Input-to-State Stability, Integral Input-to-State Stability, and Unbounded Level Sets

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Abstract: We provide partial Lyapunov characterizations for a recently proposed generalization of input-to-state and integral input-to-state stability (ISS and iISS, respectively). This generalization relies on the notion of stability with respect to two measures originally introduced by Movchan [1960]. We show that the two classical Lyapunov characterizations of ISS-type properties, i.e., decrease conditions in an implication or dissipative form, correspond to ISS and iISS, respectively. We also demonstrate via an example that, for the generalization considered here, ISS does not necessarily imply iISS.

1. INTRODUCTION

Input-to-State Stability (ISS) introduced by Sontag [1989] has proven to be a valuable tool in the study of systems subject to disturbances. In particular, when it is possible to analyze or design systems via a modular approach, many tools such as small-gain theorems (Jiang et al. [1994]) and different characterizations of the ISS property (Sontag and Wang [1996], Sontag and Wang [1995]) are available.

The utility of ISS subsequently led to many derivative concepts such as Input-to-Output Stability (Sontag and Wang [1999]), incremental ISS (Angeli [2002]), and other notions (see the survey by Sontag [2007] for some of these other notions). In addition, to account for a nonlinear detectability condition, Krichman et al. [2001] introduced the notion of Input-Output-to-State Stability.

Ingalls et al. [2002] presented a first attempt at deriving a generalization that would subsume some of the varied ISS-type properties into a single concept. There they considered systems without inputs and produced a Lyapunov characterization for partial detectability where measurements (a function of the state) and a transient term (dependent on a function of the initial condition) provide an upper bound on errors (another function of the state). By particular selections for the various functions required in this concept of *measurement-to-error stability*, one recovers the original property of output-to-state stability (Sontag and Wang [1997]).

In Kellett and Dower [2012] we proposed an alternate approach to generalizing ISS in order to gather various ISS-type notions in one framework. This approach relies on the concept of stability with respect to two measures. Stability with respect to two measures was first proposed by Movchan [1960] for systems without inputs. Similar to the original ISS investigations of Sontag [1989], we extended a known stability concept to systems with external inputs. A general treatment of stability with respect to two measures can be found in Lakshmikantham and Liu [1993]. Teel and Praly [2000] made use of the modern application of comparison functions in stability analysis to define $\mathcal{KL}$-*stability with respect to two measures*, and the generalization of ISS in Kellett and Dower [2012] starts from this stability concept and derives a Lyapunov characterization of ISS with respect to a single measure.

This paper continues the work begun in Kellett and Dower [2012]. In Section 1.1 we provide precise statements for the systems and properties we will consider. In Section 2 we present a Lyapunov characterization for ISS with respect to two measures. In Section 3 we define integral ISS with respect to two measures and provide a Lyapunov characterization that implies this property. In Section 3.2 we highlight a difficulty with demonstrating the converse result. In Section 4 we formulate conditions that guarantee the equivalence of the Lyapunov characterization of ISS with respect to two measures and the Lyapunov characterization of iISS with respect to two measures. We also provide an example that demonstrates the interesting result that ISS with respect to two measures does not always imply integral ISS with respect to two measures. While initially surprising, this result is consistent with the recent result of Angeli [2009] that, in the case of incremental ISS, incremental integral ISS in fact implies incremental ISS. We conclude in Section 5.

1.1 Mathematical Preliminaries and Definitions

Consider the system

$$\frac{d}{dt}x(t) = f(x(t), u(t)), \qquad x(0) = x \tag{1}$$

where $x \in \mathbb{G} \subset \mathbb{R}^n$ and $u \in \mathbb{R}^m$. In what follows, we denote by $\mathcal{U}$ the set of admissible (measurable and essentially bounded) input functions. Note that, by a slight abuse of notation, we will generally use $u \in \mathbb{R}^m$ and $u \in \mathcal{U}$ where u being a vector or function, respectively, will be clear from context. We denote the essential supremum of the function $u \in \mathcal{U}$ by $\|u\|_\infty$. We denote solutions to (1) by $\phi : \mathbb{R}_{\geq 0} \times \mathbb{G} \times \mathcal{U} \to \mathbb{R}^n$. We make the standing assumptions that $f(\cdot, \cdot)$ is locally Lipschitz in $x \in \mathbb{G}$, locally uniformly in $u \in \mathbb{R}^m$, and that system (1) is forward complete on $\mathbb{G}$; i.e., for every $x \in \mathbb{G}$ and $u \in \mathcal{U}$, solutions $\phi(t, x, u)$ exist and remain in $\mathbb{G}$ for all $t \geq 0$ (see Angeli and Sontag [1999]). We make use of the standard function classes $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ (see Hahn [1967] or Kellett [2012]). We will denote the class of positive definite functions $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $\mathcal{PD}$.

2. INPUT-TO-STATE STABILITY WITH RESPECT TO TWO MEASURES

In (Kellett and Dower [2012]), we proposed the following generalization of ISS:

Definition 1. Let $\omega_i : \mathbb{G} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$ be continuous, nonnegative functions. System (1) is said to be *input-to-state stable (ISS) with respect to* (ω_1, ω_2) if it is forward complete on $\mathbb{G}$ and if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for all $t \geq 0$,

$$\omega_1(\phi(t, x, u)) \leq \max\{\beta(\omega_2(x), t), \gamma(\|u\|_\infty)\}. \tag{2}$$

By appropriate selection of the measurement functions $\omega_i : \mathbb{G} \to \mathbb{R}_{\geq 0}$ we clearly subsume many ISS-type notions, including standard ISS ($\omega_1(x) = \omega_2(x) = |x|$), Input-to-Output Stability ($\omega_1(x) = |h(x)|$, $\omega_2(x) = |x|$, where $y = h(x)$ defines a system output), and a form of incremental ISS ($\omega_1(x_1, x_2) = \omega_2(x_1, x_2) = |x_1 - x_2|$).

It is straightforward to see that the above definition of ISS with respect to two measures involving a maximum for the upper bound is qualitatively equivalent to an upper bound involving the sum of a transient and gain bound. See [Dower et al., 2012, Section II.C] for a discussion of qualitative equivalence.

Definition 2. Let $\omega_i : \mathbb{G} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$ be continuous, nonnegative functions. An *implication-form, two measure, ISS-Lyapunov function* for (1) is a (smooth) function $V : \mathbb{G} \to \mathbb{R}_{\geq 0}$ such that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\chi \in \mathcal{K}$ so that the following hold for all $x \in \mathbb{G}$, $u \in \mathbb{R}^m$:

$$\alpha_1(\omega_1(x)) \leq V(x) \leq \alpha_2(\omega_2(x)) \tag{3}$$

$$V(x) \geq \chi(|u|) \quad \Rightarrow \quad \langle \nabla V(x), f(x, u) \rangle \leq -V(x). \tag{4}$$

This definition matches that of a State-Independent Input-Output Stable (SI-IOS) Lyapunov function when $\omega_i(\cdot) = |h(\cdot)|$, $i = 1, 2$ (see Sontag and Wang [2000]).

In (Kellett and Dower [2012]) we restricted attention to a single measure; i.e., where $\omega : \mathbb{G} \to \mathbb{R}_{\geq 0}$ is continuous and $\omega_1(\cdot) = \omega_2(\cdot) = \omega(\cdot)$. We make the following assumption on commensurability of the two measurement functions in order to provide a Lyapunov characterization of ISS with respect to two measures.

Assumption 1. There exists a function $\alpha_u \in \mathcal{K}_\infty$ such that, for all $x \in \mathbb{G}$,

$$\omega_2(x) \leq \alpha_u(\omega_1(x)). \tag{5}$$

One consequence of ISS with respect to (ω_1, ω_2) and the above assumption on the measurement functions is that their zero-value sets coincide; i.e.,

$$\{x \in \mathbb{G} : \omega_1(x) = 0\} = \{x \in \mathbb{G} : \omega_2(x) = 0\}.$$

Theorem 3. Existence of an implication-form, two measure, ISS-Lyapunov function implies that system (1) is ISS with respect to (ω_1, ω_2). Conversely, under Assumption 1, if system (1) is ISS with respect to (ω_1, ω_2), then there exists an implication-form, two measure, ISS-Lyapunov function.

The proof follows that presented in (Kellett and Dower [2012]) for a single measurement function which, in turn, follows the original proof of Sontag and Wang [1995], where especial care is required to account for possible noncompactness of level sets of the measurement function. The commensurability of the measurement functions proposed in Assumption 1 adds some straightforward inequalities to the proof in Kellett and Dower [2012], which we consequently omit.

We note that if system (1) is augmented with an output function $h : \mathbb{G} \to \mathbb{R}^m$ such that $z(t) = h(\phi(t, x, u))$ and if either of the measurement functions is defined to be $\omega_i(\cdot) \doteq |h(\cdot)|$, then it is straightforward to see that ISS with respect to (ω_1, ω_2) and Assumption 1 imply SI-IOS.

2.1 IOS-Lyapunov functions

A natural question is: what converse-type result is possible in the absence of Assumption 1? That is, without restricting the measurement functions to be comparable as defined by (5), does ISS with respect to two measures imply the existence of an implication-form, two measure, ISS-Lyapunov function satisfying (3)-(4)?

An indication of the answer to this question comes from Lyapunov-type results for Input-to-Output State Stability, Sontag and Wang [2000]. With system (1) augmented by an output

$$y = h(x) \tag{6}$$

where $h : \mathbb{G} \to \mathbb{R}^p$ is continuous, the system given by (1) and (6) is termed *Input-to-Output Stable (IOS)* if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$|h(\phi(t, x, u)| \leq \max\{\beta(|x|, t), \gamma(\|u\|_\infty)\}$$

for all $x \in \mathbb{G}$ and $u \in \mathcal{U}$. This can be seen as a special case of ISS with respect to (ω_1, ω_2) when the measurement functions are $\omega_1(x) \doteq |h(x)|$ and $\omega_2(x) \doteq |x|$.

Sontag and Wang [2000] defined IOS-Lyapunov functions and showed that the existence of such functions is equivalent to the IOS property. While the upper and lower bounds (3) are consistent with those of an IOS-Lyapunov function; i.e.,

$$\alpha_1(|h(x)|) \leq V(x) \leq \alpha_2(|x|), \quad \forall x \in \mathbb{G},$$

Sontag and Wang [2000] demonstrated that, in general, an IOS system does not admit an IOS-Lyapunov function satisfying (4). Rather, the required decrease condition is

$$V(x) \geq \chi(|u|) \;\Rightarrow\; \langle \nabla V(x), f(x, u) \rangle \leq -\kappa(V(x), |x|), \tag{7}$$

for all $x \in \mathbb{G}$ and $u \in \mathcal{U}$, where $\kappa \in \mathcal{KL}$.

This can be seen through the example (presented in Sontag and Wang [2000])

$$\dot{x}_1 = 0; \qquad \dot{x}_2 = \frac{-2x_2 + u}{1 + x_1^2},$$

where the output is taken to be the second state; i.e., $h(x) = x_2$. Intuitively, finding a Lyapunov function for this system with a decrease condition of the form (4) is impossible since the decrease rate would need to be independent of x_1. However, choosing, for example, an initial condition $x_1(0)$ very large results in a decrease rate of x_2 that is very small.

On the basis of this discussion, we anticipate a more general implication-form, two measure, ISS-Lyapunov function to be a function $V : \mathbb{G} \to \mathbb{R}_{\geq 0}$ satisfying the bounds (3) and a decrease condition similar to (7).

3. INTEGRAL INPUT-TO-STATE STABILITY WITH RESPECT TO TWO MEASURES

3.1 Lyapunov Characterization

Analogous to the above defined ISS with respect to two measures, we propose the following generalization of integral ISS, where again the maximum is qualitatively equivalent to a summation:

Definition 4. Let $\omega_i : \mathbb{G} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$ be continuous, nonnegative functions. System (1) is said to be *integral input-to-state stable (iISS) with respect to* (ω_1, ω_2) if it is forward complete on $\mathbb{G}$ and there exist functions $\alpha \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$, and $\gamma \in \mathcal{K}$ such that, for all $t \geq 0$, $x \in \mathbb{G}$, and $u \in \mathcal{U}$,

$$\alpha\left(\omega_1(\phi(t, x, u))\right) \leq \max\left\{\beta(\omega_2(x), t), \int_0^t \gamma(u(\tau)))d\tau\right\}. \quad (8)$$

In the two measure context, we define a dissipative-form ISS-Lyapunov function as follows:

Definition 5. Given continuous, nonnegative functions $\omega_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, a *dissipative-form, two measure, ISS-Lyapunov function* for (1) is a (smooth) function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathcal{PD}$, and $\sigma \in \mathcal{K}$ so that, for all $x \in \mathbb{G}$, $u \in \mathbb{R}^m$, the following hold:

$$\alpha_1(\omega_1(x)) \leq V(x) \leq \alpha_2(\omega_2(x)) \quad (9)$$

$$\langle \nabla V(x), f(x, u) \rangle \leq -\rho(V(x)) + \sigma(|u|). \quad (10)$$

It is reasonably straightforward to show that the above dissipative-form ISS-Lyapunov function implies integral ISS with respect to two measures.

Theorem 6. If system (1) admits a dissipative-form, two measure, ISS-Lyapunov function for fixed continuous, nonnegative functions $\omega_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, then it is iISS with respect to (ω_1, ω_2).

Proof: Applying a comparison principle [Angeli et al., 2000, Corollary IV.3] ([Kellett, 2012, Lemma 12]) to the differential inequality (10) yields suitable functions $\beta \in \mathcal{KL}$ and $\tilde{\sigma} \in \mathcal{K}$ such that

$$V(\phi(t, x, u)) \leq \beta(V(x), t) + \int_0^t 2\tilde{\sigma}(|u(s)|)ds.$$

Making use of the upper and lower bounds (9), and with $\sigma \doteq 2\tilde{\sigma}$ we then have

$$\alpha_1(\omega_1(\phi(t, x, u))) \leq \beta(\alpha_2(\omega_2(x)), t) + \int_0^t \sigma(|u(s)|)ds. \quad \blacksquare$$

Similar to the previous generalizations of ISS and iISS, we define 0-GAS with respect to two measures.

Definition 7. Let $\omega_i : \mathbb{G} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$ be continuous, nonnegative functions. System (1) is said to be *zero-input globally asymptotically stable (0-GAS) with respect to* (ω_1, ω_2) if it is forward complete on $\mathbb{G}$ and there exists $\beta \in \mathcal{KL}$, such that, for all $t \geq 0$, and $x \in \mathbb{G}$

$$\omega_1(\phi(t, x, 0)) \leq \beta(\omega_2(x), t). \quad (11)$$

If $\omega_1(x) = \omega_2(x) = |x|$ then the above is the usual definition of 0-GAS. We make the obvious observation that iISS with respect to (ω_1, ω_2) implies 0-GAS with respect to (ω_1, ω_2). This follows directly from (8) along with the comparison function properties that the inverse of a class-$\mathcal{K}_\infty$ function always exists and is itself a class-$\mathcal{K}_\infty$ function, and also that the composition of a class-$\mathcal{K}_\infty$ function and a class-$\mathcal{KL}$ function is again a class-$\mathcal{KL}$ function.

3.2 The Converse of Theorem 6

One of the key steps in Angeli et al. [2000] to prove that iISS implies the existence of an iISS-Lyapunov function involves a characterization of 0-GAS. Unfortunately, a simple example shows that this characterization will not hold in the arbitrary measure case. To be precise, [Angeli et al., 2000, Lemma IV.10] is:

Lemma 8. The system (1) is 0-GAS if and only if there exist a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_i \in \mathcal{K}_\infty$ for $i = 1, 2, 3$, and $\lambda, \delta \in \mathcal{K}$ such that, for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \text{and} \quad (12)$$

$$\langle \nabla V(x), f(x, u) \rangle \leq -\alpha_3(|x|) + \lambda(|x|)\delta(|u|). \quad (13)$$

However, the 0-GAS characterization required to generalize the proof in Angeli et al. [2000] to obtain the converse to Theorem 6 would involve a decrease condition of the form

$$\langle \nabla V(x), f(x, u) \rangle \leq -\alpha_3(V(x)) + \lambda(V(x))\delta(|u|). \quad (14)$$

The following example shows that the argument of λ cannot, in general, be $V(x)$.

Example 1. Consider the following second-order system

$$\dot{x}_1 = -x_1|u|(1 - \varphi(u)); \qquad \dot{x}_2 = -x_2 + x_1 u \quad (15)$$

where $\varphi : \mathbb{R} \to [0, 1]$ is smooth, has compact support on $[\frac{1}{2}, \frac{3}{2}]$, and attains a maximum of 1 at 1. Consider the measurement functions $\omega_1(x) = \omega_2(x) = \omega(x) = |x_2|$. This system is clearly 0-GAS with respect to $\omega(\cdot)$. Suppose we had an appropriate Lyapunov function; i.e., a smooth function $V : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\lambda, \delta \in \mathcal{K}$ satisfying

$$\alpha_1(|x_2|) \leq V(x) \leq \alpha_2(|x_2|), \quad \text{and} \quad (16)$$

$$\langle \nabla V(x), f(x, u) \rangle \leq -\alpha_3(V(x)) + \lambda(V(x))\delta(|u|). \quad (17)$$

For $x_2 = 1$ and $u = 0$ we see that (16)-(17) imply

$$\langle \nabla V(x), f(x, 0) \rangle = -\left.\frac{\partial V}{\partial x_2}\right|_{x_2=1}$$
$$\leq -\alpha_3(V(x)) + \lambda(V(x))\delta(0) \leq -\alpha_3 \circ \alpha_1(1) < 0,$$

for all $x_1 \in \mathbb{R}$. Let $\varepsilon = \alpha_3 \circ \alpha_1(1) > 0$. Consequently,

$$\left.\frac{\partial V}{\partial x_2}\right|_{x_2=1} \geq \varepsilon, \quad \forall x_1 \in \mathbb{R}. \tag{18}$$

Then, for any $x_1 \in \mathbb{R}$ satisfying

$$x_1 > \frac{\lambda\left(\alpha_2(1)\right)\delta(1)}{\varepsilon} + 1 \tag{19}$$

with $u = 1$ and $x_2 = 1$, using the definition of φ, (18), (19), and the upper inequality of (16), we have

$$\begin{aligned}\langle \nabla V(x), f(x,1)\rangle = &\left.\frac{\partial V}{\partial x_2}\right|_{x_2=1} (-1+x_1) \geq \varepsilon(-1+x_1)\\ &> \varepsilon\,\frac{\lambda\left(\alpha_2(1)\right)\delta(1)}{\varepsilon} \geq \lambda(V(x))\delta(1)\\ &\geq -\alpha_3(V(x)) + \lambda(V(x))\delta(1).\end{aligned}$$

Therefore, for $u = 1$, $x_2 = 1$, and $x_1 \in \mathbb{R}$ sufficiently large, it is impossible to find functions satisfying (16)-(17). Consequently, proving the converse of Theorem 6 requires an alternate approach to that in Angeli et al. [2000].

4. IMPLICATION AND DISSIPATION FORM LYAPUNOV FUNCTIONS

4.1 Conditions for Equivalence

We now turn our attention to the relationship between implication-form ISS-Lyapunov functions and dissipative-form ISS-Lyapunov functions.

In the standard ISS/iISS framework, it is known that the existence of an implication-form ISS-Lyapunov function implies the existence of a dissipative-form ISS-Lyapunov function ([Sontag and Wang, 1995, Remark 2.4]) and the converse holds when the function $\rho \in \mathcal{PD}$ of (10) is in fact of class-$\mathcal{K}$ and is ultimately larger than the gain function ([Ito, 2006, Remark 4]). This latter result can be shown to hold true in the general two measure case, and we formalize this result here.

Proposition 9. Given a dissipative-form, two measure, ISS-Lyapunov function satisfying (9)-(10) such that $\rho \in \mathcal{K}$ satisfies $\sup_{s\geq 0}\rho(s) > \sup_{s\geq 0}\sigma(s)$ there exists an implication-form, two measure, ISS-Lyapunov function satisfying (3)-(4).

Note that the same function $V : \mathbb{G} \to \mathbb{R}_{\geq 0}$ will not, in general, serve as both a dissipative-form and implication-form, two measure, ISS-Lyapunov function.

Proof: Following the argument of the proof in [Rüffer et al., 2010, Proposition 2.2], we can construct a dissipative-form two measure ISS-Lyapunov function where the decrease function is of class-$\mathcal{K}_\infty$. This, in turn, provides us with an implication-form, two measure, ISS-Lyapunov function where the right-hand side of the decrease condition (4) is not explicitly $-V(x)$. Following the argument in [Praly and Wang, 1996, Lemma 11] to obtain the desired exponential decrease (i.e., (4)) completes the proof. ■

In order to address the case where the implication-form implies the dissipative-form we require a definition. We denote a sequence of points $x_n \in \mathbb{G}$ converging to a point on the boundary of $\mathbb{G}$ by $x_n \to \partial\mathbb{G}^\infty$. If $\mathbb{G}$ is unbounded, then $x_n \to \partial\mathbb{G}^\infty$ subsumes the case $|x_n| \to \infty$.

Definition 10. Let $\mathbb{A} \subset \mathbb{G}$ be compact. A continuous function $V : \mathbb{G} \to \mathbb{R}_{\geq 0}$ is a *proper indicator for* $\mathbb{A}$ *on* $\mathbb{G}$ if $V(x) = 0$ if and only if $x \in \mathbb{A}$ and $\lim_{x\to\partial\mathbb{G}^\infty} V(x) = \infty$.

We may now state a sufficient condition for the existence of a dissipative-form, two measure, ISS-Lyapunov function.

Proposition 11. Let $\mathbb{A} \subset \mathbb{G}$ be a compact set and let $V : \mathbb{G} \to \mathbb{R}_{\geq 0}$ be an implication-form, two measure, ISS-Lyapunov function satisfying (3)-(4). Furthermore, suppose $V : \mathbb{G} \to \mathbb{R}_{\geq 0}$ is a proper indicator for the compact set $\mathbb{A}$ on $\mathbb{G}$. Then there exists a dissipative-form, two measure, ISS-Lyapunov function satisfying (9)-(10).

Note that, as a consequence of the upper and lower class-$\mathcal{K}_\infty$ bounds (3), a two measure ISS-Lyapunov function will be a proper indicator for a compact set if the measurement functions are equal (i.e., $\omega_1(x) = \omega_2(x) = \omega(x)$) and if this single measurement function is a proper indicator for a compact set. This case was proved in Kellett and Dower [2012]. The proof of Proposition 11 is a simple modification of the proof of [Kellett and Dower, 2012, Proposition 1].

4.2 The Impact of Compactness

A natural question following Proposition 11 is whether or not the assumption of having a proper indicator for a compact set is necessary. In this section, we provide an example system that admits an implication-form, two measure, ISS-Lyapunov function but for which no dissipative-form, two measure, ISS-Lyapunov function exists.

Example 2. Define the system

$$\dot{x}_1 = x_1|u|(1-\varphi(u)) \tag{20}$$

$$\dot{x}_2 = -x_2 + u + x_1\kappa(x_2)\varphi(u) \tag{21}$$

where $\varphi : \mathbb{R} \to [0,1]$ is smooth, has compact support on $[\frac{1}{2}, \frac{3}{2}]$, and attains a maximum of 1 at 1, and $\kappa : \mathbb{R} \to [0,1]$ is smooth with support on $(-3,3)$ and satisfies $\kappa(x_2) = 1$ for $|x_2| \leq 2$. Fix $\omega_1(x) = \omega_2(x) = \omega(x) \doteq |x_2|$. Note that the set $\{(x_1,x_2) \in \mathbb{R}^2 : \omega(x) = |x_2| = 0\}$ is not compact.

Take $V(x) = \frac{1}{2}x_2^2$ and $\chi(u) = 2|u|$ in (3) and (4). Then the bounds (3) are trivially satisfied and (4) is

$$\begin{aligned}\langle \nabla V(x), f(x,u)\rangle &= -x_2^2 + x_2u + x_2x_1\kappa(x_2)\varphi(u)\\ &\leq -\frac{3}{4}x_2^2 + u^2 + x_2x_1\kappa(x_2)\varphi(u).\end{aligned}$$

We see that when $|x_2| \geq \chi(|u|)$, the above becomes

$$\langle \nabla V(x), f(x,u)\rangle \leq -V(x) + x_2x_1\kappa(x_2)\varphi(u).$$

By definition of φ, when $|u| \notin \left[\frac{3}{2}, 3\right]$, φ is identically zero and consequently, in this range, we have

$$\omega(x) \geq \chi(|u|) \quad \Rightarrow \quad \langle \nabla V(x), f(x,u)\rangle \leq -V(x). \tag{22}$$

On the other hand, for $|u| \in \left[\frac{3}{2}, 3\right]$ we see that the condition $\omega(x) \geq \chi(|u|)$ implies $|x_2| \geq 3$. On this range of x_2, κ is identically zero and this again implies (22). Consequently, we see that $V(x) = \frac{1}{2}x_2^2$ is an implication-form, two measure, ISS-Lyapunov function and, by Theorem 3, system (20)-(21) is ISS with respect to ω.

To see that no function $W : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ satisfying (9)-(10) can exist, we pursue a contradiction argument. In particular, we assume that such a W does exist and focus on the line $x_2 = 1$ where $\kappa(x_2) = 1$. Let $\varepsilon > 0$ satisfy $0 < \varepsilon \leq \rho(W(x))$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, $x_2 = 1$. That such an $\varepsilon > 0$ exists follows from the fact that $\rho \in \mathcal{PD}$ and $\alpha_1(|x_2|) = \alpha_1(1) \leq W(x)$ where $\alpha_1 \in \mathcal{K}_\infty$ is from (9). First, consider $u = 0$ which implies $\varphi(u) = 0$ and also $\dot{x}_1 = 0$. Then (10) implies W must satisfy

$$-x_2 \frac{\partial W}{\partial x_2} \leq -\rho(W(x)) \tag{23}$$

which implies

$$\left.\frac{\partial W}{\partial x_2}\right|_{x_2=1} \geq \rho(W(x)) \geq \varepsilon > 0. \tag{24}$$

On the other hand, consider $u = 2$ which implies $\varphi(u) = 1$ and so $\dot{x}_1 = 0$ again. In this case, satisfying (10) requires

$$\frac{\partial W}{\partial x_2}(-x_2 + 2 + x_1) \leq -\rho(W(x)) + \sigma(2)$$

which, for $x_1 > -1$, implies

$$\left.\frac{\partial W}{\partial x_2}\right|_{x_2=1} \leq \frac{-\rho(W(x)) + \sigma(2)}{1 + x_1}. \tag{25}$$

Therefore, combining (24) and (25), we see that the partial derivative of the second component of W evaluated at $x_2 = 1$ must satisfy

$$0 < \varepsilon \leq \rho(W(x)) \leq \left.\frac{\partial W}{\partial x_2}\right|_{x_2=1} \leq \frac{-\rho(W(x)) + \sigma(2)}{1 + x_1}.$$

However, since the right-hand side of the upper inequality goes to zero as $x_1 \to \infty$, such a W cannot exist. In other words, the system (20)-(21) does not admit a dissipative-form, two measure, ISS-Lyapunov function.

Furthermore, it is possible to show that the system defined by (20)-(21) is not integral ISS with respect to ω. To see this, fix a constant input value $\bar{u} > 0$ such that $\varphi(\bar{u}) = \frac{1}{2}$. In this case, we see that system (20)-(21) is given by

$$\dot{x}_1 = \frac{1}{2}x_1\bar{u}; \qquad \dot{x}_2 = -x_2 + \bar{u} + \frac{1}{2}x_1\kappa(x_2).$$

We immediately have $\phi_1(t, x, \bar{u}) = e^{\bar{u}\frac{t}{2}}x_1$. Note that, for a fixed $\bar{u}$, the right-hand side of the integral ISS estimate (8) can be made arbitrarily small by choosing both $\omega(x) > 0$ and $t > 0$ to be sufficiently small. However, from the definition of κ, as long as $|\phi_2(t, x, \bar{u})| < 2$ we have

$$\phi_2(t, x, \bar{u}) = e^{-t}x_2 + \int_0^t e^{-(t-\tau)}\left(\bar{u} + \frac{1}{2}x_1 e^{\bar{u}\frac{\tau}{2}}\right) d\tau.$$

Since we are free to choose the initial condition $x_1 > -1$, we restrict attention to $x_1 > 0$ In this case, for arbitrarily large $x_1 > 0$, it follows that $\phi_2(t, x, \bar{u}) = 2$ is possible for arbitrarily small $t > 0$ and arbitrarily small but positive x_2, making it impossible to satisfy a bound of the form (8). Consequently, we see that system (20)-(21) is ISS with respect to ω but is *not* integral ISS with respect to ω.

Remark 1. One possible consequence of the nonequivalence of implication and dissipative-form ISS-Lyapunov functions is that, in the general two measure case considered in this paper, ISS may not be equivalent to integral Input to integral State Stability (iIiISS) as demonstrated in the classical case in Sontag [1998]. In particular, to demonstrate that ISS implies iIiISS, Sontag used a dissipative-form ISS-Lyapunov function which, as we have just demonstrated, may fail to exist in the general case. It remains an open question as to whether or not the equivalence of ISS and iIiISS holds in this general case.

4.3 Incremental Stability

The above example demonstrating a system that is ISS with respect to two measures but that is not integral ISS with respect to two measures is initially surprising since, in the classical case, ISS systems form a strict subset of integral ISS systems. However, the example presented above is consistent with recent results on incremental ISS derived in Angeli [2009].

Angeli [2002] defined incremental ISS based on two copies of a single system; i.e., for $f : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ and

$$\begin{aligned} \dot{x}_1 &= f(x_1, u_1), \quad x_1 \in \mathbb{R}^n,\ u_1 \in \mathbb{U}, \\ \dot{x}_2 &= f(x_2, u_2), \quad x_2 \in \mathbb{R}^n,\ u_2 \in \mathbb{U}, \end{aligned} \tag{26}$$

incremental ISS is the property that there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that solutions of the above system satisfy

$$\begin{aligned} &|\phi_1(t, x_1, u_1) - \phi_2(t, x_2, u_2)| \\ &\qquad \leq \max\{\beta(|x_1 - x_2|, t), \gamma(\|u_1 - u_2\|). \end{aligned} \tag{27}$$

In other words, the difference between trajectories starting from different initial conditions and subject to different inputs is bounded by a transient term depending on the distance between initial conditions and a steady-state term that depends on the worst-case difference between the input signals. An integral form of incremental ISS was similarly defined in Angeli [2009].

The surprising result of Angeli [2009] is that incremental integral ISS implies incremental ISS. However, it remains an open question whether or not there is in fact a gap; in other words, are incremental integral ISS systems a strict subset of incremental ISS systems? The example in the previous section provides an indication that this may be true. In particular, the lack of compactness of level sets of the measurement function is a crucial element in constructing a system that is ISS with respect to two measures but not integral ISS with respect to two measures. This lack of compactness is also a fundamental element of the incremental stability notions.

We note that the incremental ISS of (26) as defined by (27) is close to, but not precisely the same as, ISS with respect to two measures of (26) when the measurement functions are $\omega_1(x_1, x_2) = \omega_2(x_1, x_2) = |x_1 - x_2|$. In particular, (27) looks at the worst-case difference between two different input signals while (2) depends on the worst-case value of a single input. Consequently, the states of (27) appear in the same manner, but inputs must be considered differently.

Finally, [Angeli, 2009, Theorem 1] presents a converse theorem for incremental integral ISS where a *continuous* dissipative-form Lyapunov function is demonstrated but the existence of a smooth Lyapunov function is cited as an open problem. This provides a further indication that the impediments mentioned in Section 3.2 with regards to obtaining a (smooth) Lyapunov characterization of iISS with respect to two measures are nontrivial.

5. CONCLUSIONS

In this paper we explored Lyapunov characterizations of a generalization of ISS and iISS based on the concept of stability with respect to two measures. This generalization subsumes many ISS-type notions including standard ISS, Input-to-Output Stability, and a form of incremental ISS. Partial Lyapunov characterizations are provided for both ISS (Theorem 3) and integral ISS (Theorem 6).

Of particular interest is the fact that the Lyapunov characterization of ISS with respect to two measures relies on an implication-form ISS-Lyapunov function while that for integral ISS relies on a dissipative-form ISS-Lyapunov function. Additionally, we demonstrated that, in general, neither type of ISS-Lyapunov function implies the other. It is known that a dissipative-form ISS-Lyapunov function implying an implication-form ISS-Lyapunov function requires a strengthening of the decrease rate, and this carries over to the two measure case. It is surprising, however, that an implication-form ISS-Lyapunov function implying a dissipative-form ISS-Lyapunov function relies on the compactness of level sets of the ISS-Lyapunov function - a property which does not hold in general.

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