Minimum-time control of a class of nonlinear systems with partly unknown dynamics and constrained input

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Abstract: The minimum-time problem frequently arises in the design of control for actuators, and is usually solved assuming to know the correct model of the system. In industrially important cases, however, important parts of the dynamics, like friction forces or disturbances by exosystems, are hardly known or even unknown. Against this background, this paper presents an iterative approach to achieve the minimum-time control for a nonlinear, single input second-order system with constrained input and partly unknown dynamics, effectively removing the requirement of perfect knowledge of the system and its parameters to achieve the minimum-time solution in application. First it is shown that, under reasonable assumptions about the unknown part of the dynamics, the optimal control exists for the presented class of systems and that it is a bang-bang control, with at most one switch. Then this property is exploited in the proposed algorithm, that finds the single optimal switching time by an iterative method, without involving any kind of identification of the unknown system parts.

1. INTRODUCTION

One of the more natural optimization objectives is the minimization of the time needed to complete a specific task, and one can find this requirement in various situations. In a technical context this can be translated for instance into the problem of opening/closing a valve as fast as possible, or more generally to drive an actuator to a specified position in minimum-time. While the basic structure of such systems may not be challenging, the fact, that important parts of the dynamics like friction or disturbances by exosystems are hardly known or even unknown, is challenging, as it prevents to find the (exact) optimal control for the minimum-time problem. This motivates to investigate the minimum-time problem of a nonlinear second order system, where some parts of the dynamics are unknown.

Apart from the mentionend area of actuators, the minimum-time problem can also be found in various additional technical applications, from wastewater treatment Moreno [1999], where the time needed to treat pollutants is minimized, over the field of robotics, that sparked a lot of publications (Slotine and Yang [1989], Sontag and Sussmann [1986], Bobrow [1988], Huang and McClamroch [1988], Shin and McKay [1985], Chernousko et al. [1989]), to special applications like the change of the linked-flux in a PMSM in minimum-time in Li and Xu [2001].

A similar system structure is treated in the work of Shen and Andersson [2010] as it treats the minimum-time problem of a second-order system, but with the difference of the system being assumed to be linear, time invariant and stable. The proposed optimal-time solution is then found by exploiting this linear setup and the availability of an analytic solution of the differential equations. They set up a system of equations to calculate the optimal switching time and the resulting final time, and propose to solve these equations by a numerical algorithm, which is motivated by some geometrical considerations in the phase plane. The comparison of the theoretical result with the application to a real plant showed that small model-mismatches (the plant was identified before) have a negative effect on the determined optimal control.

Another way to treat the minimum-time problem for second order systems is used in Rapaport and Dochain [2011] and Moreno [1999] in the minimum-time problem of biological reactors. They use a method based on Miele [1962], where the relative optimality of two trajectories in the reachable set can be compared using Green's Theorem. In the problem setting of Rapaport and Dochain [2011] the resulting minimum-time control is either bang-bang or bang-bang with singular arcs, depending on the properties of the involved biological reactions.

The system class treated in this work is similar to the classical double integrator with some additional dynamics. Contrary to some similar examples in Lee and Markus [1967], other referenced work and other available work on optimal control in general, we do not rely on the perfect knowledge of the system, but assume an important part of the system dynamics to be unknown. The assumptions which have to be made about the unknown system part are quite natural for a technical system and do not prevent the application of the proposed iterative method, that achieves the minimum-time control using information gathered by various experiments done with the system.

We present the system and the problem formulation in section 2, show the existence (section 3) and the bang-bang property (section 4) of the optimal control, and use that to derive an iterative algorithm in section 5 to achieve the optimal control. At the end we present a simulation example in section 6, including a comparison with the results of a numerical solver for a plausibility check, and in section 7 we draw some conclusions and give an outlook to possible extensions of the problem setting.

2. PROBLEM FORMULATION

We want to investigate the minimum time transition problem for a single input second order system

$$\dot{x}_1(t) = x_2(t) \tag{1a}$$
$$\dot{x}_2(t) = -F(x_2(t), t) + bu(t) \tag{1b}$$

where the control input u is limited to the interval $\mathscr{U}$

$$u \in \mathscr{U}, \qquad \mathscr{U} := [\underline{u}, \overline{u}], \qquad \underline{u} < 0, \overline{u} > 0$$

and the function F is unknown. The initial condition is arbitrary but fixed, $x(t_0) = x_0 \in \mathscr{X}_0$ and the objective is to achieve a state transition in minimum time to the origin $x(t_f) = (0,0)^{\mathrm{T}}$. It is assumed that by a suitable control u the system state can be brought back into x_0 and there exist time instants t_k such that

$$x(t_k) = x_0$$
$$F(x_2, t - t_k) = F(x_2, t) \quad \forall k \in \mathbb{N}_0, t \in [t_k, t_{k+1})$$

holds. This makes the state transition from x_0 to the origin a repeatable process, with t_k being the starting time for an iteration k. This allows us to consider the system (1) being defined at iteration k relatively to the starting time t_k, using as time parameter $t^k = t - t_k$. For simplicity of notation we will omit using the iteration index k whenever possible and continue to write t instead of t^k.

To solve the minimum time problem, we seek to minimize the cost function J by a proper sequence $u^*(t)$:

$$u^*(t) = \arg \min_{u(t)\in\mathscr{U}} J, \quad J = \int_0^{t_f} \mathrm{d}t. \tag{2}$$

We make the following assumptions about F:

A 1. The function is strictly increasing with x_2:

$$\frac{\partial F(x_2,t)}{\partial x_2} > 0$$

A 2. For every $x_2 \in \mathbb{R}$ the force F is bounded from above resp. from below for $x_2 \le 0$ resp. $x_2 \ge 0$. This prevents losing control authority over the system, when u is limited to the interval $\mathscr{U}$:

$$-F(x_2,t) + b\overline{u} > 0, \quad x_2 \le 0$$
$$-F(x_2,t) + b\underline{u} < 0, \quad x_2 \ge 0$$

A 3. F is continuous in x and t and continuously differentiable in x.

Note that the system defined in (1) contains an unknown term F that is time variant, but for the application of an iterative method the system has to be repeatable. Therefore the function F is time variant within one iteration but time invariant from one iteration to the next iteration – it shows the same time variance in iteration $k+1$ as in k. Systems that can be casted into this scheme are systems with unknown disturbances from exosystems that are resetable or systems with periodic disturbances with a period of $1/n \cdot (t_{k+1} - t_k), n \in \mathbb{N}$.

It is important to stress that A1 only applies to functions F that do depend on x_2 (an example would be $F = 0$, for which the system is the double integrator and for which the results in this paper holds too, though an analytic solution is available), and that A2 guarantees that there exists a control $\hat{u}(t) \in \mathscr{U}$, such that for any $x_0 \in \mathscr{X}_0$ the origin will be reached at a time $t_f > 0$.

3. EXISTENCE OF OPTIMAL CONTROL

The theorem of Filippov [1962] gives conditions under which an optimal control exists for systems

$$\dot{x} = f(t,x,u) \tag{3}$$

with x and f being n-dimensional vectors and the control input $u \in U(t,x)$. The optimal control is a function $u(t) \in U(t,x)$ such that the solution $x(t)$ of (3), with $x(0) = x_0$, $u = u(t)$ attains the point x^* in the least possible time. The following conditions must hold:

C 1. The vector function $f(t,x,u)$ is continuous in t,x,u and is continuously differentiable in x

C 2. The following inequality holds for all t,x and all $u \in U(t,x)$,

$$x^{\mathrm{T}} f(t,x,u) \le C\left(|x|^2 + 1\right)$$

with $|\cdot|$ being the length of a vector.

C 3. $U(t,x)$ shall be closed and bounded, and upper semicontinous in t and x with respect to inclusion.

When u describes $U(t,x)$, $f(t,x,u)$ describes a set $R(t,x)$, which is needed by following theorem:

Theorem 1. (Filippov [1962]). Suppose that conditions C1 – C3 stated above are satisfied, and that the set $R(t,x)$ is convex for every t and x. Also suppose that there exists at least one measureable function $\tilde{u} \in U(t,\tilde{x})$ such that the solution $\tilde{x}(t)$ of (3), with $u = \tilde{u}(t)$, and initial condition $\tilde{x}(0) = x_0$, attains x^* for some $t^* > 0$. Then there also exists an optimal control, i.e. a measurable function $u(t) \in U(t,x(t))$ for which the solution $x(t)$ of (3), with initial condition $x(0) = x_0$, attains x^* in the least possible time.

Using A3, system (1) satisfies condition C1, and condition C3 holds, as $\mathscr{U}$ is independent of t,x and it is a compact and convex set. Condition C2 is with system (1):

$$x_1\, x_2 + x_2(-F(x_2,t) + bu) \le C\left(x_1^2 + x_2^2 + 1\right). \tag{4}$$

Under A2 the left hand side of (4) can be expressed as

$$x_1\, x_2 + x_2(-F(x_2,t) + bu) \le x_1\, x_2 + |x_2|\,|b(\overline{u} - \underline{u})|$$

and substituting $|b(\overline{u} - \underline{u})|$ by a constant $\tilde{C}$ allows us to express (4) as

$$Cx_1^2 + Cx_2^2 + C - x_1 x_2 - \tilde{C}|x_2| \ge 0.$$

Denoting the left hand side of the inequality $h(x_1,x_2)$, we can see that $h(x_1, x_2 = 0) = C\left(x_1^2 + 1\right)$, which is strictly positive. For each of both cases, $(x_1,x_2) \in \mathbb{R} \times \mathbb{R}_-$ and $(x_1,x_2) \in \mathbb{R} \times \mathbb{R}_+$ we can find a single minimum of $h(x_1,x_2)$ by setting the first derivative of h to zero, which is

$$\nabla h(x_1,x_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{cases} \left(2Cx_1 - x_2, -x_1 + 2Cx_2 + \tilde{C}\right)^{\mathrm{T}}, & x_2 < 0 \\ \left(2Cx_1 - x_2, -x_1 + 2Cx_2 - \tilde{C}\right)^{\mathrm{T}}, & x_2 > 0 \end{cases}$$

The minimum is at $(x_{1,m}, x_{2,m}) = \left(-\frac{\tilde{C}}{4C^2-1}, -\frac{2C\tilde{C}}{4C^2-1}\right)$ respectively $(x_{1,m}, x_{2,m}) = \left(\frac{\tilde{C}}{4C^2-1}, \frac{2C\tilde{C}}{4C^2-1}\right)$ for $(x_1,x_2) \in \mathbb{R} \times \mathbb{R}_-$ respectively $(x_1,x_2) \in \mathbb{R} \times \mathbb{R}_+$. For both,

$$h(x_{1,m}, x_{2,m}) = \tilde{C}^2\left(-4C^3 + C\right) + C\left(4C^2 - 1\right)^2$$

and is strictly positive for any

$$C > \max\left(1, \sqrt{(\tilde{C}+1)/4}\right). \tag{5}$$

Calculating the Hessian confirms that both local extremals are global minima within their respective domains,

$$H(h) = \begin{bmatrix} 2C & -1 \\ -1 & 2C \end{bmatrix}, \quad (x_1, x_2) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}$$

as H is positive definite on the domain $\mathbb{R} \times \mathbb{R}_-$ and $\mathbb{R} \times \mathbb{R}_+$ for any C satisfying inequality (5), and it shows that $h(x_1, x_2)$ is strictly convex in both domains, $\mathbb{R} \times \mathbb{R}_-$ and $\mathbb{R} \times \mathbb{R}_+$. Therefore inequality (4) holds and condition C2 is satisfied.

The convexity condition on the set $R(t,x)$ means, see Cesari [1965], that for every (t,x) the set $\tilde{f}(t,x,U(t,x))$ is a convex subset of the $n+1$-dimensional Euclidean space, with $\tilde{f}(t,x,u) = (f_0(t,x,u), f(t,x,u))^{\mathrm{T}}$ where f_0 defines the cost function $J = \int_{t_1}^{t_2} f_0(t,x,u)\,\mathrm{d}t$. As of (2) $f_0 = 1$ and from (1) $f(t,x,u) = \begin{pmatrix} x_2 \\ -F(x_2,t) + bu \end{pmatrix}$, we can see that $R(t,x) = \tilde{f}(t,x,\mathscr{U})$ is a convex set for every t,x, as $\mathscr{U}$ is convex. As the conditions C1-C3 are satisfied as well, this shows, by theorem 1, that there exists an optimal control for the minimum time transition problem of system (1).

4. NATURE OF OPTIMAL CONTROL

The Hamiltonian of the optimization problem is (Stengel [1994], Bryson and Ho [1975]),

$$\mathscr{H} = 1 + \lambda_1 x_2 + \lambda_2 (-F(x_2,t) + bu)$$

and the evolution of the lagrangian multipliers is defined by $\dot{\lambda} = -\frac{\partial \mathscr{H}}{\partial x}$, giving

$$\dot{\lambda}_1 = 0 \tag{6a}$$

$$\dot{\lambda}_2 = -\lambda_1 + \lambda_2 \frac{\partial F}{\partial x_2} \tag{6b}$$

We cannot get $u^*(t)$ from the condition $\mathscr{H}_u = \frac{\partial \mathscr{H}}{\partial u} \overset{!}{=} 0$ as $\mathscr{H}_u$ is independent of u:

$$\mathscr{H}_u = \lambda_2 b$$

We can use in this case Pontryagin's minimum principle (Pontryagin et al. [1962], Stengel [1994]) to show the switching behaviour of the optimal solution $u^*(t)$

$$\begin{aligned} \mathscr{H}(x^*, u^*, \lambda^*, t) &\le \mathscr{H}(x^*, u, \lambda^*, t) \\ 1 + \lambda_1^* x_2^* - \lambda_2^* (F(x_2^*, t) + bu^*) &\le 1 + \lambda_1^* x_2^* \\ &\quad - \lambda_2^* (F(x_2^*, t) + bu) \\ \to \lambda_2^* u^* &\le \lambda_2^* u \end{aligned}$$

$$u^*(t) = \begin{cases} \underline{u} & \lambda_2^* > 0 \\ \overline{u} & \lambda_2^* < 0 \end{cases} \tag{7}$$

From (7) the optimal solution u^* is changing sign if λ_2 changes sign. For $\lambda_2^* = 0$ (7) does not provide any information about u^* and the control input can become any value in the interval $\mathscr{U}$. For the optimal solution u^* being of bang-bang type, the time interval where $\lambda_2^* = 0$ has to be infinitely short. This is not the case if $\dot{\lambda}_2^*$ and higher derivatives are zero in this time intervals. Note that from (6a) $\lambda_1^* = $ const., and that by (6b), we have:

$$\dot{\lambda}_2^* = -\lambda_1^* + \lambda_2^* \frac{\partial F}{\partial x_2}(x_2^*, t)$$

By setting $\lambda_2 = 0$ it is obvious that the first derivative of λ_2 is only zero, if $\lambda_1^* = 0$. But if $\lambda_1^* = 0$, the adjoint value λ_2^* will never become zero for any $\lambda_2^*(0) \neq 0$:

$$\dot{\lambda}_2^* = \lambda_2^* \frac{\partial F}{\partial x_2}(x_2^*, t) = \begin{cases} > 0 & \lambda_2(0) > 0 \\ < 0 & \lambda_2(0) < 0 \end{cases}$$

because of A1. The only case where λ_2 is zero for a non-infinitely short time interval, is if $\lambda_1 = 0$ and $\lambda_2(0) = 0$, which results in $\lambda_2(t) = 0, \forall t \in [0, t_f]$. In this case the Hamiltonian becomes

$$\mathscr{H} = 1$$

but we know that for the time optimal solution

$$\mathscr{H}(t = t_f) = 0$$

must hold, by which we know that $\lambda(t) = (0,0)^{\mathrm{T}}$ can not be the optimal solution. From that we conclude that for the time optimal solution λ_2^* is only zero for infinitely short time intervals and u^* is of bang-bang type.

To fix the number of necessary switches we have a look at the adjoint equations (6). As already mentionend above from (6a) we know that $\lambda_1 = $ const.$, \forall t \in [0, t_f]$. The evolution of λ_2 decides the number of switches because of (7), a change in the sign of λ_2^* causes a switch in u^*.

We can distinguish four different possible evolutions of λ_2, depending on the value of λ_1 and the initial value of λ_2:

I.) $\lambda_1^* > 0, \lambda_2^*(0) < 0$
In this case $\dot{\lambda}_2 < 0$ and $\lambda_2(t) < 0, \quad \forall t \in [0, t_f]$. No switch in u^* will occur.

II.) $\lambda_1^* < 0, \lambda_2^*(0) > 0$
In this case $\dot{\lambda}_2 > 0$ and $\lambda_2(t) > 0, \quad \forall t \in [0, t_f]$. No switch in u^* will occur.

III.) $\lambda_1^* > 0, \lambda_2^*(0) > 0$
If the condition $\lambda_2 \frac{\partial F}{\partial x_2} < \lambda_1$ holds for a sufficient long time interval, λ_2 will become negative, and a switch in u^* will occur. After that we will have the same situation as in case I and no further switch will occur.

IV.) $\lambda_1^* < 0, \lambda_2^*(0) < 0$
If the condition $\lambda_2 \frac{\partial F}{\partial x_2} > \lambda_1$ holds for a sufficient long time interval, λ_2 will become positive, and a switch in u^* will occur. After that we will have the same situation as in case 2 and no further switch will occur.

With that analysis it has been shown that at most *one* switch is sufficient for the time optimal state transition from an initial state to the origin. To solve the problem of minimum time state transition of an actuator with partly unknown dynamics it is sufficient to find out if the switch is necessary and what the switching time is. Due to the unknown part of the dynamics we propose to solve this by an iterative process.

5. ITERATIVE SOLUTION

The proposed algorithm is composed by two main parts – the first part detects the necessary sequence of input values, using the information about the initial condition and, if necessary, a first experiment. The second part uses a bisection method to find iteratively the optimal switching time $\bar{t}^*$. For derivation and better presentation of the iterative algorithm we will restrict F to time invariant functions $F(x_2)$ for now, and will have a look at the cases of $F(x_2,t)$ afterwards. Figure 1 shows schematically the time optimal trajectories of a hypothetical system in the state plane for different initial conditions $x_0^{(1)} \div x_0^{(6)}$. The full

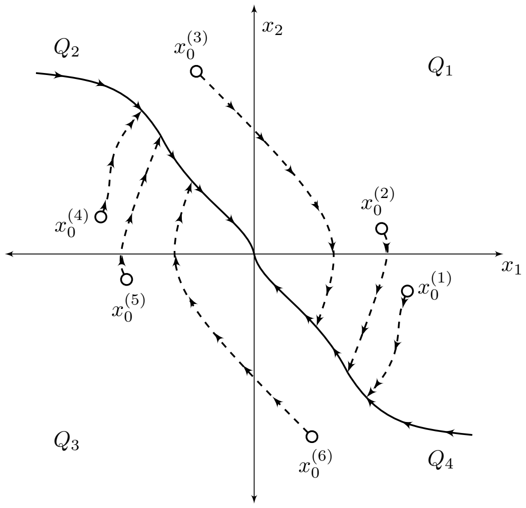

Fig. 1. The switching curve (solid line) and typical time optimal trajectories for different initial conditions $x_0^{(1)} \div x_0^{(6)}$ of a hypothetical system

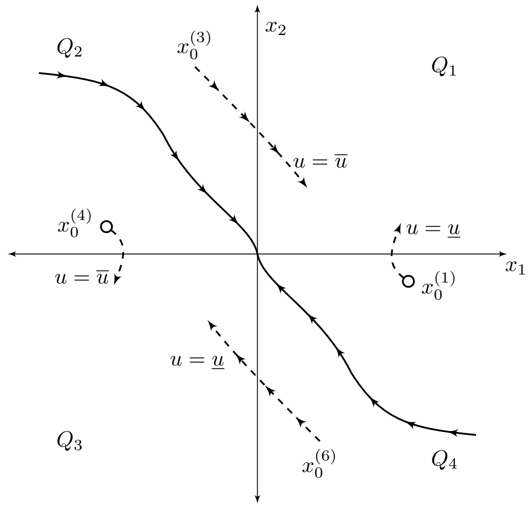

Fig. 2. Schematically the result of the first experiment run to determine the position of the initial condition in the state plane. This first experiment is only necessary if $x_0 \in Q_2 \cup Q_4$.

state space $\mathbb{R}^2$ is segmented into the four quadrants $Q_1 \dots Q_4$. The solid line is the switching curve Γ,

$$\Gamma = \left\{x_0 \in \mathbb{R}^2 \mid \exists t_f \geq 0 \,\text{s.t.}\, x(t_f) = (0,0)^{\mathrm{T}}, x(t) = \phi\left(0,t,x_0,\underline{u}\right)\right\}$$
$$\cup \left\{x_0 \in \mathbb{R}^2 \mid \exists t_f \geq 0 \,\text{s.t.}\, x(t_f) = (0,0)^{\mathrm{T}}, x(t) = \phi\left(0,t,x_0,\overline{u}\right)\right\}$$

where $\phi\left(0,t,x\left(t_k\right),u\left(t\right)\right)$ is the flow of system (1) from $t = 0$ to t. Trajectories laying on the right hand side of the switching curve are trajectories where $u^*\left(t\right) = \underline{u}$ with $t \in [0, \overline{t}^*[$ and trajectories on the left hand side are the result of the input sequence starting with $\overline{u}$ until the optimal switching time $\overline{t}^*$. When a trajectory hits Γ, the input has to be switched to the second extremal value of u, by which the trajectory will follow Γ to the origin.

Due to $\dot{x}_1 = x_2$ the switching curve only can be approached via the quadrant Q_4 or via Q_2, trajectories starting in Q_1 and Q_3 therefore first have to enter one of these two quadrants in the state plane to be able to approach the switching curve. The trajectories starting in Q_2 but on the right hand side of the switching curve, do have to traverse first Q_1 and then enter Q_4 to be able to approach the switching curve – approaching the switching curve in Q_2 is not possible due to the large initial velocity. The same holds analogously for trajectories starting in Q_4 on the left hand side of the switching curve.

5.1 Optimal input sequence

The switching curve seperates the state plane into two regions Γ_r, Γ_l,

$$\Gamma_l = \left\{ x \in \mathbb{R}^2 \mid (x - x_\gamma) \in Q_3 \atop x_\gamma = \arg\min_{\overline{x}_\gamma \in \Gamma} \left(d\left(x, \overline{x}_\gamma\right)\right) \right\} \cup \left(Q_2 \cap \Gamma\right)$$

$$\Gamma_r = \left\{ x \in \mathbb{R}^2 \mid (x - x_\gamma) \in Q_1 \atop x_\gamma = \arg\min_{\overline{x}_\gamma \in \Gamma} \left(d\left(x, \overline{x}_\gamma\right)\right) \right\} \cup \left(Q_4 \cap \Gamma\right)$$

with $d\left(a,b\right)$ being the Euclidean distance between two vectors $a, b \in \mathbb{R}^2$. The optimal initial input value $u^*(0)$ depends only on whether x_0 being in Γ_r or in Γ_l. As the main part of the system dynamics is unknown, the switching curve is unknown. The proposed algorithm sets the optimal input sequence according to the following rules:

- If $x_0 \in Q_1 \cup Q_3$: The membership of x_0 to one of the sets Γ_l, Γ_r is entirely defined by the initial condition itself:
$$x_0 \in \begin{cases} \Gamma_r & x_0 \in Q_1 \\ \Gamma_l & x_0 \in Q_3 \end{cases}$$
- If $x_0 \in Q_2 \cup Q_4$: The initial condition may be on either side of the unknown switching curve. To determine the relative position of x_0 a first experiment run is used. The experiment starts at x_0 and stops at time t_Q, which is the time when the state trajectory leaves the initial quadrant, $x\left(t_Q\right) \notin Q_{initial}$. Depending on the quadrant the trajectory enters, the relative position of the initial condition can be determined, see Figure 2. From that the following rules can be found, which allows to determine the set the initial condition belongs to:
$$\begin{aligned} \text{If } x_0 \in Q_2: \quad & u\left(t\right) = \underline{u}, \quad 0 \leq t \leq t_Q \\ & x\left(t\right) \in Q_2, \quad 0 \leq t < t_Q \\ & x\left(t\right) \notin Q_2, \quad t_Q \leq t \\ \text{If } x_0 \in Q_4: \quad & u\left(t\right) = \overline{u}, \quad 0 \leq t \leq t_Q \\ & x\left(t\right) \in Q_4, \quad 0 \leq t < t_Q \\ & x\left(t\right) \notin Q_4, \quad t_Q \leq t \end{aligned}$$
$$x_0 \in \begin{cases} \Gamma_r & x\left(t_Q\right) \in Q_1 \\ \Gamma_l & x\left(t_Q\right) \in Q_3 \end{cases}$$
- If the set membership of x_0 to either Γ_l or Γ_r is determined, the ideal input sequence can be defined as follows:
$$u^*\left(t\right) = \begin{cases} \left. \begin{cases} \underline{u} & 0 \leq t < \overline{t}^* \\ \overline{u} & \overline{t}^* \leq t \leq t_f \end{cases} \right\} & x_0 \in \Gamma_r \\ \\ \left. \begin{cases} \overline{u} & 0 \leq t < \overline{t}^* \\ \underline{u} & \overline{t}^* \leq t \leq t_f \end{cases} \right\} & x_0 \in \Gamma_l \end{cases} \tag{8}$$

5.2 Optimal switching time

By (8) the ideal input sequence is defined, but the ideal switching time $\bar{t}^*$ is unknown. As already shown in Figure 1 the switch between the two extremal values of the input sequence only occurs in Q_2 or in Q_4, depending on the relative position of the initial condition to the switching curve. Therefore all trajectories are brought into the according switching quadrant first, that is, the first value of the ideal input sequence is applied at least until the trajectory enters the switching quadrant. The time t_{SQ} denotes the time the switching quadrant is entered, that is:

$$\begin{array}{lll} x(t) \in Q_4 & t_{SQ} \le t \le t_f & \forall x_0 \in \Gamma_r \\ x(t) \in Q_2 & t_{SQ} \le t \le t_f & \forall x_0 \in \Gamma_l \end{array}$$

The ideal switching time $\bar{t}^*$ is then

$$\bar{t}^* = t_{SQ} + t_S^*$$

where t_S^* is the part of the switching time that needs to be found.

The ideal switching time may be found by an iterative process, using an initial guess t_S^0, applying the optimal input sequence $\hat{u}(t)$ with the non-optimal switching time, and determining the resulting state $x(t_f)$, where $x(t_f)$ in the non-optimal case is defined as the state at time t_f, when the state trajectory leaves the switching quadrant:

$$\begin{array}{llll} x_0 \in \Gamma_r: & x(t) \in Q_4, & t_{SQ} \le t < t_f \\ & x(t) \notin Q_4, & t_f \le t \\ x_0 \in \Gamma_l: & x(t) \in Q_2, & t_{SQ} \le t < t_f \\ & x(t) \notin Q_2, & t_f \le t \end{array}$$

Using the information $x(t_f)$ in the non-optimal case, the switching time t_S^{k+1} for the $k+1$th repetition of the experiment can be adapted, using a bisection method.

Algorithm 1 Bisection algorithm

$k \leftarrow 0$
$t_S^k \leftarrow$ initial value
run experiment with t_S^k
while $d\left(x^k(t_f), (0,0)^{\mathrm{T}}\right) > \varepsilon$ **do**
  if $x^k(t_f) \in Q_1$ **then**
    $t_{lb} \leftarrow t_S^k$
    if t_{ub} undefined **then**
      $t_S^{k+1} \leftarrow 2t_S^k$
    else
      $t_S^{k+1} \leftarrow \frac{t_{lb}+t_{ub}}{2}$
    end if
  else $\{x^k(t_f) \in Q_3\}$
    $t_{ub} \leftarrow t_S^k$
    if t_{lb} undefined **then**
      $t_S^{k+1} \leftarrow \frac{t_S^k}{2}$
    else
      $t_S^{k+1} \leftarrow \frac{t_{lb}+t_{ub}}{2}$
    end if
  end if
  $k \leftarrow k+1$
  run experiment with t_S^k
end while

Algorithm 1 describes the bisection method for $x_0 \in \Gamma_r$ ($x_0 \in \Gamma_l$ can be treated analogously), where k denotes the current iteration. The time t_{lb} respectively t_{ub} denote the lower respectively the upper bound on t_S^*. Those bounds are unknown at the initialzation stage of the algorithm, but after the experiment with the initial switching time t_S^0 either t_{lb} or t_{ub} are set to this initial value, leaving one bound undefined. Depending on the outcome of the initial experiment, the next iteration of the switching time is either $t_S^1 = 2t_S^0$ or $t_S^1 = t_S^0/2$, and the experiment will be repeated using t_S^1. The doubling (or halving) of the switching time t_S^k continous until both boundary values t_{ub} and t_{lb} are defined. The subsequent iterations will halve the interval $\left[t_{lb}, t_S^k\right]$ resp. $\left[t_S^k, t_{ub}\right]$. This iterative process is guaranteed to converge to the value t_S^*.

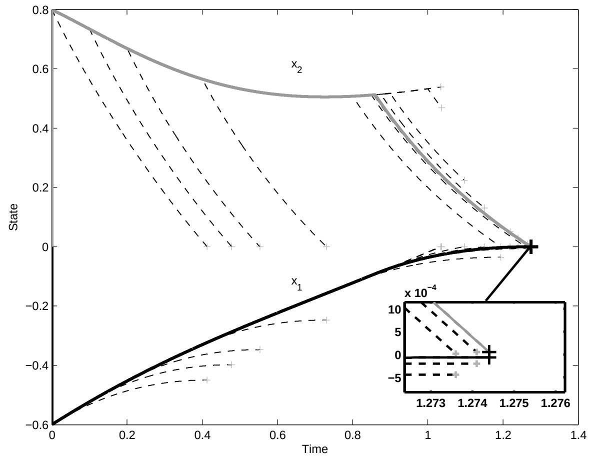

Fig. 3. The state trajectories x_1, x_2 over time, for initial condition $x_0 = (-0.6, 0.8)^{\mathrm{T}}$. The trajectories of all intermediate iterations are plotted in dashed lines with the individual final values marked by grey crosses, and the final iteration is plotted in solid lines (x_1 – black, x_2 – grey). The inset shows a detail at $t = t_f$, showing the final state $x(t_f)$ marked by black crosses. The final error is due to the termination criterion $\varepsilon = 0.1 \cdot 10^-3$.

When treating the case $F(x_2, t)$ instead of $F(x_2)$, basically the same considerations can be made in the derivation of the algorithm, although there won't be a single switching curve for all $x_0 \in \mathbb{R}^2$ but different ones for each x_0. As we do not rely on the fact that there is only a single switching curve, the algorithm developed in this section also may be applied in the time varying case.

6. SIMULATION RESULTS

The iterative algorithm was tested with the unknown (to the algorithm) term being

$$F\left(x_2\left(t^k\right), t^k\right) = {x_2}^3 + \frac{1}{2}{x_2}^2 + x_2 - 0.5\sin\left(\frac{2\pi}{t^k+1}\right),$$

with $b = 1, \mathscr{U} = [-1, 1]$ and different initial conditions. F uses the notation t^k to highlight that for this simulation example it is assumed that the time variant part is resetted at every t_k. The results for a specific x_0 are shown in Figure 3 and Figure 4.

The optimal switching time and optimal terminal time for various intial conditions x_0 are compared to a numerical solution to the same minimum time problem calculated by ACADO Houska and Ferreau [2009–2011] in Table 1. These values are presented here only for a plausibility check, as

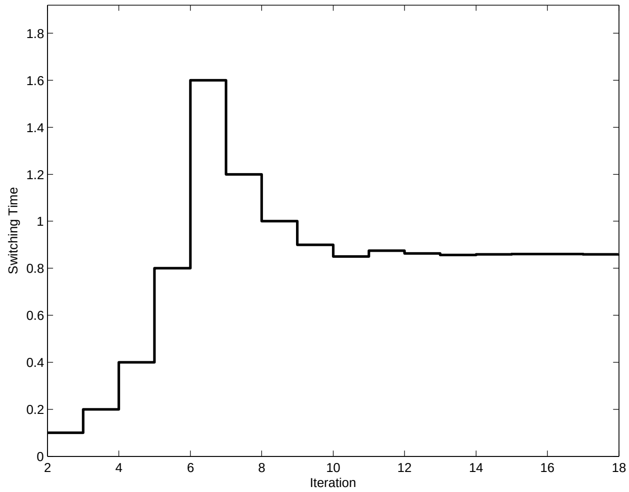

Fig. 4. The development of the switching time over the iterations k.

ACADO uses different methods and a different termination criterion, and the results also depend on the particular settings for the various parameters of the ACADO solver. But nevertheless Table 1 shows that the results of the bisection method are plausible by achieving almost the same results, with the final time always being less than or equal to the ACADO solution. It has to be stressed that the presented algorithm did not use any information about F while it was perfectly known to ACADO.

Table 1. Comparison of optimal switching and final time, **A** $\rightarrow$ ACADO, **B** $\rightarrow$ Bisection method

Initial Condition		Switching time $\bar{t}^*$		Final time t_f^*	
$x_{1,0}$	$x_{2,0}$	**A**	**B**	**A**	**B**
-0.8	-0.8	2.483	2.4838	3.289	3.2889
-0.6	0.8	0.8604	0.8600	1.2745	1.2745
0.3	-0.7	0.07466	0.0741	0.8295	0.8295
0.7	0.2	1.245	1.2437	1.6935	1.6935
0.1	-0.7	1.185	1.1850	1.539	1.5388
-0.1	0.8	0.5857	0.5853	0.8489	0.8488

7. CONCLUSION AND OUTLOOK

It has been shown that for a class of systems with partly unknown dynamics the bang-bang control is the optimal solution to the minimum-time problem, and that the optimal control can be found using an iterative method and experimental data from the system. In a future work we might extend this result to systems with a dynamic associated with the input to the second order system, which would allow to apply the method directly to a broad range of applications. Another question that may be worth some investigation is how to incorporate a (possibly known) dependency of $\dot{x}_2$ from the state x_1 into our method.

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REFERENCES

J.E. Bobrow. Optimal robot plant planning using the minimum-time criterion. *Robotics and Automation, IEEE Journal of*, 4 (4):443 –450, aug 1988. ISSN 0882-4967.

A.E. Bryson and Yu-Chi Ho. *Applied optimal control: optimization, estimation, and control; Revised Printing*. Taylor & Francis, 1975.

L. Cesari. An existence theorem in problems of optimal control. *Journal of the Society for Industrial and Applied Mathematics Series A Control*, 3(1):7–22, 1965.

F. L. Chernousko, L. D. Akulenko, and N. N. Bolotnik. Time-optimal control for robotic manipulators. *Optimal Control Applications and Methods*, 10(4):293–311, 1989. ISSN 1099-1514.

A. Filippov. On certain questions in the theory of optimal control. *Journal of the Society for Industrial and Applied Mathematics Series A Control*, 1(1):76–84, 1962.

B. Houska and H.J. Ferreau. ACADO Toolkit Users Manual. http://www.acadotoolkit.org, 2009–2011.

H.-P. Huang and N.H. McClamroch. Time-optimal control for a robotic contour following problem. *Robotics and Automation, IEEE Journal of*, 4(2):140 –149, apr 1988. ISSN 0882-4967.

E.B. Lee and L. Markus. *Foundations of optimal control theory*. The SIAM Series in Applied Mathematics. John Wiley & Sons, Inc., New York - London - Sydney, 1967.

Shengming Li and Longya Xu. Minimum-time flux-linkage transition for pmsm control with minimum number of inverter switching. In *Electrical Machines and Systems, 2001. ICEMS 2001. Proceedings of the Fifth International Conference on*, volume 2, pages 1207 –1210 vol.2, aug 2001. doi: 10.1109/ICEMS.2001.971898.

A. Miele. Extremization of linear integrals by greens theorem. *Optimization Techniques, Academic Press, New York*, pages 69–98, 1962.

Jaime Moreno. Optimal time control of bioreactors for the wastewater treatment. *Optimal Control Applications and Methods*, 20(3):145–164, 1999. ISSN 1099-1514.

L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mischenko. *The Mathematical Theory of Optimal Processes*. Wiley Interscience, 1962.

A. Rapaport and D. Dochain. Minimal time control of fed-batch processes with growth functions having several maxima. *Automatic Control, IEEE Transactions on*, 56(11):2671 – 2676, nov. 2011. ISSN 0018-9286.

Z. Shen and S.B. Andersson. Minimum time control of a second-order system. In *Decision and Control (CDC), 2010 49th IEEE Conference on*, pages 4819–4824. IEEE, 2010.

Kang Shin and N. McKay. Minimum-time control of robotic manipulators with geometric path constraints. *Automatic Control, IEEE Transactions on*, 30(6):531 – 541, jun 1985. ISSN 0018-9286.

J.-J.E. Slotine and H.S. Yang. Improving the efficiency of time-optimal path-following algorithms. *Robotics and Automation, IEEE Transactions on*, 5(1):118 –124, feb 1989. ISSN 1042-296X.

E. Sontag and H. Sussmann. Time-optimal control of manipulators. In *Robotics and Automation. Proceedings. 1986 IEEE International Conference on*, volume 3, pages 1692 – 1697, apr 1986.

R.F. Stengel. *Optimal control and estimation*. Dover Publications, 1994.

