $(\theta_1, \cdots, \theta_n)$ is the orthonormal basis of T_p^*M. Accordingly, the linear map $* \colon \Omega^k(M) \to \Omega^{n-k}(M)$ can be derived from the linear maps $*_p$ for all $p \in M$.

We can define the inner product of differential k-forms $\omega, \eta \in \Omega^k(M)$:

$$\langle\, \omega,\, \eta\, \rangle = \int_Z \eta \wedge *\omega \tag{14}$$

that can be considered as the pairing (6) in the case of manifolds with metric. Note that (14) can be regarded as the inner product of l-dimensional vector:

$$\begin{aligned}\langle\, \omega,\, \eta\, \rangle &= \int_Z (\eta^{i_1\cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\ &\qquad \wedge (\omega^{i_{k+1}\cdots i_n}\, dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_n}) \\ &= \int_Z \eta^{i_1\cdots i_k} \cdot \omega^{i_{k+1}\cdots i_n}\, dx^{i_1} \wedge \cdots \wedge dx^{i_n}, \\ &= \int_Z (\eta^{i_1\cdots i_k}, \omega^{i_{k+1}\cdots i_n})\, dx, \end{aligned} \tag{15}$$

where $l = \binom{n}{k}$, the coefficients $\eta^{i_1\cdots i_k}, \omega^{i_{k+1}\cdots i_n} \in C^\infty(M)$ are components of l-vectors, $dx \in \Omega^n(M)$ is the volume form, we defined the inner product of l-dimensional vector $(\ ,\ ) \colon (C^\infty(M))^l \times (C^\infty(M))^l \to \mathbb{R}$, and we used the summation convention regarding the indexes $i_1, \cdots, i_n$ of basic differential 1-forms dx^i.

3.2 Optimality of passivity-based controls for DPH systems

As we have seen in the previous section, optimal problems described by the pair of differential forms using the wedge product (14) can be considered as typical optimal problems described by the pair of vectors using the vector product (15).

In this section, we first describe that one of passivity-based controls, damping assignment using the port variables distributed on the whole system domain Z can be regarded as an optimal regulator.

Theorem 6. Let us consider the control input $u^i \in \Omega^{n-i}(Z)$ to the effort variable e^i of the DPH system (12), i.e., $\tilde{e}^i = e^i + u^i$, where $i \in \{p, q\}$. Then, the controls

$$u^p = (R^p)^{-1}\,(Q^p e^p - S^p \dot{e}^p)\,, \tag{16}$$

$$u^q = (R^q)^{-1}\,(Q^q e^q - S^q \dot{e}^q) \tag{17}$$

can be derived from the optimal regulator problem determined by

$$\mathscr{F} = \int_I \int_Z \mathcal{F}\, dt = \int_I \int_Z (\mathcal{L} + \lambda^p \wedge \mathcal{C}^p + \lambda^q \wedge \mathcal{C}^q)\, dt, \tag{18}$$

$$\begin{aligned}\mathcal{L} &= Q^p {*\alpha^p} \wedge \alpha^p + Q^q {*\alpha^q} \wedge \alpha^q \\ &\quad + R^p u^p \wedge *u^p + R^q u^q \wedge *u^q,\end{aligned} \tag{19}$$

where $Q^i, R^i, S^i \in C^\infty(Z)$ for $i \in \{p, q\}$ are constant functions, and we have defined the following constraints describing the DPH system:

$$\mathcal{C}^p = \dot{\alpha}^p + (-1)^r d(*\alpha^q + u^q) \in \Omega^p(Z), \tag{20}$$

$$\mathcal{C}^q = \dot{\alpha}^q + d(*\alpha^p + u^p) \in \Omega^q(Z) \tag{21}$$

by regarding e^i as $*\alpha^i$ in (12).

Proof. The integrand $\mathcal{F}$ of (18) can be transformed as follows:

$$\begin{aligned}\mathcal{F} &= \mathcal{L} + \lambda^p \wedge \{\dot{\alpha}^p + (-1)^r d(e^q + u^q)\} \\ &\quad + \lambda^q \wedge \{\dot{\alpha}^q + d(e^p + u^p)\} \\ &= \mathcal{L} - \dot{\lambda}^p \wedge \alpha^p - \dot{\lambda}^q \wedge \alpha^q \\ &\quad + (\partial/\partial t)(\lambda^p \wedge \alpha^p) + (\partial/\partial t)(\lambda^q \wedge \alpha^q) \\ &\quad - (-1)^r(-1)^{n-p} d\lambda^p \wedge (e^q + u^q) \\ &\quad + (-1)^r(-1)^{n-p} d\{\lambda^p \wedge (e^q + u^q)\} \\ &\quad - (-1)^{n-q} d\lambda^q \wedge (e^p + u^p) \\ &\quad + (-1)^{n-q} d\{\lambda^q \wedge (e^p + u^p)\},\end{aligned} \tag{22}$$

where we have used the formulas of integration by parts: $(\partial/\partial t)(\lambda \wedge \alpha) = \dot{\lambda} \wedge \alpha + \lambda \wedge \dot{\alpha}$ and $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for $\alpha, \omega \in \Omega^k(M)$ and $\lambda, \eta \in \Omega^{n-k}(M)$. By applying Stokes theorem to (22), we obtain

$$\begin{aligned}\mathcal{F} &= \mathcal{L} - \dot{\lambda}^p \wedge \alpha^p - \dot{\lambda}^q \wedge \alpha^q \\ &\quad - (-1)^{r+n-p} d\lambda^p \wedge (e^q + u^q) \\ &\quad + (-1)^{r+n-p} [\lambda^p \wedge (e^q + u^q)]_{\partial Z} \\ &\quad - (-1)^{n-q} d\lambda^q \wedge (e^p + u^p) \\ &\quad + (-1)^{n-q} [\lambda^q \wedge (e^p + u^p)]_{\partial Z},\end{aligned} \tag{23}$$

where we have eliminated constant terms on the boundary ∂I. We shall derive Euler-Lagrange equations from the variational derivative of (23) with respect to u^i and $*\alpha^i$. The Euler-Lagrange equation with respect to the variation of u^i derived from (23) is given by the stationary conditions of the following variation:

$$\left(\frac{\partial \mathcal{F}}{\partial u^i} - \frac{\partial}{\partial t}\frac{\partial \mathcal{F}}{\partial \dot{u}^i}\right) \wedge \delta u^i = \frac{\partial \mathcal{F}}{\partial u^i} \wedge \delta u^i$$
$$= \frac{\partial}{\partial u^i}\Big\{(-1)^{p(n-p)} R^p {*}u^p \wedge u^p + (-1)^{q(n-q)} R^q {*}u^q \wedge u^q$$
$$- (-1)^{r+n-p} d\lambda^p \wedge u^q + (-1)^{r+n-p}[\lambda^p \wedge u^q]_{\partial Z}$$
$$-(-1)^{n-q} d\lambda^q \wedge u^p + (-1)^{n-q}[\lambda^q \wedge u^p]_{\partial Z}\Big\} \wedge \delta u^i = 0$$
$$\longrightarrow$$
$$\Big\{(-1)^{p(n-p)} R^p {*}u^p - (-1)^{n-q} d\lambda^q + (-1)^{n-q}[\lambda^q]_{\partial Z}\Big\} \wedge \delta u^p = 0, \tag{24}$$
$$\Big\{(-1)^{q(n-q)} R^q {*}u^q - (-1)^{r+n-p} d\lambda^p + (-1)^{r+n-p}[\lambda^p]_{\partial Z}\Big\} \wedge \delta u^q = 0. \tag{25}$$

From the stationary conditions of (24) and (25), we obtain the following feedback laws:

$$\begin{aligned} {*}u^p &= (-1)^{p(n-p)}(-1)^{n-q}(R^p)^{-1}\left(d\lambda^q - [\lambda^q]_{\partial Z}\right) \\ &= (-1)^r (R^p)^{-1}\left(d\lambda^q - [\lambda^q]_{\partial Z}\right), \end{aligned} \tag{26}$$
$$\begin{aligned} {*}u^q &= (-1)^{q(n-q)}(-1)^{r+n-p}(R^q)^{-1}\left(d\lambda^p - [\lambda^p]_{\partial Z}\right) \\ &= (R^q)^{-1}\left(d\lambda^p - [\lambda^p]_{\partial Z}\right). \end{aligned} \tag{27}$$

The Euler-Lagrange equation with respect to the variation of ${*}\alpha^i$ derived from (23) is given by the stationary conditions of the following variation:

$$\left(\frac{\partial \mathcal{F}}{\partial {*}\alpha^i} - \frac{\partial}{\partial t}\frac{\partial \mathcal{F}}{\partial {*}\dot{\alpha}^i}\right) \wedge \delta {*}\alpha^i = \frac{\partial \mathcal{F}}{\partial {*}\alpha^i} \wedge \delta {*}\alpha^i$$
$$= \frac{\partial}{\partial {*}\alpha^i}\Big\{(-1)^{p(n-p)} Q^p \alpha^p \wedge {*}\alpha^p + (-1)^{q(n-q)} Q^q \alpha^q \wedge {*}\alpha^q$$
$$- (-1)^{p(n-p)} {*}\dot{\lambda}^p \wedge {*}\alpha^p - (-1)^{q(n-q)} {*}\dot{\lambda}^q \wedge {*}\alpha^q$$
$$- (-1)^{r+n-p} d\lambda^p \wedge {*}\alpha^q + (-1)^{r+n-p}[\lambda^p \wedge {*}\alpha^q]_{\partial Z}$$
$$-(-1)^{n-q} d\lambda^q \wedge {*}\alpha^p + (-1)^{n-q}[\lambda^q \wedge {*}\alpha^p]_{\partial Z}\Big\} \wedge \delta {*}\alpha^i$$
$$= 0$$
$$\longrightarrow$$
$$\Big\{(-1)^{p(n-p)} Q^p \alpha^p - (-1)^{p(n-p)} {*}\dot{\lambda}^p - (-1)^{n-q} d\lambda^q + (-1)^{n-q}[\lambda^q]_{\partial Z}\Big\} \wedge \delta {*}\alpha^p = 0, \tag{28}$$
$$\Big\{(-1)^{q(n-q)} Q^q \alpha^q - (-1)^{q(n-q)} {*}\dot{\lambda}^q - (-1)^{r+n-p} d\lambda^p + (-1)^{r+n-p}[\lambda^p]_{\partial Z}\Big\} \wedge \delta {*}\alpha^p = 0, \tag{29}$$

where we used the formulas $\eta \wedge {*}\omega = \omega \wedge {*}\eta \in \Omega^n(M)$ and $\eta \wedge {*}\omega = (-1)^{k(n-k)} {*}\omega \wedge \eta$ for any $\eta, \omega \in \Omega^k(M)$. By substituting the stationary conditions of (24) and (25) into (28) and (29), respectively, we obtain

$$Q^p \alpha^p - {*}\dot{\lambda}^p - R^p {*}u^p = 0, \tag{30}$$
$$Q^q \alpha^q - {*}\dot{\lambda}^q - R^q {*}u^q = 0, \tag{31}$$

The system of (30) and (31) can be rewritten as the port-Hamiltonian system,

$$\left\{ \begin{aligned} \begin{bmatrix} -{*}\dot{\lambda}^p \\ -{*}\dot{\lambda}^q \end{bmatrix} &= \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} \lambda^p \\ \lambda^q \end{bmatrix} - \begin{bmatrix} Q^p \alpha^p \\ Q^q \alpha^q \end{bmatrix}, \\ \begin{bmatrix} f^b \\ e^b \end{bmatrix} &= \begin{bmatrix} \lambda^p|_{\partial Z} \\ (-1)^p \lambda^q|_{\partial Z} \end{bmatrix}, \end{aligned} \right. \tag{32}$$

where $(f_d^p, f_d^q) := (Q^p \alpha^p, Q^q \alpha^q)$ are the distributed energy terms that mean dissipations. Accordingly, a trivial general solution of (32) can be given by

$$\lambda^p = S^p e^p, \quad \lambda^q = S^q e^q \tag{33}$$

for any $S^i \in C^\infty(Z)$ for $i \in \{p, q\}$. Hence, these relations with (30) and (31) yield the control laws (16) and (17). □

Remark 7. In the Lagrangian (19), we have used the volume form, e.g., $Q^i {*}\alpha^i \wedge \alpha^i$ as a cost function instead of the conventional quadratic form, e.g., $x^\top Q x$, where x is a vector and Q is a matrix. For example, in the case of $\mathcal{L} = p^2/(2m)\, dx$, where p is a function and m is a constant, we can define

$$\alpha^i = p\, dx^{(i)}, \quad {*}\alpha^i = e^i = \frac{\partial \mathcal{L}}{\partial \alpha^i} = \frac{p}{m}\, dx^{(n-i)}, \tag{34}$$

where $dx \in \Omega^n(Z)$, $dx^{(i)} \in \Omega^i(Z)$, note that $dx^{(i)}$ and $dx^{(n-i)}$ have the same dimension as a vector space spanned by basic 1-forms, and we have assumed that the size of the vector p is equivalent to that of the number of the basis of the space $\Omega^i(Z)$. Then, the cost function $\mathcal{H}$ can be given by

$$\frac{1}{2} {*}\alpha^p \wedge \alpha^p = \frac{p^2}{2m}\, dx, \tag{35}$$

which is a quadratic form on Riemannian manifolds. Note that the Hodge star $*$ depends on a metric of the manifolds.

3.3 Boundary controls and distributed controls

In the previous section, we discussed the optimality of the distributed controls. This section discusses the limitation of passivity-based boundary controls.

The general expression of distributed controls for the DPH system (12) is given by

$$f^p = (-1)^r d(e^q + v^q) - w^p, \tag{36}$$
$$f^q = d(e^p + v^p) - w^q, \tag{37}$$
$$f^b = (e^p + v^p)|_{\partial Z}, \quad e^b = (-1)^p (e^q + v^q)|_{\partial Z}, \tag{38}$$

where v^p, v^q, w^p, and w^q are controls. In this case, the power balance (13) is changed into

$$\int_Z \{(e^p + v^p) \wedge f^p + (e^q + v^q) \wedge f^q\} + \int_Z \{(e^p + v^p) \wedge w^p + (e^q + v^q) \wedge w^q\} + \int_{\partial Z} e^b \wedge f^b = 0. \tag{39}$$

The pair of v^p and v^q means boundary integrable, because they can be considered as portions of e^p and e^q. However, w^p and w^q means boundary non-integrable, and they cannot be transformed into the boundary term $e^b \wedge f^b$.

For practical purposes, we are interested in controlling the above pair of v^p and v^q by using their boundary value. That is, let us consider the boundary input $u^i \in \Omega^{n-i}(\partial Z)$ for $i \in \{p, q\}$ such that

$$f^{p\prime} = (-1)^r d(e^q + \tilde{u}^q), \quad f^{q\prime} = d(e^p + \tilde{u}^p), \tag{40}$$
$$f^{b\prime} = e^p|_{\partial Z} + u^p, \quad e^{b\prime} = (-1)^p (e^q|_{\partial Z} + u^q), \tag{41}$$

where $\tilde{u}^i \in \Omega^{n-i}(Z)$ is an admissible differential $(n-i)$-form satisfying $\tilde{u}^i|_{\partial Z} = u^i$. Actually, $\tilde{u}^i$ satisfies the power balance

$$\int_Z \{(e^p + \tilde{u}^p) \wedge f^{p\prime} + (e^q + \tilde{u}^q) \wedge f^{q\prime}\} + \int_{\partial Z} e^{b\prime} \wedge f^{b\prime} = 0. \tag{42}$$

Then, $\tilde{u}^i$ is naturally determined by the boundary condition given by u^iunder preserving the power balance (42). In other words, we cannot always assign desired states to $\tilde{u}^i$ distributed on Z by controlling u^i defined on ∂Z.

In this paper, we are considering geometric controls; therefore, we cannot investigate $\tilde{u}^i$ from the analytical viewpoint. Hence, we consider instead discretized optimal distributed controls assigned to equally-spaced points.

4. NUMERICAL EXPERIMENTATION

This section illustrates applications of passivity-based control methods for the flexible beam with large deformations [9, 10]. The DPH system of the flexible beam with large deformations that has been presented in [8]. Moreover, we discuss the relationship between the passivity-based control methods for general PDEs and the strain feedback methods for flexible beams.

4.1 Strain Feedback Methods

Let us consider the Euler-Bernoulli equation model

$$w_{tt} + \frac{EI}{A_\rho} w_{xxxx} = -x\tau_{tt}, \tag{43}$$

where $w(t,0) = w_x(t,0) = 0$, $w_{xx}(t,L) = w_{xxx}(t,L) = 0$, $x \in Z = [0,L] \in \mathbb{R}$ is the spatial coordinate along the beam, $w = w(t,x)$ is the shearing position at x, A_ρ is the mass per unit length, EI is the flexural stiffness, τ is the rotation of the electrical motor at the base of the beam, and the subscripts mean partial derivatives. Then, the strain feedback

$$\tau_{tt} = K_1 w_{txx}(t,0) \tag{44}$$

and the integral strain feedback

$$\tau_{tt} = K_2 w_{xx}(t,0) \tag{45}$$

are defined, where K_1 and K_2 are constants. The strain feedback (44) generates damping in (43). The integral strain feedback (45) changes the potential term $(EI/A_\rho)w_{xxxx}$ of (43).

4.2 System Model

In the previous section, the beam was modeled by the linear PDE, i.e., the Euler-Bernoulli beam from which analytical solutions can be derived in terms of semigroup theory. The Euler-Bernoulli beam model can be extended to a nonlinear PDE, i.e. it can be derived as a reduction model from the equations of beams with large deformations

$$\begin{cases} A_\rho \begin{bmatrix} y_{tt} \\ w_{tt} \end{bmatrix} - \dfrac{\partial}{\partial x}(\varLambda C \varGamma) = \begin{bmatrix} 0 \\ u \end{bmatrix}, \\ I_\rho \theta_{tt} - EI\theta_{xx} - \varPsi \varLambda C \varGamma = 0, \end{cases} \tag{46}$$

where $x \in Z = [0,L] \in \mathbb{R}$ is the spatial coordinate along the beam, $y = y(t,x) \in \mathbb{R}$ is the axial position along the equilibrium position, $(x+y)$ is the axial position, $w = w(t,x)$ is the shearing position, $\theta = \theta(t,x)$ is the rotation of the cross section along the unchangeable length of the beam, $u = u(t,x)$ is the distributed control input, A_ρ is the mass per unit length, I_ρ is the mass moment of the inertia of the cross section, EI is the flexural stiffness, EA is the axial stiffness, GA is the shear stiffness, and we defined the following variables:

$$C = \begin{bmatrix} EA & 0 \\ 0 & GA \end{bmatrix}, \ \varLambda = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{47}$$

$$\varGamma = \begin{bmatrix} \varGamma_1 \\ \varGamma_2 \end{bmatrix} = \varLambda^\top \begin{bmatrix} 1 + y_x - \cos\theta \\ w_x - \sin\theta \end{bmatrix}, \varPsi = \begin{bmatrix} -w_x \\ 1 + y_x \end{bmatrix}^\top. \tag{48}$$

The model (46) can be expressed by the port representation. From the total energy of the model:

$$\begin{aligned} \mathcal{H} = \frac{1}{2} \int_Z \{ & A_\rho {y_t}^2 + A_\rho {w_t}^2 + I_\rho {\theta_t}^2 \\ & + EA{\varGamma_1}^2 + GA{\varGamma_2}^2 + EI{\theta_x}^2 \} \, dx, \end{aligned} \tag{49}$$

the following port variable pairs are obtained:

$$\begin{cases} f^p = \begin{bmatrix} A_\rho y_{tt}\, dx \\ A_\rho w_{tt}\, dx \\ I_\rho \theta_{tt}\, dx \end{bmatrix}, & e^p = \begin{bmatrix} -y_t \\ -w_t \\ -\theta_t \end{bmatrix}, \\ f^r = -\theta_t \, dx, & e^r = -\varPsi \varLambda C \varGamma, \\ f^q = \begin{bmatrix} -y_{xt}\, dx \\ -w_{xt}\, dx \\ -\theta_{xt}\, dx \end{bmatrix}, & e^q = \begin{bmatrix} (\varLambda C \varGamma)_1 \\ (\varLambda C \varGamma)_2 \\ EI\theta_x \end{bmatrix}, \\ f^d = \begin{bmatrix} \mu A_\rho y_t \, dx \\ \mu A_\rho w_t \, dx\, dx \\ \mu I_\rho \theta_t \, dx \end{bmatrix}, \end{cases} \tag{50}$$

where $(\varLambda C \varGamma)_i$ means the i-th element of $\varLambda C \varGamma$. Accordingly, the DPH system is as follows:

$$\begin{cases} \begin{bmatrix} f^p \\ f^r \\ f^q \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & -* & 0 & 0 & d \\ 0 & 0 & * & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{e}^p \\ e^r \\ \tilde{e}^q \end{bmatrix} + \begin{bmatrix} f^d \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} f^b \\ e^b \end{bmatrix} = \begin{bmatrix} \tilde{e}^p|_{\partial Z} \\ -\tilde{e}^q|_{\partial Z} \end{bmatrix}, \end{cases} \tag{51}$$

where f^d is a weak viscosity damping term for numerical stabilizations, $\tilde{e}^i = e^i + u^i$ for $i \in \{p,q\}$, u^i is the control input, and note that the pair (f^r, e^r) does not yield boundary port variables.

4.3 Passivity-based controls

The formal form of the passivity-based controls for (51) is given by

$$u^p = (R^p)^{-1}(Q^p e^p - S^p \dot{e}^p), \tag{52}$$

$$u^q = (R^q)^{-1}(Q^q e^q - S^q \dot{e}^q), \tag{53}$$

where Q^i, R^i and S^i for $i \in \{p,q\}$ are constant diagonal $(3,3)$-matrices. In (53), the term $(R^p)^{-1}Q^p e^p$ corresponds to damping assignment by a direct velocity feedback. In (53), the term $-(R^q_3)^{-1}S^q_3 \dot{e}^q_3$ corresponds to the strain feedback (44) that generates damping, and the term $(R^q_3)^{-1}Q^q_3 e^q_3$ corresponds to the integral strain feedback (44) that changes the flexural stiffness, i.e., this is one of energy shaping methods, where k-elements or (k,k)-elements are denoted by the subscript k, and note that $\theta \approx w_x$ under small strains. Note that u^q is realized by using a pair of port variables at the boundary of each discretized domain; therefore, it can be considered as a discretized optimal distributed control assigned to equally-spaced points.

4.4 Numerical Results

Finally, we shall check the effects of the passivity-based controls in (53) in terms of numerical calculations with the method in [10]. We set $A_\rho = 1$, $I_\rho = 10$, $EI = 20$, $EA = 500$, and $GA = 500$. The base of the beam is constrained by $x = 0$ and $\theta = 0$ at the origin of a 2-dimensional space. The tip of the beam was initially actuated by a clockwise external force during 2 seconds. The left graph in Figure 1 and the solid line in Figure 3 illustrate a free motion of the beam.

We first used the gains $(R_1^q)^{-1}Q_1^p = (R_2^q)^{-1}Q_2^p = 0.01$ and $(R_3^q)^{-1}Q_3^p = 0.1$ for damping assignment at the points $x = 0.1L, 0.2L, \cdots, 0.9L$. The right graph in Figure 1 illustrates the controlled motion. It was stabilized.

We next used $\mu A_\rho = 0.001$ and $\mu I_\rho = 0.01$, which cause a very weak damping. We applied potential shaping to the beam at the boundary points of the intervals $[0, 0.1L]$, $[0.3L, 0.5L]$, and $[0.7L, 0.9L]$. The graphs in Figure 2 illustrate the motion controlled by the energy shaping with $(R_3^p)^{-1}Q_3^q = -5.0$ and $(R_3^p)^{-1}Q_3^q = -10.0$ during $t \in [0, 20]$, where the smaller gains make the beam stiffer. In Figure 3, the chained line shows the response in the second case, and the dashed line is the same setting with the damping assignment.

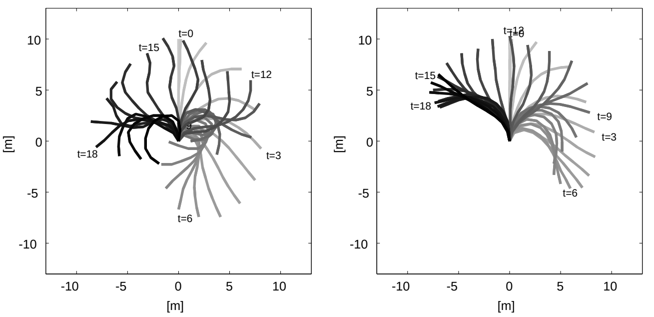

Fig. 1. Free motion (left), Damping assignment (right)

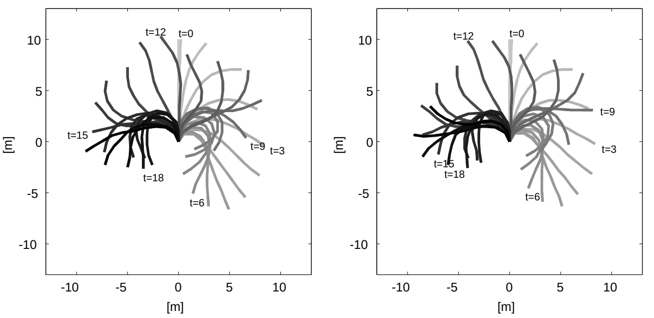

Fig. 2. Energy shaping 1 (left), Energy shaping 2 (right)

5. CONCLUSIONS

This paper derived passivity-based control methods from the optimal problem on DPH systems. We illustrated the numerical applications of passivity-based controls for the DPH system of the flexible beam with large deformations from which the analytical solutions has not been derived.

This work was supported by Japan Society for the Promotion of Science Grant Number No. 22360167.

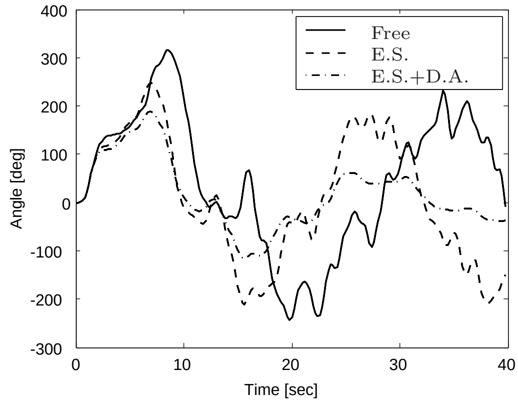

Fig. 3. Response of roations of tips

REFERENCES

[1] Z.H. Luo and B.Z. Guo, *Stability and Stabilization of Infinite Dimensional Systems with Applications*, Springer-Verlag, London, 1999.

[2] A. J. van der Schaft and B. M. Maschke, "Hamiltonian formulation of distributed-parameter systems with boundary energy flow", *J. of Geometry and Physics*, Vol. 42, pp. 166–194, 2002.

[3] V. Duindam, A. Macchelli, S. Stramigioli and H. Bruyninckx (Eds.): *Modeling and Control of Complex Physical Systems - The Port-Hamiltonian Approach*, Springer, 2009.

[4] A. J. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*, 2nd revised and enlarged edition, Springer-Verlag, London, 2000.

[5] M. Takegaki and S. Arimoto, "A new feedback method for dynamic control of manipulators", *J. of Dynamic Systems, Measurement, and Control*, Vol. 103, No. 2, pp. 119–125, 1981.

[6] J.C. Willems, "Terminals and Ports", *IEEE circ. and sys. magazine*, (2010), pp. 8–26.

[7] R. Sepulchre, M. Janković and P.V. Kokotović, *Constructive Nonlinear Control*, Springer, 1997.

[8] G. Nishida, K. Takagi, B. Maschke, and T. Osada, "Multi-scale distributed parameter modeling of ionic polymer-metal composite soft actuator", *Control Engineering Practice*, Vol.19, Issue 4, pp.321-334, 2011.

[9] J. C. Simo and L. Vu-Quoc, "On the Dynamics of Flexible Beams Under Large Overall Motions - The Plane Case: Part I", *J. of Applied Mechanics*, Vol. 53, Dec., pp. 849–854, 1986.

[10] J. C. Simo and L. Vu-Quoc, "On the Dynamics of Flexible Beams Under Large Overall Motions - The Plane Case: Part II", *J. of Applied Mechanics*, Vol. 53, Dec., pp. 855–863, 1986.

[11] Z. H. Luo, "Direct Strain Feedback Control of Flexible Robot Arms: New Theoretical and Experimental Results", IEEE Trans. on Automatic Control, Vol. 38, No. 11, pp. 1610–1622, 1993.

[12] S. Morita, *Geometry of Differential Forms*, American Mathematical Society, 2001.

[13] D. Liberzon, *Calculus of Variations and Optimal Control Theory: A Concise Introduction*, Princeton Univ. Press, 2011.

