Approximate Finite-Horizon Optimal Control for Input-Affine Nonlinear Systems with Input Constraints

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Abstract: The *finite-horizon optimal control problem with input constraints* consists in controlling the state of a dynamical system over a finite time interval (possibly unknown) minimizing a cost functional, while satisfying hard constraints on the input. For linear systems the solution of the problem often relies upon the use of bang-bang control signals. For nonlinear systems the "shape" of the optimal input is in general not known. The control input can be found solving an Hamilton-Jacobi-Bellman (HJB) partial differential equation (pde): it typically consists of a combination of bang-bang arcs and singular arcs. In the paper a methodology to approximate the solution of the HJB pde arising in the finite-horizon optimal control problem with input constraints is proposed. This approximation yields a dynamic state feedback law. The theory is illustrated by means of an example: the minimum time optimal control problem for an industrial wastewater treatment plant.

Keywords: Optimal control; Minimum-time control; Nonlinear systems; Partial differential equations.

1. INTRODUCTION

Given a system with known initial state and with hard constraints on the amplitude of the control input, the aim of the finite-horizon optimal control problem is to determine the control signal which minimizes the cost functional while satisfying the control constraints. Various practical problems, such as the minimum time and the maximum range optimal control problems, can be formulated in this framework. For linear systems the finite-horizon optimal control problem with input constraints has a well-known solution which relies upon the use of bang-bang controls, see e.g. the recent book of Liberzon [2012]. For nonlinear systems the solution can be found solving the associated HJB pde. However, it might be difficult or even impossible to solve the equation analytically.

In Sassano and Astolfi [2012] and Sassano and Astolfi [2013] a new method to solve, approximatively, classes of optimal control problems has been developed. The method relies upon the use of dynamic state feedback and does not require the solution of any pde. The goal of this work is to extend the results in Sassano and Astolfi [2013] providing approximate solutions for general finite-horizon optimal control problems with a more general class of cost functionals and in the presence of input constraints. The extension of the ideas in Sassano and Astolfi [2013] is not straightforward. First of all, since different costs are considered, it is not possible to exploit the solution of the associated algebraic Riccati equation (ARE) in the construction of an approximate solution of the problem. Then the non-differentiability of the value function which is inherent to problems in which there is a hard constraint on the input leads to a feedback that may not be everywhere differentiable.

We illustrate the theory solving the approximate minimum time optimal control problem for a bioreactor. The paper is organized as follows. In Section 2 the formulation of the problem is given together with some additional definitions. In Section 3 the main result is presented, i.e., a dynamic control law that approximatively solves the optimal control problem. In Section 4, the case study is discussed, and in Section 5 conclusions are drawn.

2. PRELIMINARIES

2.1 Problem formulation

Consider an input-affine, nonlinear, system described by the equation[1]

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system and $u(t) \in \mathbb{R}^m$ is the control input subject to the constraints

$$u_{min} \leq u_j(t) \leq u_{max},\ j = 1, 2, ..., m,\ \forall t \in \mathbb{R}, \tag{2}$$

with $u_{min} < u_{max}$. The mappings f and g are assumed sufficiently smooth. With minor loss of generality we

[1] For simplicity, the arguments of the functions are dropped whenever this does not cause confusion.

assume $u_{min} = -1$ and $u_{max} = 1$. Finally, consider the additional equation

$$Cx(T) = 0, \tag{3}$$

with $C \in \mathbb{R}^{r\times n}$ constant, which is used to model r constraints on the final state.

Aim of the constrained finite-horizon optimal control problem is to find a control input u that minimizes the cost functional

$$J(x(t), u(t)) = \int_0^T [\phi(x(t)) + \gamma(x(t))u(t)]dt, \tag{4}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$, $\gamma : \mathbb{R}^n \to \mathbb{R}^{1\times m}$ while satisfying the constraints (1), (2) and (3). Note that T may or may not be a priori assigned. Hence, the optimal control problem has running cost

$$\mathcal{L}(x, u) = \phi(x) + \gamma(x)u, \tag{5}$$

and terminal cost

$$\mathcal{T}(x(T)) = 0. \tag{6}$$

The dynamic programming solution of this problem is based on the solution of the HJB pde

$$\phi(x) + V_t + V_x f(x) - |\gamma(x) + V_x g(x)| = 0, \tag{7}$$

where

$$|\gamma(x) + V_x g(x)| = \sum_{j=1}^{m} |\gamma_j(x) + \sum_{i=1}^{n} V_{x_i} g_{ij}(x)|, \tag{8}$$

subject to the condition $V(x,T) = 0$.

In what follows, similarly to Sassano and Astolfi [2013], we define a regional version of the problem.

Problem 1. Consider the system (1), the constraints (2), (3) and the cost (4). The *regional dynamic constrained finite-horizon optimal control* problem consists in finding an integer $\tilde{n} \geq 0$, a dynamic control law described by equations of the form

$$\begin{aligned} \dot{\xi} &= \alpha(x, \xi, t), \\ u &= \beta(x, \xi, t), \end{aligned} \tag{9}$$

with $\xi(t) \in \mathbb{R}^{\tilde{n}}$, $\alpha : \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \times \mathbb{R} \to \mathbb{R}^{\tilde{n}}$ and $\beta : \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \times \mathbb{R} \to \mathbb{R}^m$ smooth mappings with $|\beta_i(x, \xi, t)| \leq 1$ for all $i \in [1, m]$, and a set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ such that the closed-loop system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\beta(x, \xi, t), \\ \dot{\xi} &= \alpha(x, \xi, t), \end{aligned} \tag{10}$$

satisfies the conditions

$$J((x_0, \xi_0), \beta) \leq J((x_0, \xi_0), \tilde{u}),$$
$$Cx(T) = 0,$$

for any $\tilde{u}$ and any (x_0, ξ_0) for which the trajectory of the system (10) remains in Ω for all $t \in [0, T]$. □

The solution of this problem is still hard to determine. Hence, we define an approximate version of the regional dynamic finite-horizon optimal control problem.

Problem 2. Consider the system (1), the constraints (2), (3) and the cost (4). The *approximate regional dynamic constrained finite-horizon optimal control* problem consists in finding an integer $\tilde{n} \geq 0$, a dynamic control law described by the equations (9), a set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ and functions $\rho_1 : \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \times \mathbb{R} \to \mathbb{R}_+$ and $\rho_2 : \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}_+$ such that the regional dynamic constrained finite-horizon optimal control problem is solved with respect to the running cost

$$\mathcal{L}(x, \xi, u) = \phi(x) + \gamma(x)u + \rho_1(x, \xi, t), \tag{11}$$

and the terminal cost

$$\mathcal{T}(x(T), \xi(T)) = \rho_2(x(T), \xi(T)). \tag{12}$$

□

Remark 1. The non-negativity of ρ_2 is required to avoid that the terminal cost may become unbounded. This assumption can be relaxed requiring that ρ_2 be bounded from below.

To simplify the forthcoming development we discuss the underlying linearized problem. Let x_ℓ and u_ℓ be the linearization point and define $A = \frac{\partial f}{\partial x}(x_\ell)$, $F = \frac{\partial \phi}{\partial x}(x_\ell)$, $h_{s\ell} = f(x_\ell) + g(x_\ell)u_\ell$, $h_{c\ell} = \phi(x_\ell) + \gamma(x_\ell)u_\ell$, $B = g(x_\ell)$ and $G = \gamma(x_\ell)$. Consider the linear system described by the equation

$$\dot{x} = Ax + Bu + h_{s\ell}, \tag{13}$$

and the cost functional

$$J_\ell(x(t), u(t)) = \int_0^T [Fx(t) + Gu(t) + h_{c\ell}]dt, \tag{14}$$

subject to the constraints (2) and (3). The dynamic programming solution of this problem relies on the solution of the HJB pde

$$Fx + h_{c\ell} + V_{\ell\tau} + V_{\ell x}(Ax + h_\ell) - |V_{\ell x}B + G| = 0,$$

subject to $V_\ell(x, T) = 0$, where $V_{\ell x}$ and $V_{\ell\tau}$ are the partial derivatives of the value function V_ℓ.

In what follows we assume that the underlying linearized problem has been solved and V_ℓ has been computed. Although solving the linearized problem is in general difficult, there are some significant cases in which this is possible, e.g. the minimum time optimal control problem.

2.2 Algebraic solution and value function

Similarly to Sassano and Astolfi [2013], we consider the extended state $(x^T, \tau)^T$, with $\dot{\tau} = 1$ and assume that the HJB equation (7) can be solved *algebraically*, as detailed hereafter.

Definition 1. Let $\Sigma : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times n}$, with $x^T\Sigma(x, \tau)x$ positive for all $(x, \tau) \in \mathbb{R}^n \times \mathbb{R}$, and $\sigma : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}_+$. A continuously differentiable mapping $W(x, \tau) = [V_{\ell x} + \Delta_x(x, \tau), V_{\ell\tau} + \Delta_\tau(x, \tau)]$, $\Delta_x : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{1\times n}$, $\Delta_\tau : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, is an algebraic $\widehat{W}$ solution of (7) if

$$\begin{aligned} 0 = \phi &+ [V_{\ell\tau} + \Delta_\tau(x, \tau)] + [V_{\ell x} + \Delta_x(x, \tau)]f \\ &- |[V_{\ell x} + \Delta_x(x, \tau)]g + \gamma| + x^T\Sigma x + \tau^2\sigma. \end{aligned} \tag{15}$$

□

Using the algebraic $\widehat{W}$ solution of the equation (15), define the function

$$\begin{aligned} V(x, \tau, \xi, s) = V_\ell(x, \tau) &+ \Delta_x(\xi, s)x + \Delta_\tau(\xi, s)\tau \\ &+ \frac{1}{2}||x - \xi||_R^2 + \frac{1}{2}b||\tau - s||^2, \end{aligned} \tag{16}$$

where $\xi \in \mathbb{R}^n$, $s \in \mathbb{R}$, $b > 0$, $R = R^T > 0$ and $||v||_R^2 = v^T Rv$.

Remark 2. We use a different notion of algebraic solution than the one in Sassano and Astolfi [2013]. Therein, the solution of the ARE associated to the linear problem is exploited to ensure that the function V is locally positive definite. Since in our case the cost functional is not quadratic, it is not possible to use the ARE. To guarantee that the function V is locally positive definite we also exploit the solution of the associated linearized problem, but compute the "correction" terms in a different way.

3. DYNAMIC CONTROL LAW

Consider the nonlinear system (1), the constraints (2), (3) and the cost functional (4). Let

$$\Lambda(\xi, s) = \Psi(\xi, s)R^{-1}, \qquad \lambda(\xi, s) = \psi(\xi, s)R^{-1},$$

where $\Psi(\xi, s) \in \mathbb{R}^{n\times n}$ and $\psi(\xi, s) \in \mathbb{R}^{1\times n}$ are the Jacobian matrices of the mapping $\Delta_x(\xi, s)$ and $\Delta_\tau(\xi, s)$ with respect to ξ. Let $P(x)$, $\Phi(x, \xi, s)$, $\ell(x, \tau, s)$, $H(x, \xi, s)$, $\Pi(x, \tau, s)$, $W_1(x, \xi, s)$, $W_2(x, s)$, $D_1(x, \xi, s)$ and $D_2(x, s)$ be continuous mappings such that

$$\begin{aligned} f(x) &= P(x)x, \\ \Delta_x(x, s) - \Delta_x(\xi, s) &= (x-\xi)^T\Phi(x, \xi, s)^T, \\ \Delta_\tau(x, s) - \Delta_\tau(x, \tau) &= \ell(x, \tau, s)(s-\tau), \\ \Delta_\tau(\xi, s) - \Delta_\tau(x, s) &= (x-\xi)^T H(x, \xi, s)(x-\xi), \\ \Delta_x(x, s) - \Delta_x(x, \tau) &= x^T\Pi(x, \tau, s), \\ \frac{\partial\Delta_x(\xi, s)}{\partial s} - \frac{\partial\Delta_x(x, s)}{\partial s} &= W_1(x, \xi, s)(x-\xi), \\ \frac{\partial\Delta_x(x, s)}{\partial s} &= W_2(x, s)x, \\ \frac{\partial\Delta_\tau(\xi, s)}{\partial s} - \frac{\partial\Delta_\tau(x, s)}{\partial s} &= D_1(x, \xi, s)(x-\xi), \\ \frac{\partial\Delta_\tau(x, s)}{\partial s} &= D_2(x, s)x. \end{aligned}$$

We are now ready to state the main result of the paper.

Proposition 1. Consider the system (1), the constraints (2) and (3), and the cost (4). Let W be an algebraic $\widehat{W}$ solution of (15). Let $R = R^T > 0$ and $b > 0$ be such that the function V defined in (16) is positive definite. In addition suppose there exists a set $\Omega \subseteq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ in which

$$\begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix} < \begin{bmatrix} \Sigma & 0 \\ 0 & \sigma \end{bmatrix}, \tag{17}$$

and

$$|L_4| \le |L_4 + (x-\xi)^T Rg|, \tag{18}$$

for all $(x, \tau, \xi, s) \in \Omega$, where

$$\begin{aligned} L_1 &= \Pi P + \eta W_2 + \Lambda Y^T + Y\Lambda^T + \Lambda H\Lambda^T, \\ L_2 &= Y\lambda^T + \Lambda H\lambda^T + \frac{\eta}{2}D_2^T + \frac{\eta}{2}\Lambda D_1^T, \\ L_3 &= \lambda H\lambda^T - \frac{\eta}{2}D_1\lambda^T - \frac{\eta}{2}\lambda D_1^T, \\ L_4 &= (V_{\ell x} + \Delta_x(x, \tau))g + \gamma, \\ Y &= \frac{1}{2}(R-\Phi)^T P + \frac{\eta}{2}W_1, \end{aligned}$$

and $\eta = 1 - \dfrac{\ell}{b}$. Suppose finally that

$$\frac{\partial}{\partial u_i}\left[\frac{d}{dt}(V_x g_i + \gamma_i)\right] \neq 0, \text{ if } \{V_x g_i + \gamma_i\} = 0. \tag{19}$$

Then there exists $\bar{k}$ such that for all $k > \bar{k}$ the function V satisfies the HJB inequality

$$\mathcal{HJBI} \triangleq \phi + V_x f + V_\tau + V_\xi\dot{\xi} + V_s\dot{s} - |V_x g + \gamma| \le 0, \tag{20}$$

for all $(x, \tau, \xi, s) \in \Omega$, with $\dot{\xi} = -kV_\xi^T$ and $\dot{s} = \eta$. Hence

$$\begin{aligned} \dot{s} &= \eta, \\ \dot{\xi} &= -k(\Psi^T x - R(x-\xi) + \psi^T\tau), \\ u_i &= \begin{cases} -\operatorname{sign}\{V_x g_i + \gamma_i\}, & \text{if } \{V_x g_i + \gamma_i\} \neq 0, \\ u_{i_{singular}}, & \text{if } \{V_x g_i + \gamma_i\} = 0, \end{cases} \end{aligned} \tag{21}$$

with $u_{i_{singular}}$ such that the derivative with respect to time of $\{V_x g_i + \gamma_i\}$ is identically equal to zero, solves Problem 2 with $\rho_1 = -\mathcal{HJBI} \ge 0$ and $\rho_2 = V(x(T), \tau(T), \xi(T), s(T))$. □

Remark 3. Similarly to Sassano and Astolfi [2013], it is possible to define the gain k in the dynamics of ξ as a function of (x, ξ, τ, s) to reduce the absolute value of μ_1, hence the additional running cost ρ_1 in (11).

Remark 4. The gain k may be selected to satisfy condition (18) as explained hereafter. Suppose that there exists a set Ω in which $V_\xi V_\xi^T$ is nonzero, then selecting

$$k(x, \tau, \xi, s) = (V_\xi V_\xi^T)^{-1}\left[x^T \ (x-\xi)^T \ \tau\right] M \begin{bmatrix} x \\ (x-\xi) \\ \tau \end{bmatrix} - \mu_3 \tag{22}$$

yields $\mu_1 = -\mu_3$, thus providing a solution to Problem 2 with the additional running cost $\rho_1 = 0$. Note that there is no condition on the sign of the gain k, since in this context stability requirements are not imposed.

Remark 5. Note that the left hand side of (17) is zero at zero. Then, if $\Sigma(0,0) = \bar{\Sigma} > 0$ and $\sigma(0,0) = \bar{\sigma} > 0$, by continuity, there exists a region $\bar{\Omega}$ containing the origin such that (17) is satisfied for all $(x, \tau, \xi, s) \in \bar{\Omega}$.

4. EXAMPLE: MINIMUM TIME OPTIMAL CONTROL OF A BIOREACTOR

In this section the theory is illustrated by means of a classical example: the minimum time optimal control problem for a bioreactor. Proposition 1 is used to design a dynamic control law solving an approximate problem. In the example we compare the optimal control law (which can be explicitly computed) with the control law obtained from the optimal solution of the linearized problem and the dynamic control law obtained from the solution of the approximate problem.

Several results have been obtained in the control of bioreactors, see e.g. Moreno [1999], Antonelli and Astolfi [2000], Betancur et al. [2006], Rapaport and Dochain [2011] and references therein. In Moreno [1999] an optimal policy for a class of bioreactors has been proposed, while in Rapaport and Dochain [2011] the optimal policy has been developed for a more general class of bioreactors and it has been demonstrated that the optimal policy may contain singular arcs.

A simple model that globally describes the dynamic behavior of a bioreactor is given by the differential equations, see Moreno [1999],

$$\dot{s} = \theta(s, v) + \pi(s, v)u, \qquad \dot{v} = u, \tag{23}$$

where

$$\theta(s(t), v(t)) = -\mu(s(t))\left[s_{in} - s(t) + \frac{\rho(z_0)}{yv(t)}\right],$$
$$\pi(s(t), v(t)) = \frac{s_{in} - s(t)}{v(t)},$$
$$\mu(s) = \frac{\mu_0 s}{k_s + s + \frac{s^2}{k_i}}, \tag{24}$$

and $\rho(z_0) = v_0(x_0 + ys_0) - ys_{in}v_0$. The variables $x(t)$ and $s(t)$ describe, respectively, the biomass and the substrate concentrations in the tank, $v(t)$ the volume of water in the tank and $u(t)$ the input water flow. In addition y is the constant yield coefficient, s_{in} the constant substrate concentration in the input flow, k_s the affinity constant, k_i the inhibition constant and μ_0 the constant maximum specific growth rate. The function $\mu(s)$ models the microbiological growth rate, which is described using the Haldane law (24). The control u takes values in the interval $[0, u_{max}]$, with $u_{max} > 0$, and the variables x, s and v take non-negative values. Furthermore, when the maximum level of water in the tank v_{max} is reached the control u is set to zero. Differently from Moreno [1999], s_{in} is assumed constant.

The purpose of the bioreactor is to control the concentration s of the substrate in the tank below a specified level s_{min}, before the volume reaches v_f, where $0 < v(0) < v_f \leq v_{max}$. The cost functional to be minimized is the reaction time T, i.e., in (4) we have $\phi(x) = 1$ and $\gamma(x) = 0$. Thus, this is a minimum time optimal control problem.

The first step to develop the dynamic control law consists in computing the solution of the minimum time optimal control problem for the linearized system. The linearized system around (s_f, v_f) is described by equations of the form

$$\dot{x}_1 = -ax_1 + bx_2 + cu + d, \qquad \dot{x}_2 = u, \tag{25}$$

where $x_1 = s - s_f$, $x_2 = v - v_f$ and a, b, c and d are constants. Note that d is zero if the state (s_f, v_f) is an equilibrium point. To compute the value function for the linear problem, the system is integrated and the time t is eliminated, since the value function is exactly the time needed to reach the final state. From general results on linear systems it is known that the optimal control has at most one switching time and the control input assumes only the extreme values. With these observations only two non-trivial situations are possible: the control input is 0 until the switching time and u_{max} until the final time, or, the control input is u_{max} until the switching time and 0 thereafter. A simple analysis reveals that the first candidate optimal control does not yield positive switching times, whereas the second candidate optimal control yields a positive switching time for all states. Let V_ℓ be the value function of the linearized problem and $V_{\ell s}$ and $V_{\ell v}$ its partial derivatives with respect to s and v, respectively.

Note that

$$1 + V_{\ell v}u_{max} + V_{\ell s}\dot{s}_\ell = 0, \tag{26}$$

where $\dot{s}_\ell$ is the first of the equations (25) with $u = u_{max}$. Consistently with Definition 1 the solution for this problem relies on the solution of the algebraic equation

$$1 + (V_{\ell v} + \Delta_v)u_{max} + (V_{\ell s} + \Delta_s)[\theta + \pi u_{max}] = 0. \tag{27}$$

One $\widehat{W}$ solution is given by

$$\Delta_s = -V_{\ell s}, \qquad \Delta_v = \frac{V_{\ell s}\dot{s}_\ell}{u_{max}}.$$

According to (16) the modified value function is

$$V(s, v) = V_\ell(s, v) + \frac{V_{\ell s}(\xi_1, \xi_2)\dot{s}_\ell(\xi_1, \xi_2)}{u_{max}}v - V_{\ell s}(\xi_1, \xi_2)s + \frac{1}{2}R_1(s - \xi_1)^2 + \frac{1}{2}R_2(v - \xi_2)^2. \tag{28}$$

Assuming that there exist $R_1 > 0$ and $R_2 > 0$ such that $V(s, v)$ is locally positive definite, we compute the partial derivatives of V, namely V_{ξ_1}, V_{ξ_2}, V_s and V_v. Proposition 1 yields the dynamic control law [2]

$$\dot{\xi}_1 = -kV_{\xi_1}, \qquad \dot{\xi}_2 = -kV_{\xi_2},$$
$$u = \begin{cases} [1 - \text{sign}(V_s\pi + V_v)]\dfrac{u_{max}}{2}, & \text{if } V_s\pi + V_v \neq 0, \\ u_{singular}, & \text{if } V_s\pi + V_v = 0, \end{cases} \tag{29}$$

with $u_{singular}$ such that the derivative with respect time of $V_s\pi + V_v$ is identically equal to zero (note that condition (18) holds).

The closed-loop system (23)-(29) has been simulated with the parameters (see Moreno [1999] and Buitrón [1993]): $v_f = 50m^3$, $s_f = 1mgl^{-1}$, $u_{max} = 0.013888889m^3s^{-1}$, $K_s = 2mgl^{-1}$, $K_i = 50mgl^{-1}$, $\mu_0 = 0.00002s^{-1}$, $y = 0.5$ and $s_{in} = 300mgl^{-1}$. The initial conditions (s_0, v_0) have been selected in the regions $45 \leq s_0 \leq 55mgl^{-1}$ and $4.5 \leq v_0 \leq 5m^3$. The biomass concentration x_0 has been selected as $x_0 = 13000 + 3400(5 - v_0)mgl^{-1}$ to recover the case studied in Moreno [1999]. The gain of the dynamic control law has been defined as

$$k = \begin{cases} k_1 = \dfrac{1 + V_s\theta + (V_s\pi + V_v)u}{V_{\xi_1}^2 + V_{\xi_2}^2}, & \text{if } V_{\xi_1}^2 + V_{\xi_2}^2 \neq 0, \text{ and } k_1 > 0, \\ k_2 = \dfrac{10}{1 + V_{\xi_1}^2 + V_{\xi_2}^2}, & \text{otherwise.} \end{cases} \tag{30}$$

The selection $k = k_1$ is in accordance with Remark 4 and guarantees that Proposition 1 holds with $\rho_1 = 0$. Finally, the initial condition for the dynamic extension has been set to $(\xi_1(0), \xi_2(0)) = (1, v_0)$ and $R_1 = R_2 = 340$.

Simulations have been carried out with the optimal control law u^*, the optimal control law for the linearized system u_ℓ and the dynamic control law u_D. As already discussed the optimal control law for the linearized problem gives a *batch-strategy*: the reactor is filled as fast as possible and the reaction phase is stopped when $s(t)$ reaches s_f.

Fig. 1 shows the time histories of the control signals $u^*(t)$, $u_\ell(t)$ and $u_D(t)$ for the initial state $(s_0, v_0) = (50, 5)$. Note that the dynamic control approximately recovers the

[2] The different definition of u is to adapt the proposition, developed when there is a symmetric bound on the control, to this problem in which the bound is asymmetric.

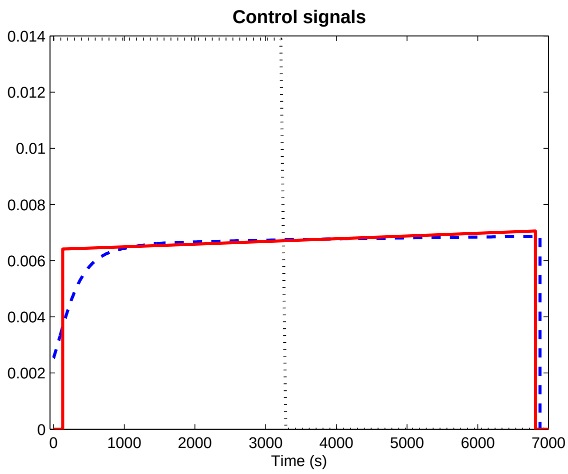

Fig. 1. Time histories of the control signals u^* (solid line), u_ℓ (dotted line) and u_D (dashed line) for $(s_0, v_0) = (50, 5)$.

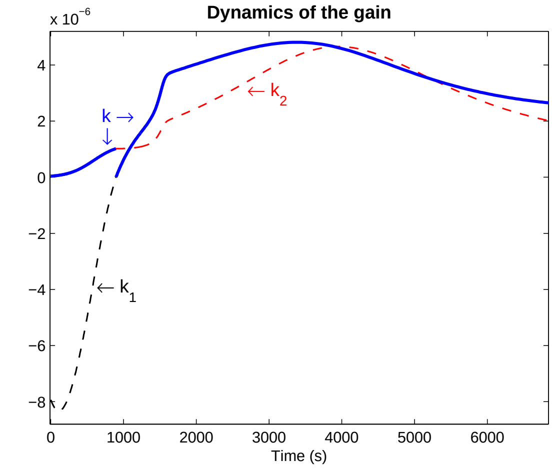

Fig. 2. Time history of the gain k (solid line) for the initial condition $(s_0, v_0) = (50, 5)$.

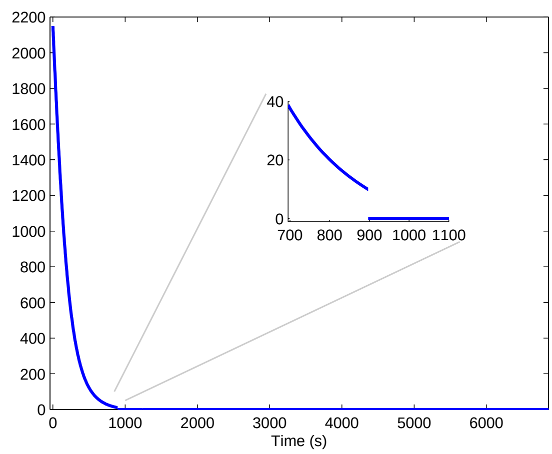

Fig. 3. Time history of the additional running cost ρ_1 for the initial condition $(s_0, v_0) = (50, 5)$.

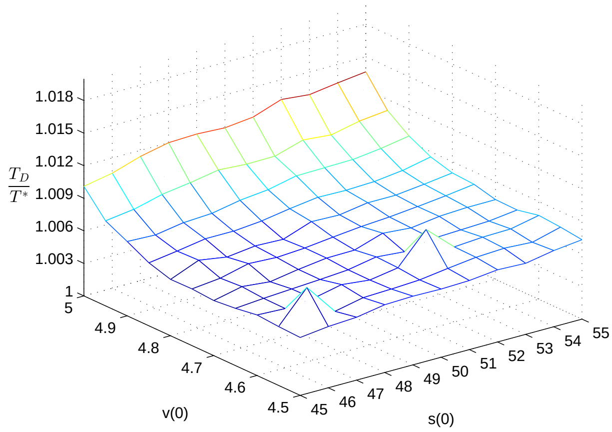

Fig. 4. Ratio between the final time T_D, resulting from the application of the dynamic control law, and the optimal final time T^* for $s_0 \in [45, 55]$ and $v_0 \in [4.5, 5]$.

optimal strategy. The final time for the optimal policy is 7071 seconds, for the dynamic control is 7154 seconds, and for the batch-strategy is 18524 seconds. Note the presence of the singular arc for both the optimal and the dynamic control law. Fig. 2 displays the time evolution of the gain k. Note that k is initially equal to k_2 and at approximatively 900 seconds it switches to k_1, as demonstrated by the discontinuity in the graph. This behavior is consistent with the one shown in Fig. 3 where the value of ρ_1 is positive when $k = k_2$ and identically equal to zero when $k = k_1$.

Fig. 4 shows the ratio $\dfrac{T_D}{T^*}$ for a range of initial conditions. Note that the dynamic control law yields performance similar to the optimal one and that it significantly outperforms the batch-strategy which gives an average ratio of 2.6.

5. CONCLUSION

The finite-horizon optimal control problem with input constraints for input-affine nonlinear systems has been studied. The problem has been solved by means of a dynamic extension yielding a combination of bang-bang signals and singular arcs. Simulations on a model of an industrial wastewater treatment plant have shown the performances of the dynamic control law, which are similar to the optimal strategies.

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