Positive Definiteness of Generalized Homogeneous Functions

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Abstract: Identification of positive definiteness of functions is crucial in control theory. However for generalized homogeneous functions, there does not exist an effective method to identify the positive definiteness. In this paper, we consider Lipschitz continuous generalized homogeneous functions. For the functions, we propose a new method to identify the positive definiteness of the functions. Moreover, we apply our proposed method to an optimal homogeneous finite-time control problem. We confirm the effectiveness of the proposed method through the example.

1. INTRODUCTION

Generalized-homogeneity based control discovered by Bhat and Bernstein [1] currently attracts much attention in nonlinear control theory [2, 3, 4, 5, 6, 7, 8, 9]. As the most important result of the generalized homogeneity based control, Zubov proved that every asymptotically stable generalized homogeneous systems permits a generalized homogeneous Lyapunov function in 1958 [10]. The result was rediscovered by Rosier [11] in 1992.

However, generalized homogeneous control systems design remains a difficult problem. In particular, analysis of generalized homogeneous function is difficult. One of major reasons of difficulty is the fact that the quadratic function analysis (including SOS tools) may not be applied to homogeneous systems. Worse still, Ahmadi recently proved that it is strongly NP-hard to decide whether a classical homogeneous polynomial of degree 4 is positive definite [12].

Hence, we need another positive definiteness identification method for generalized homogeneous systems. In this paper, we consider Lipschitz continuous generalized homogeneous functions. For the functions, we propose a new method for identification of the positive definiteness of the functions. Moreover, we apply our proposed method to an optimal homogeneous finite-time control problem. We confirm the effectiveness of the proposed method through an example.

2. HOMOGENEOUS SYSTEMS AND FINITE-TIME STABILITY

This paper considers a position control problem for a robot manipulator. The robot manipulator is modelled by a control-affine nonlinear system. In this section, we summarize locally homogeneous finite-time stabilization of general control-affine nonlinear systems.

In this section, we consider the following control-affine nonlinear system:

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$$\dot{x} = f(x) + g(x)u, \tag{1}$$

where $x \in \mathbb{R}^n$ is a state, $u \in \mathbb{R}^m$ is an input, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \mathbb{R}^{n\times m}$ are continuous mappings.

Throughout the paper, $\|\cdot\|$ denotes a Euclidean norm, $\mathbb{R}_{\geq 0} := [0, +\infty) \subset \mathbb{R}$, $B^n_\delta := \{x \in \mathbb{R}^n | \|x\| \leq 1\}$, and $S^{n-1} := \{x \in \mathbb{R}^n | \|x\| = 1\}$.

2.1 Generalized Homogeneity

In this paper, we discuss the problem of positive definiteness of generalized homogeneous functions. The most important notions; the dilation and homogeneity are defined as follows.

Definition 1. (Dilation). The mapping $\Delta^r_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ defined as follows is said to be a dilation:

$$\Delta^r_\varepsilon x := (\varepsilon^{r_1} x_1, \cdots, \varepsilon^{r_n} x_n), \tag{2}$$

where $\varepsilon > 0$ and $r = (r_1, r_2, \cdots, r_n) \in \mathbb{R}^n$ $(r_i > 0,\ 1 \leq i \leq n)$.

Note that we often refer r in the dilation mappings as "dilation exponent."

Definition 2. (Generalized Homogeneous Function). A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a generalized homogeneous function of degree $k \in \mathbb{R}$ with respect to a dilation exponent r if the following equality holds for all $\varepsilon \geq 0$:

$$V(\Delta^r_\varepsilon x) = \varepsilon^k V(x). \tag{3}$$

If dilation exponent $r = (1, .., 1)$, the function V is said to be a classical homogeneous function.

Note that we simply refer to a generalized homogeneous function as a homogeneous function in this paper.

Definition 3. (Generalized Homogeneous System). Consider an input affine nonlinear system (1). (1) is said to be homogeneous of degree $\tau \in \mathbb{R}$ with respect to a dilation exponent (r, s) if the following equality holds for all $\varepsilon > 0$:

$$f(\Delta^r_\varepsilon x) + g(\Delta^r_\varepsilon x)\Delta^s_\varepsilon u = \varepsilon^\tau \Delta^r_\varepsilon [f(x) + g(x)u]. \tag{4}$$

Note that the homogeneous functions and homogeneous systems have the following property [5]:

Lemma 1. Let $\alpha > 0$ be an arbitrary constant. Consider a homogeneous function $V : \mathbb{R}^n \to \mathbb{R}$ (resp. a homogeneous system) of degree $k \in \mathbb{R}$ (of degree $\tau \in \mathbb{R}$) with respect to a dilation exponent r ((r, s)). Then, V (resp. the system) is homogeneous of degree αk ($\alpha\tau$) with respect to dilation exponent αr ($(\alpha r, \alpha s)$).

According to Lemma 1, we suppose all $r_i > 1$ without any loss of generality in this paper.

Definition 4. (Generalized Homogeneous Feedback). Consider a homogeneous system (1) of degree τ with respect to dilation exponent (r, s). Then, the feedback $u(x)$ such that the following equality holds is said to be a homogeneous feedback:

$$u(\Delta_\varepsilon^r x) = \Delta_\varepsilon^s u(x). \tag{5}$$

2.2 Finite-time stability and convergence rates

Stability of the origin of (1) and the convergence rates are defined as follows.

Definition 5. (Stability). The origin of (1) is said to be

(1) stable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq 0; \tag{6}$$
(2) globally asymptotically stable if the origin is stable and all solutions $x(t)$ satisfy the following equation:
$$\lim_{t\to\infty} \|x(t)\| = 0. \tag{7}$$

Definition 6. (Convergence Rate). The origin of (1) is said to be

(1) rationally stable if the origin is stable and there exist positive constants $\delta, b_1, b_2 > 0$ and $0 < \eta \leq 1$ such that
$$\|x(t)\| \leq b_1(1 + \|x_0\|^{b_2} t)^{1/b_2} \|x(0)\|^\eta, \quad \forall t \geq 0, \forall x(0) \in B_\delta; \tag{8}$$
(2) exponentially stable if the origin is stable and there exist positive constants $\delta, b_1, b_2 > 0$ such that
$$\|x(t)\| \leq b_1 e^{-b_2 t} \|x(0)\|, \quad \forall t \geq 0, \forall x(0) \in B_\delta; \tag{9}$$
(3) finite-time stable if the origin is stable and there exist a positive constant $\delta > 0$ and a function $T : B_\delta \backslash \{0\} \to \mathbb{R}_{>0}$ such that
$$\lim_{t\to T(x(0))} x(t) = 0, \quad \forall x(0) \in B_\delta. \tag{10}$$

The following lemma plays a central role in the homogeneity-based control systems design.

Lemma 2. Consider a homogeneous equation $\dot{x} = f(x)$ of degree τ with respect to dilation exponent r. Assume that the origin is asymptotically stable. Then,

(1) if $k > 0$ the origin is rationally stable;
(2) if $k = 0$ the origin is exponentially stable;
(3) if $k < 0$ the origin is finite-time stable.

3. POSITIVE DEFINITENESS OF GENERALIZED HOMOGENEOUS FUNCTIONS

In this section, we analyze positive definiteness of generalized homogeneous functions. Note that for every generalized homogeneous function can be transformed into a classical homogeneous function by a homeomorphism as follows:

Lemma 3. Consider a homogeneous function $V : \mathbb{R}^n \to \mathbb{R}$ of degree k with respect to a dilation exponent r and the following homeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$:

$$\phi = \left(|x_1|^{\frac{1}{r_1}} \operatorname{sgn} x_1, |x_2|^{\frac{1}{r_2}} \operatorname{sgn} x_2, ..., |x_n|^{\frac{1}{r_n}} \operatorname{sgn} x_n\right). \tag{11}$$

Then, the following function $\tilde{V}$ is a classical homogeneous function of degree k:

$$\tilde{V}(x) = V(\phi^{-1}(x)). \tag{12}$$

Proof. Note that

$$\phi^{-1}(x) = \left(|x_1|^{r_1} \operatorname{sgn} x_1, |x_2|^{r_2} \operatorname{sgn} x_2, ..., |x_n|^{r_n} \operatorname{sgn} x_n\right). \tag{13}$$

Then,

$$\begin{aligned}\tilde{V}(\varepsilon x) &= V(\varepsilon^{r_1}|x_1|^{r_1} \operatorname{sgn} x_1, ..., \varepsilon^{r_n}|x_n|^{r_n} \operatorname{sgn} x_n) \\ &= \varepsilon^k V(|x_1|^{r_1} \operatorname{sgn} x_1, ..., |x_n|^{r_n} \operatorname{sgn} x_n) \\ &= \varepsilon^k \tilde{V}(x).\end{aligned} \tag{14}$$

□

By Lemma 3, the positive definiteness of generalized homogeneous functions is equivalent to one of classical homogeneous functions as follows:

Proposition 1. A homogeneous function $V : \mathbb{R}^n \to \mathbb{R}$ of degree k with respect to a dilation exponent r is positive definite if and only if the classical homogeneous function $\tilde{V} = V(\phi^{-1}) : \mathbb{R}^n \to \mathbb{R}$ is positive definite, where ϕ is defined in (11).

Proof. Note that $\phi(0) = 0$. Therefore, $\tilde{V}(0) = 0$ if $V(0) = 0$. Since ϕ is a global homeomorphism, if $V(x) = 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$, $\tilde{V}(x) > 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$, vice versa. □

The identification problem of positive definiteness of a generalized homogeneous function is reduced to that of the corresponding classical homogeneous function. However, identification of classical homogeneous function itself is a difficult problem. In the following section, we propose a method for identification of positive definiteness of classical homogeneous functions.

4. POSITIVE DEFINITENESS OF CLASSICAL HOMOGENEOUS FUNCTIONS

Identification of positive definiteness of classical homogeneous functions is also a difficult problem. The difficulty can be found through the following fact.

Fact 1. Consider a continuous classical homogeneous function V of degree k. Then, the following conditions are equivalent.

(1) V is positive definite.
(2) $V(x) > 0$ for all $x \in S^{n-1} = \{x | \sum_{i=1}^n x_i^2 = 1\}$
(3) There exists $x \in \mathbb{R}^n$ such that $V(x) > 0$ and a set $\{x | V(x) = 0\}$ is 0-dimensional.
(4) There exists $x \in \mathbb{R}^n$ such that $V(x) > 0$ and the following simultaneous equations have no solution:
$$V(x) = 0 \tag{15}$$
$$\sum_{i=1}^n x_i^2 = 1 \tag{16}$$

Proof. (1) ⇒ (2)-(4) is obvious. Then, we prove the converse. In the following discussion, note that $k > 0$, since V is continuous.

(2) ⇒ (1): Since V is homogeneous, $V(\varepsilon x) = \varepsilon^k V(x)$ for every $\varepsilon \geq 0$. Substituting $k = 0$, $V(0) = 0$. Note that every $x \in \mathbb{R}^n$ can be written as $x = \varepsilon x_0$, where $\varepsilon \geq 0$ and $x_0 \in S^{n-1}$. Therefore, if $V(x_0) > 0$ for all $x_0 \in S^{n-1}$, $V(x) = \varepsilon^k V(x_0) > 0$ for all $x \in \mathbb{R}^n$.

(3) ⇒ (1): Assume there exists an x_a such that $V(x_a) = 0$. Then for all x such that $\{x | x = \varepsilon x_a, \varepsilon \geq 0\}$, $V(x) = 0$. This is a contradiction to the assumption that the set $\{x | V(x) = 0\}$ is 0-dimensional. Accordingly, there exists no x such that $V(x) = 0$ except at $x = 0$. By the assumption that there exists $x \in \mathbb{R}^n$ such that $V(x) > 0$ and the mean value theorem, $V(x) > 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$. Therefore, V is positive definite.

(4) ⇒ (1): By the same discussion of the above, there exists no x such that $V(x) = 0$ except at $x = 0$.Therefore, V is positive definite. □

The condition (4), the existence of solutions of the simultaneous equations seems not to be a difficult problem. However for real functions, the existence of equations is an NP-hard problem [13]. Thus, we focus on the condition (1).

In this section, we propose a method for identification of positive definiteness of locally Lipschitz continuous classical homogeneous functions. To analyze function V on S^{n-1}, we parameterize S^{n-1} by generalized polar coordinates $\theta \in \mathbb{R}^{n-1}$ as $x = \eta(\theta)$, where a mapping $\eta : \mathbb{R}^{n-1} \to S^{n-1} \subset \mathbb{R}^n$ is defined as follows:

$$\begin{aligned}
x_1 &= \sin\theta_{n-1}\sin\theta_{n-2}\cdots\sin\theta_2\sin\theta_1,\\
x_2 &= \sin\theta_{n-1}\sin\theta_{n-2}\cdots\sin\theta_2\cos\theta_1,\\
x_3 &= \sin\theta_{n-1}\sin\theta_{n-2}\cdots\sin\theta_3\cos\theta_2,\\
&\vdots\\
x_i &= \sin\theta_{n-1}\sin\theta_{n-2}\cdots\sin\theta_{i-2}\cos\theta_{i-1},\\
&\vdots\\
x_{n-1} &= \sin\theta_{n-1}\cos\theta_{n-2},\\
x_n &= \cos\theta_{n-1}.
\end{aligned} \tag{17}$$

Note that for generalized polar coordinates, the following lemma holds:

Lemma 4. Consider generalized polar transformation (17). Then, the following inequality holds in $\mathbb{R}^n$

$$\|\eta(\theta_a) - \eta(\theta_b)\| \leq (n-1)\|\theta_a - \theta_b\| \tag{18}$$

for all $\theta_a, \theta_b \in \mathbb{R}^{n-1}$.

Proof. If $n = 2$, the arc length from $\eta(\theta_a)$ to $\eta(\theta_b)$ is $|\theta_a - \theta_b|$ if $|\theta_a - \theta_b| \leq \pi$. Therefore, $\|\eta(\theta_a) - \eta(\theta_b)\| \leq \|\theta_b - \theta_a\|$.

Consider $n = 3$. Let $\theta_a = (\theta_{a1}, \theta_{a2})$ and $\theta_b = (\theta_{b1}, \theta_{b2})$. Then the following inequality holds by the triangular inequality (See Fig. 1):

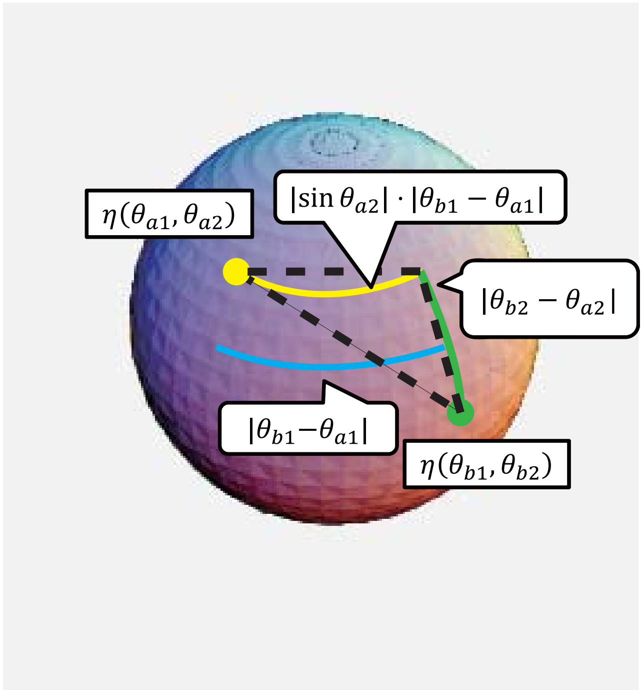

Fig. 1. Proof of Lemma 4

$$\begin{aligned}
\|\eta(\theta_a) - \eta(\theta_b)\| &\leq \|\eta(\theta_{a1}, \theta_{a2}) - \eta(\theta_{b1}, \theta_{a2})\|\\
&\quad + \|\eta(\theta_{b1}, \theta_{a2}) - \eta(\theta_{b1}, \theta_{b2})\| &(19)\\
&\leq |\sin\theta_{a2}| \cdot |\theta_{a1} - \theta_{b1}| + |\theta_{a2} - \theta_{b2}| &(20)\\
&\leq |\theta_{a1} - \theta_{b1}| + |\theta_{a2} - \theta_{b2}| &(21)\\
&\leq 2\|\theta_a - \theta_b\|. &(22)
\end{aligned}$$

By the same discussion as above, if the statement is true for n, the case of $n + 1$ is also true. This concludes the proof. □

Then, the following lemma holds for the local Lipschitz continuity of the function.

Lemma 5. Consider a locally Lipschitz continuous classical homogeneous function $V : \mathbb{R}^n \to \mathbb{R}$ of degree k. Suppose Lipschitz constant in the set $D^n = \{x | \sum_{i=1}^n x_i^2 \leq 1\}$ be K such that the following holds for all $x_a, x_b \in D^n$:

$$|V(x_a) - V(x_b)| \leq K\|x_a - x_b\|. \tag{23}$$

Then, function $V(\eta(\cdot)) : \mathbb{R}^{n-1} \to \mathbb{R}$ is locally Lipschitz and the following inequality holds in D^n:

$$|V(\eta(\theta_a)) - V(\eta(\theta_b))| \leq K(n-1)\|\theta_a - \theta_b\|. \tag{24}$$

Proof. According to Lemma 4,

$$\begin{aligned}
|V(\eta(\theta_a)) - V(\eta(\theta_b))| &\leq K\|\eta(\theta_a) - \eta(\theta_b)\| &(25)\\
&\leq K(n-1)\|x_a - x_b\|. &(26)
\end{aligned}$$

□

The following theorem holds by the above lemmas:

Theorem 1. Consider a locally Lipschitz continuous classical homogeneous function $V : \mathbb{R}^n \to \mathbb{R}$ of degree k and . Suppose Lipschitz constant in the set $D^n = \{x | \sum_{i=1}^n x_i^2 \leq 1\}$ be K.

Let $\delta > 0$ be a constant. Consider finitely many elements $\theta_a, \theta_b, \ldots, \theta_m \in [0, 2\pi]^{n-1}$ such that for each $\theta \in [0, 2\pi]^{n-1}$ there exists θ_r $(r = a, \ldots, m)$ such that $\|\theta - \theta_r\| \leq \delta$.

Then, the function V is positive defenite if the following condition holds for all $r = a, \ldots, m$:

$$V(\eta(\theta_r)) > K(n-1)\delta \tag{27}$$

Remark 1. In Theorem 1, S^{n-1} is parameterized by $[0, 2\pi]^{n-1} \subset \mathbb{R}^{n-1}$. Note that $[0, \pi]$ is sufficient for $n-2$ parameters instead of $[0, 2\pi]$. However to simplify the discussion, we employ $[0, 2\pi]^{n-1}$.

By Theorem 1, we can identify the positive definiteness of classical homogeneous function by analyzing finitely many elements in S^{n-1}. This implies that we can theoretically guarantee the positive definiteness by numerical computation.

5. IDENTIFICATION OF POSITIVE DEFINITENESS OF GENERALIZED HOMOGENEOUS FUNCTIONS

The preceding section gives the sufficient condition for positive definiteness of classical homogeneous systems. In this section, we show a sufficient condition for one of generalized homogeneous functions. Moreover, we propose a positive definiteness identification method for generalized homogeneous functions.

Lemma 6. Consider a locally Lipschitz continuous homogeneous function $V : \mathbb{R}^n \to \mathbb{R}$ of degree k with respect to dilation exponent $r = (r_1, \ldots, r_n)$. Suppose α be a constant such that $0 < \alpha < r_i$ for all $i = 1, \ldots, n$, and Lipschitz constant in the set $D^n = \{x | \sum_{i=1}^n x_i^2 \le 1\}$ be K such that the following holds for all $x_a, x_b \in D^n$:

$$|V(x_a) - V(x_b)| \le K\|x_a - x_b\|. \tag{28}$$

Then, the function $\tilde{V}(x) = V(\phi^{-1}(x))$ is a locally Lipschitz continuous classical homogeneous function of degree αk. Moreover, the following inequality holds for all $x_a, x_b \in D^n$:

$$|\tilde{V}(x_a) - \tilde{V}(x_b)| \le r_{\max} K\|x_a - x_b\|, \tag{29}$$

where $r_{\max} = \max\{r_1, \ldots, r_n\}$.

Proof. The classical homogeneity of $\tilde{V}$ is clear. Note that the following implication holds:

$$\{\phi^{-1}(x) | x \in D^n\} \subset D^n. \tag{30}$$

Moreover for every $r > 1$, the following relation holds for all $x_{ai}, x_{bi} \in [-1, 1]$:

$$\begin{aligned} \big||x_{ai}|^{r_i} \operatorname{sgn} x_{ai} - |x_{bi}|^{2r_i} \operatorname{sgn} x_{bi}\big| &= \left|\int_{x_{ai}}^{x_{bi}} \frac{\partial |x|^{r_i} \operatorname{sgn} x}{\partial x}(x) dx\right| \\ &\le \left|\int_{x_{ai}}^{x_{bi}} r_i |x|^{r_i - 1} dx\right| \\ &\le r_i |x_{ai} - x_{bi}| \end{aligned} \tag{31}$$

Therefore,

$$\begin{aligned} |V(\phi^{-1}(x_a)) - V(\phi^{-1}(x_b))| &\le K\|\phi^{-1}(x_a) - \phi^{-1}(x_b)\| \quad (32) \\ &\le K r_{\max} \|x_a - x_b\|. \quad (33) \end{aligned}$$

□

Remark 2. Every locally Lipschitz generalized homogeneous function can be turned into a locally Lipschitz classical homogeneous function. However, the converse is not true; there exists a function that a generalized homogeneous function that is not locally Lipschitz but $V(\phi^{-1}(x))$ is a locally Lipschitz. An example of such functions are presented in section 6.

Then, the following theorem holds:

Theorem 2. Consider a locally Lipschitz continuous homogeneous function $V : \mathbb{R}^n \to \mathbb{R}$ of degree k with respect to dilation exponent $r = (r_1, \ldots, r_n)$. Suppose α be a constant such that $0 < \alpha < r_i$ for all $i = 1, \ldots, n$, and Lipschitz constant in the set $D^n = \{x | \sum_{i=1}^n x_i^2 \le 1\}$ be K such that the following holds for all $x_a, x_b \in D^n$:

$$|V(x_a) - V(x_b)| \le K\|x_a - x_b\|. \tag{34}$$

Let $\delta > 0$ be a constant. Consider finitely many elements $\theta_a, \theta_b, \ldots, \theta_m \in [0, 2\pi]^{n-1}$ such that for each $\theta \in [0, 2\pi]^{n-1}$ there exists θ_r $(r = a, \ldots, m)$ such that $\|\theta - \theta_r\| \le \delta$.

Then, the function V is positive defintie if the following condition holds for all $r = a, \ldots, m$:

$$V \circ \phi^{-1} \circ \eta(\theta_r) > r_{\max} K(n-1)\delta, \tag{35}$$

where $r_{\max} = \max\{r_1, \ldots, r_n\}$.

According to Theorem 2, we propose the following algorithm for positive definiteness analysis.

(1) Choose sufficiently large number $l \in \mathbb{N}$. Divide $[0, 2\pi]^{n-1}$ into l^{n-1}, and we can obtain l^{n-1} grid points in $[0, 2\pi]^{n-1}$. Then, $\delta = 2\pi/l$.
(2) If the following inequality holds for all grid points θ_r, the function V is positive definite:
$$V \circ \phi^{-1} \circ \eta(\theta_r) > r_{\max} K(n-1)\frac{2\pi}{l}. \tag{36}$$
(3) If there exists a grid θ_r such that $V \circ \phi^{-1} \circ \eta(\theta_r) \le 0$, V is not positive definite.
(4) Substitute $l = 2l$ and return to (1).

The method is numerical calculation; however, note that the positive definiteness is analytically guaranteed.

6. OPTIMAL HOMOGENEOUS FINITE-TIME CONTROL

6.1 Optimal Homogeneous Finite-time Control of Double Integrator

Nonlinear finite-time control attracts attention in nonlinear control theory. However in the best of our knowledge, the optimal finite-time control problem has not been discussed yet.

In this section, we consider an optimal finite-time stabilization problem for the following double integrator system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned} \tag{37}$$

(37) is homogeneous system of degree $1 - r_1$ with respect to dilation exponent $r = (r_1, 1)$ and $s = 2 - r_1$. Note that if the homogeneous feedback controller $u(x)$ asymptotically stabilizes the origin and $1 < r_1 < 2$, the origin of (37) is finite-time stable.

Consider the following cost functional J inspired by one of Hermes [15]:

$$J = \int_0^{+\infty} \left[\frac{r_1}{2} |k_1 x_1 + k_2 |x_2|^{r_1} \operatorname{sgn} x_2|^{\frac{2}{r_1}} - k_0 |x_1|^{\frac{1}{r_1}} \operatorname{sgn} x_1 \cdot x_2 - k_1 x_2^2 + \frac{2-r_1}{2} |u|^{\frac{2}{2-r_1}} \right] dt, \quad (38)$$

where $k_0, k_1, k_2 \in \mathbb{R}$ are appropriate parameters such that the following homogeneous function $L(x)$ of degree $2r_2$ with respect to dilation exponent $r = (r_1, 1)$ is positive definite:

$$L(x) := \frac{r_1}{2} |k_1 x_1 + k_2 |x_2|^{r_1} \operatorname{sgn} x_2|^{\frac{2}{r_1}} - k_0 |x_1|^{\frac{1}{r_1}} \operatorname{sgn} x_1 \cdot x_2 - k_1 x_2^2. \quad (39)$$

Then, Hamilton-Jacobi-Bellman (HJB) equation is obtained as follows:

$$\frac{\partial V}{\partial x_1} x_2 + L(x) - \frac{r_1}{2} \left| \frac{\partial V}{\partial x_2} \right|^{\frac{2}{r_1}} = 0. \quad (40)$$

For the equation, we obtain the following lemma:

Lemma 7. The following function $V : \mathbb{R}^2 \to \mathbb{R}$ is a solution of HJB equation (40).

$$V(x) = \frac{r_1}{1+r_1} k_0 |x_1|^{\frac{1+r_1}{r_1}} + k_1 x_1 x_2 + \frac{1}{1+r_1} k_2 |x_2|^{1+r_1} \quad (41)$$

The proof follows direct calculation. Then, the following theorem holds.

Theorem 3. Consider system (37) and cost functional (38). Assume functions $L(x)$ defined by (39) and $V(x)$ defined by (41) are positive definite.

Then, the following input $u^*(x)$ globally asymptotically stabilizes the origin and minimizes the cost functional (38).

$$u^*(x) = -|k_1 x_1 + k_2 |x_2|^{r_1} \operatorname{sgn} x_2|^{\frac{2-r_1}{r_1}} \operatorname{sgn}(k_1 x_1 + k_2 |x_2|^{r_1} \operatorname{sgn} x_2) \quad (42)$$

We analyze positive definiteness of functions $V(x)$ and $L(x)$. Note that both $V(x)$ and $L(x)$ are not a sum-of-square function; SOS tools cannot be applied to the functions.

On one hand, $V(x)$ is a differentiable function and the following inequality holds in D^2:

$$\left\| \frac{\partial V}{\partial x}(x) \right\| \le |k_0| + 2|k_1| + |k_2| \quad (43)$$

Hence, the following inequality holds in D^2:

$$\|V(x_a) - V(x_b) \le \| (|k_0| + 2|k_1| + |k_2|) \|x_a - x_b\| \quad (44)$$

for all $x_a, x_b \in D^2$.

On the other hand, $L(x)$ is not a locally Lipschitz function ; however, $L(\phi^{-1}(x))$ is a C^1 function as follows:

$$L(\phi^{-1}(x)) = \frac{r_1}{2} |k_1 |x_1|^{r_1} \operatorname{sgn} x_1 + k_2 |x_2|^{r_1} \operatorname{sgn} x_2|^{\frac{2}{r_1}} - k_0 x_1 x_2 - k_1 |x_2|^2. \quad (45)$$

Hence, we can apply our proposed method to $L(\phi^{-1}(x))$. The following inequality holds in D^2:

$$\|L(\phi^{-1}(x_a)) - L(\phi^{-1}(x_b))\| \le \left(2r_1 |k_1| \cdot (|k_1| + |k_2|)^{\frac{2-r_1}{r_1}} + 2|k_0| + |k_1| \right) \|x_a - x_b\|. \quad (46)$$

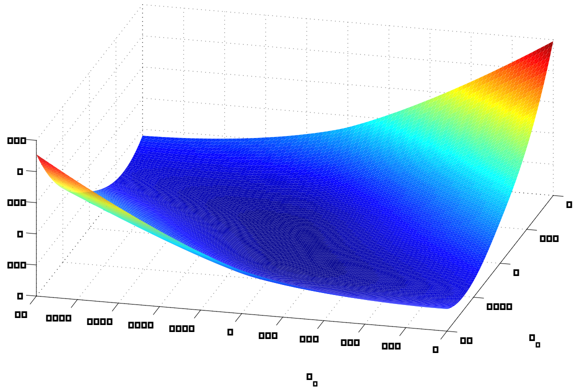

Fig. 2. Lyapunov Function

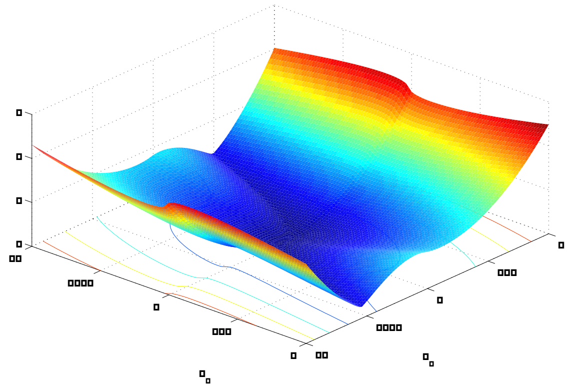

Fig. 3. Cost Functional

6.2 Positive Definiteness Analysis

If both V and L are positive definite, $u^*(x)$ is an optimal finite-time controller. In this subsection, we consider the case that dilation exponent $r = (5/3, 1)$ and $s = 1/3$. We choose $k_0 = 1, k_1 = 1, k_2 = 2$.

In this case, local Lipschitz constants are 5 for $V(x)$ and 10 (with approximation) for $L(x)$, respectively. For $V(x)$, the minimum of the function on grid points with $l = 1000$ is $0.1228 > 0.05$. Hence V is positive definite. For $L(x)$, one with $l = 1000$ is $0.156 > 0.10$. Hence L is also positive definite.

Thus by the proposed method, we can confirm that $V(x)$ and $L(x)$ are positive definite. Figure 2 depicts the shape of $V(x)$, and Figure 3 illustrates the shape of $L(x)$. We can confirm that those functions are positive definite, indeed.

Then, the optimal homogeneous finite-time controller is obtained as follows:

$$u^*(x) = -\left(x_1 + 2|x_2|^{5/3} \operatorname{sgn} x_2 \right)^{1/5}. \quad (47)$$

Figure 4 illustrates time history of the state, and figure 5 shows that of the input. We can confirm that the state converges to the origin in finite-time by the optimal homogeneous finite-time controller.

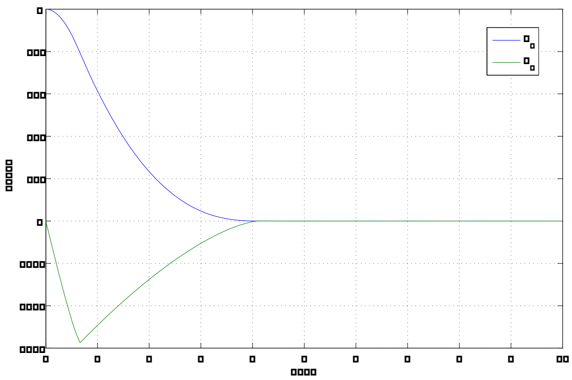

Fig. 4. Simulation Result: State

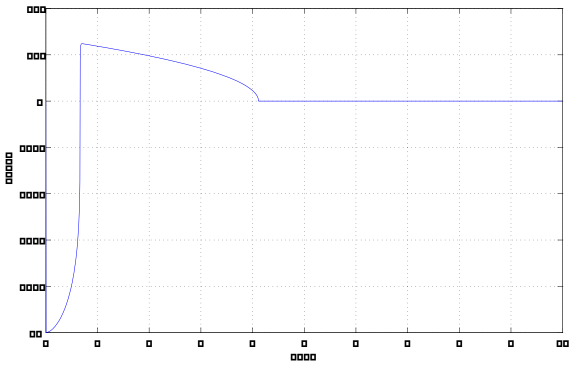

Fig. 5. Simulation Result: Input

7. CONCLUSION

In this paper, we propose a new method to identify the positive definiteness of locally Lipschitz generalized homogeneous functions. Moreover, we apply our proposed method to an optimal finit-time control problem.

However, the proposed method needs local Lipschitz constant. Automatic calculation of the Lipschitz constant remains a future work.

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